Quadratic approximation for derivatives at end points

Three points x0, x0+h, x0+2h with values of function u(x) u1=u(x0), u2=u(x0+h), u3=u(x0+2h). Consider the polynomial $p(x) = ax^2 + bx + c$ such that p(x0)=u1, p(x0+h)=u2, p(x0+2h)=u3. Then a, b, c are the solutions of:

In[8]:=
```
sols = Solve[{u1 == a x0^2 + b x0 + c, u2 == a (x0 + h)^2 + b (x0 + h) + c,
    u3 == a (x0 + 2 h)^2 + b (x0 + 2 h) + c}, {a, b, c}] // Simplify
```

Out[8]=
$$\left\{\left\{a \to \frac{u1 - 2\,u2 + u3}{2\,h^2},\ b \to -\frac{1}{2\,h^2}\left(h\,(3\,u1 - 4\,u2 + u3) + 2\,(u1 - 2\,u2 + u3)\,x0\right),\ c \to \frac{1}{2\,h^2}\left(2\,h^2\,u1 + h\,(3\,u1 - 4\,u2 + u3)\,x0 + (u1 - 2\,u2 + u3)\,x0^2\right)\right\}\right\}$$

The constant 2a is the value of p’’(x) for all values of x. The first derivative p’(x)=2ax+b can be calculated at each point.

First Derivative at middle point:

In[9]:=
```
2 a (x0 + h) + b /. sols // Simplify
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Out[9]=
$$\left\{\frac{-u1 + u3}{2\,h}\right\}$$

First derivative at right most point x0+2h

In[10]:=
```
2 a (x0 + 2 h) + b /. sols // Simplify
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Out[10]=
$$\left\{\frac{u1 - 4\,u2 + 3\,u3}{2\,h}\right\}$$

First derivative at leftmost point x0

In[11]:=
```
2 a (x0) + b /. sols // Simplify
```

Out[11]=
$$\left\{-\frac{3\,u1 - 4\,u2 + u3}{2\,h}\right\}$$

The approximation is $O\left(h^2\right)$ for both end points:

In[15]:=
$$\text{Series}\left[-\frac{3\,u[x0] - 4\,u[x0 + h] + u[x0 + 2\,h]}{2\,h}, \{h, 0, 3\}\right]$$

Out[15]=
$$u'[x0] - \frac{1}{3}\,u^{(3)}[x0]\,h^2 - \frac{1}{4}\,u^{(4)}[x0]\,h^3 + O[h]^4$$

In[16]:=
$$\text{Series}\left[\frac{u[x0 - 2\,h] - 4\,u[x0 - h] + 3\,u[x0]}{2\,h}, \{h, 0, 3\}\right]$$

Out[16]=
$$u'[x0] - \frac{1}{3}\,u^{(3)}[x0]\,h^2 + \frac{1}{4}\,u^{(4)}[x0]\,h^3 + O[h]^4$$

For the second derivative, the approximation is $O\left(x^2\right)$ only for the middle point. For the left/ right points are only $O(h)$.

In[17]:= $\text{Series}\left[\frac{u[x0-2h]-2u[x0-h]+u[x0]}{h^2},\{h,0,3\}\right]$

Out[17]= $u''[x0]-u^{(3)}[x0]\,h+\frac{7}{12}u^{(4)}[x0]\,h^2-\frac{1}{4}u^{(5)}[x0]\,h^3+O[h]^4$

In[18]:= $\text{Series}\left[\frac{u[x0-h]-2u[x0]+u[x0+h]}{h^2},\{h,0,3\}\right]$

Out[18]= $u''[x0]+\frac{1}{12}u^{(4)}[x0]\,h^2+O[h]^4$

In[19]:= $\text{Series}\left[\frac{u[x0]-2u[x0+h]+u[x0+2h]}{h^2},\{h,0,3\}\right]$

Out[19]= $u''[x0]+u^{(3)}[x0]\,h+\frac{7}{12}u^{(4)}[x0]\,h^2+\frac{1}{4}u^{(5)}[x0]\,h^3+O[h]^4$

Look for $O\left(h^2\right)$ solutions for the left point:

In[24]:= $\text{Series}\left[\frac{1}{h^2}\,\alpha\,u[x0]+\beta\,u[x0+h]+\gamma\,u[x0+2h]+\delta\,u[x0+3h],\{h,0,3\}\right]$

Out[24]= $\frac{\alpha\,u[x0]+\beta\,u[x0]+\gamma\,u[x0]+\delta\,u[x0]}{h^2}+\frac{\beta\,u'[x0]+2\gamma\,u'[x0]+3\delta\,u'[x0]}{h}+$
$\frac{1}{2}(\beta\,u''[x0]+4\gamma\,u''[x0]+9\delta\,u''[x0])+\frac{1}{6}\left(\beta\,u^{(3)}[x0]+8\gamma\,u^{(3)}[x0]+27\delta\,u^{(3)}[x0]\right)h+$
$\frac{1}{24}\left(\beta\,u^{(4)}[x0]+16\gamma\,u^{(4)}[x0]+81\delta\,u^{(4)}[x0]\right)h^2+$
$\frac{1}{120}\left(\beta\,u^{(5)}[x0]+32\gamma\,u^{(5)}[x0]+243\delta\,u^{(5)}[x0]\right)h^3+O[h]^4$

In[29]:= $\text{sols2} = \text{Solve}\left[\{\alpha+\beta+\gamma+\delta==0,\ \beta+2\gamma+3\delta==0,\ \beta/6+8\gamma/6+27\delta/6==0\},\{\alpha,\beta,\gamma,\delta\}\right]$

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[29]= $\left\{\left\{\beta\to-\frac{5\alpha}{2},\ \gamma\to2\alpha,\ \delta\to-\frac{\alpha}{2}\right\}\right\}$

In[32]:= $\text{Series}\left[\frac{1}{h^2}\,\alpha\,u[x0]+\beta\,u[x0+h]+\gamma\,u[x0+2h]+\delta\,u[x0+3h]\ /.\ \text{sols2}\ /.\ \alpha\to2,\{h,0,3\}\right]$

Out[32]= $\left\{u''[x0]-\frac{11}{12}u^{(4)}[x0]\,h^2-u^{(5)}[x0]\,h^3+O[h]^4\right\}$

In[33]:= $\text{sols2}\ /.\ \alpha\to2$

Out[33]= $\{\{\beta\to-5,\ \gamma\to4,\ \delta\to-1\}\}$

In[37]:=

```
secondl = (α u[x0] + β u[x0 + h] + γ u[x0 + 2 h] + δ u[x0 + 3 h]) / h^2 /. sols2 /. α → 2
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Out[37]=

$$\left\{\frac{2\,u[x0]-5\,u[h+x0]+4\,u[2\,h+x0]-u[3\,h+x0]}{h^2}\right\}$$

Verify:

In[38]:=

```
Series[secondl, {h, 0, 3}]
```

Out[38]=

$$\left\{u''[x0]-\frac{11}{12}\,u^{(4)}[x0]\,h^2-u^{(5)}[x0]\,h^3+O[h]^4\right\}$$

Look for $O\left(h^2\right)$ solutions for the right point: We see that the solution is completely symmetric by taking $h \to -h$

In[36]:=

```
Series[(α u[x0] + β u[x0 - h] + γ u[x0 - 2 h] + δ u[x0 - 3 h]) / h^2, {h, 0, 3}]
```

Out[36]=

$$\frac{\alpha\,u[x0]+\beta\,u[x0]+\gamma\,u[x0]+\delta\,u[x0]}{h^2}+\frac{-\beta\,u'[x0]-2\,\gamma\,u'[x0]-3\,\delta\,u'[x0]}{h}+$$
$$\frac{1}{2}\left(\beta\,u''[x0]+4\,\gamma\,u''[x0]+9\,\delta\,u''[x0]\right)+\frac{1}{6}\left(-\beta\,u^{(3)}[x0]-8\,\gamma\,u^{(3)}[x0]-27\,\delta\,u^{(3)}[x0]\right)h+$$
$$\frac{1}{24}\left(\beta\,u^{(4)}[x0]+16\,\gamma\,u^{(4)}[x0]+81\,\delta\,u^{(4)}[x0]\right)h^2+$$
$$\frac{1}{120}\left(-\beta\,u^{(5)}[x0]-32\,\gamma\,u^{(5)}[x0]-243\,\delta\,u^{(5)}[x0]\right)h^3+O[h]^4$$

```
sols3 = Solve[{α + β + γ + δ == 0, β - 2 γ - 3 δ == 0, β / 6 + 8 γ / 6 + 27 δ / 6 == 0}, {α, β, γ, δ}]
```