

The Tensor Track X: Renormalization and Tensorial Complexity

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Workshop on Non-commutative and Generalised Geometries and their Physical Applications



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Introduction

During this talk, I shall present the thesis of my last PhD student, Parham Radpay, entitled “Two Approaches towards Tensorial Complexity: From Functional Renormalization to Kac-Rice”.

I thank also Vincent Lahoche, who has been not only my former student, but also a supervisor during Parham’s PhD in all but name.

Assuming some familiarity with quantum field theory and the tensor track approach presented in the previous series, “Tensor Track I-IX”, I introduce the basics of renormalization à la Wetterich.

Next, I presented a summary of the PhD of Radpay.

But during the preparation of my talk, I realized that the PhD of Radpay is all about condensed matter, and more specifically, various models of spin-glasses.

It's a bit far from the topic of this workshop...

Therefore, as a conclusion, I give you a brief overview of the articles I'm still working on, which all deal with the concept of cumulants from a perspective of constructive quantum field theory.

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Wilson-Polchinski

In the Wilsonian approach, one defines a scale-dependent Wilsonian effective action $S_k[\phi]$ in which modes above k have been integrated out. The flow with k of S_k can be written as an exact functional differential equation.

In the Polchinski formulation, the flow is expressed directly for the effective interaction functional, as high-momentum shells are progressively eliminated.

Conceptually, the Wilsonian action thus emphasizes the effective measure for the remaining slow modes and is adapted to coarse-graining procedures, in which one simply integrates out degrees of freedom.

Functional viewpoint

In the functional renormalization group (FRG), the central object is a scale-dependent one particle irreducible (1PI) effective action $S_k[\phi]$, called the effective average action (EAA).

It interpolates smoothly between a microscopic (UV) action and the full effective action: at the UV scale, fluctuations are suppressed and Γ_k is close to the microscopic action, whereas, as $k \rightarrow 0$, all fluctuations are included and $\Gamma_{k \rightarrow 0}$ becomes the standard effective action.

Wetterich equation

Wetterich introduces an infrared (IR) regulator term

$$\Delta S_k[\phi] = \frac{1}{2} \int_q \phi(-q) R_k(q) \phi(q)$$

where a convenient way to implement this is (Litim regulator):

$$R_k(q) = (k^2 - q^2) \theta(k^2 - q^2).$$

The Wetterich equation is

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{Tr} [(\Gamma_k^2[\phi] + R_k)^{-1} \partial_k R_k],$$

where $\Gamma_k^2[\phi]$ is the second functional derivative of Γ_k and the trace includes momentum and any internal indices. The crucial structural feature is that this equation has a one-loop form, but with **full scale-dependent inverse propagator**, and FRG remains usable **deep in non-perturbed regimes**.

Spin glasses

Standard examples of spin glasses can be approached using classical tools, primarily the mean field techniques of statistical physics.

This is justifiable, as in ferromagnetic theory, because the transition temperature and the corresponding typical energy scale are far from the typical energy scale of quantum fluctuations.

Therefore, quantum transition physics typically operates at zero temperature, where quantum fluctuations can disrupt long-range order.

Parisi and co have introduced, in their work about spin glasses, the $N \rightarrow 0$ limit.

Classical versus Quantum

The transition between the classical and quantum spin glasses can be depicted by the following Figure:

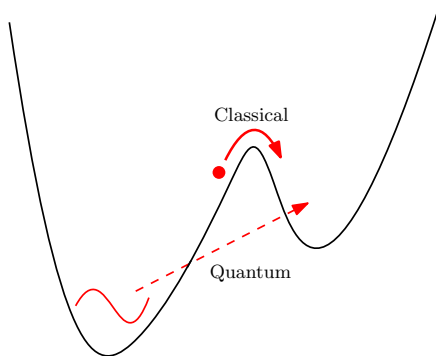


Figure: Quantum tunneling versus thermal annealing.

FRG in disordered systems

Disordered systems introduce an additional layer of structure because the physically relevant free energy involves a quenched average over disorder realizations.

In the EAA framework this means that Γ_k becomes a functional of replicated fields $\{\phi_a\}_{a=1}^n$.

Next we shall expand Γ_k in the number of free replica sums, which corresponds to an expansion in disorder cumulants:

$$\begin{aligned}\Gamma_k[\{\phi_a\}] &= \sum_a \Gamma_{1,k}[\phi_a] - \frac{1}{2} \sum_{a,b} \Gamma_{2,k}[\phi_a, \phi_b] \\ &\quad + \frac{1}{3!} \sum_{a,b,c} \Gamma_{3,k}[\phi_a, \phi_b, \phi_c] + \dots\end{aligned}$$

Here $\Gamma_{1,k}$ is the disorder-averaged effective action for one copy, and so on.

Large N

Large-N limits are a natural setting for FRG.

For glassy and disordered problems, large-N can clarify the relationship between mean-field landscapes (with exponentially many metastable states) and spatial coarse-graining.

In particular, large-N limits can exhibit sharp transitions to complexity, i.e. the structure of solutions to the flow for U_k and related objects changes qualitatively, while still allowing for explicit computation within FRG.

But this large-N limit could clash with an other limit, that of Parisi. If this issue concerns you, you should listen the rest of this talk and, far more more importantly, you should reading Parham's PhD.

Random Matrices

Random matrix theory (RMT) studies the spectral and geometric properties of large matrices whose entries are random.

One of the main principles in RMT is that many spectral observables are universal, i.e. once the matrix size is large and the entries are weakly dependent, the fine details of the distribution matter far less than symmetry class, scaling, and a few macroscopic parameters.

This makes random matrices a natural language for disorder, complexity, and the large- N limits that repeatedly appear in spin glasses and modern high-dimensional inference.

Random Matrices, continued

Here, random matrices serve two additional conceptual roles.

First, they provide an example of self-averaging spectra, which justifies treating eigenvalue distributions as deterministic objects at large N and motivates spectral coarse-graining.

Second, they suggest a way to reinterpret eigenvalues as generalized momenta, thus connecting spectral information to propagators and to **renormalization-group ideas**.

Wigner Flow

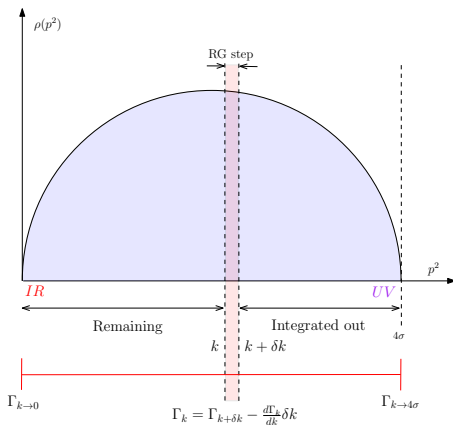


Figure: A summary of the coarse-graining scheme defining the Wigner RG flow. Modes are partially integrated from UV to IR, and the effective average action changes at each step

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Beyond matrices: tensors

The move from random matrices to random tensors is central for PhD of Radpay. Conceptually, it amounts to replacing linear operators by multilinear forms.

Analytically, the canonical spectral toolbox for Hermitian matrices (eigenvalues, resolvents, Stieltjes transforms) **no longer functions**.

Fix an integer $p \geq 3$ and $N \in \mathbb{N}$. A (real) order- p tensor is an array

$$T = (T_{i_1, \dots, i_p})_{1 \leq i_1, \dots, i_p \leq N} \in (\mathbb{R}^N)^{\otimes p}$$

which may be viewed as a multilinear form

$$T(x^{(1)}, \dots, x^{(p)}) = \sum_{i_1, \dots, i_p=1}^N T_{i_1, \dots, i_p} x_{i_1}^{(1)} \cdots x_{i_p}^{(p)}$$

The p -spin glass model

We consider the dynamics of an N -dimensional quantum particle moving through a rugged landscape represented by a rank- p Gaussian random tensor, $\mathbf{J}$. The components $J_{i_1, \dots, i_p}$ of the Gaussian tensor are independent random variables with zero mean, and their variance is defined as follows:

$$\overline{J_{i_1 \dots i_p} J_{i'_1 \dots i'_p}} = \left(\frac{\lambda p!}{N^{p-1}} \right) \prod_{\ell=1}^p \delta_{i_\ell i'_\ell}.$$

The classical variable $\mathbf{x} \in \mathbb{R}^N$ corresponds to the classical position of the particle, which is described in the point of view of quantum theory by the wave function $\Psi(\mathbf{x}, t)$, satisfying the Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{x}, t) = \hat{H}_{SG} \Psi(\mathbf{x}, t).$$

The p -spin glass model

The Hamiltonian $\hat{H}_{\text{SG}}$ is furthermore assumed to be of the form:

$$\hat{H}_{\text{SG}} = \frac{\hat{\mathbf{p}}^2}{2m} + \sum_{i_1 < i_2 \dots < i_p} J_{i_1 i_2 \dots i_p} \hat{x}_{i_1} \hat{x}_{i_2} \dots \hat{x}_{i_p} + V(\hat{\mathbf{x}}^2).$$

Note that as for the Hamiltonian, we denoted the momentum and position components operators with a hat notation, to distinguish them with their classical analogue. In particular, they satisfy the non vanished canonical commutation relations $[\hat{p}_k, \hat{x}_j] = i\hbar\delta_{kj}$.

The p -spin glass model

We choose for V :

$$V(\mathbf{x}^2) = \frac{1}{2}\mu_1 \mathbf{x}^2 + \frac{\mu_2}{4N} (\mathbf{x}^2)^2,$$

such that for $\mu_1 < 0$ and $\mu_2 > 0$, there are two stables minima, and the constraint writes as:

$$\mathbf{x}_0^2(t) = -\frac{\mu_1}{\mu_2} N,$$

enforcing dynamically the spherical constraint.

The $p = 2$ model, fixed points

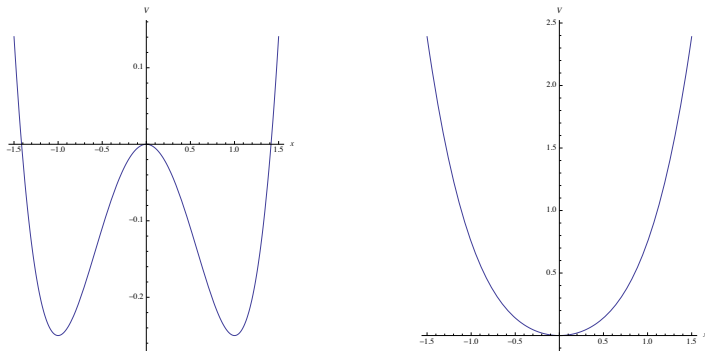


Figure: Typical shape of the potential ($N = 1$) for $\mu_1 < 0$ (on left) and $\mu_1 > 0$ (on right).

The $p = 2$ model, fixed points

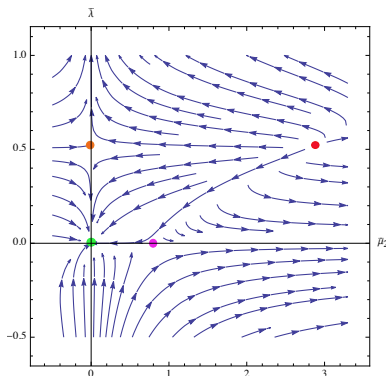


Figure: Fixed point structure of the renormalization group flow in the deep IR, fixed points FP1, FP2 and FP3 are respectively materialized by the red, purple and orange points, the green point is the Gaussian fixed point. Coupling $\bar{\mu}_2$ has been rescaled by a factor 2π for convenience.

The $p = 3$ model

Now, let us turn to the case $p = 3$, i.e. the **random tensorial spin-glass model**. In this scenario, the non-local interaction is of sixth order in the microscopic action. In the symmetric phase, and based on similar reasoning as before, non-local quartic interactions cannot be generated in the large N limit.

Therefore, the vertex expansion for the interaction part of the effective average action, graphically, is:

$$\Gamma_{k,\text{int}} = \underbrace{\left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \right)}_{\text{Local}} + \underbrace{\left(\text{diagram 4} + \text{diagram 5} + \dots \right)}_{\text{Bi-local}}.$$

The diagrammatic expansion shows the vertex $\Gamma_{k,\text{int}}$ as a sum of two parts. The first part, labeled 'Local', is enclosed in a large bracket and contains a series of diagrams: a circle with two horizontal lines, an oval with three horizontal lines, an oval with four horizontal lines, and an ellipsis. The second part, labeled 'Bi-local', is also enclosed in a large bracket and contains a series of diagrams: two overlapping ovals with horizontal lines, two overlapping ovals with horizontal lines, and an ellipsis.

Papers of Radpay

- Lahoche, V., Samary, D. O., & Radpay, P. (2025). Large time effective kinetics β -functions for quantum $(2+p)$ -spin glass. *Annals of Physics*, 480, 170102.
- Achitouv, I., Lahoche, V., Ousmane Samary, D., & Radpay, P. (2026). Time-translation invariance symmetry breaking hidden by finite-scale singularities. *Physica Scripta*, 101(11), 115201.
- Lahoche, V., Ousmane Samary, D., & Radpay, P. (2025). The quantum p -spin renormalization group in the large N limit as a benchmark for functional renormalization group. *Journal of Statistical Mechanics*: 2025(9), 093301.
- Natta, B. B. B., Lahoche, V., Samary, D. O., & Radpay, P. Ward identities and local potential approximation for large time quantum $(2+p)$ -spin glass dynamics. [arXiv:2411.11089](https://arxiv.org/abs/2411.11089).

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Conclusion

The sharpest results with respect to the cumulants from a constructive perspective were obtained in the work of **Gurau and Krajewski**, in their paper of 2015 “Analyticity results for the cumulants in a random matrix model” Annales de l’Institut Henri Poincaré D. For everything related to cumulants, I work in their footsteps.

- I would like to develop the variational approach of Vasily Sazonov in his paper of 2025, Variational Loop Vertex Expansion, JHEP 4, 032. (I have already made one paper, Variational Loop Vertex Expansion for Cumulants, arXiv:2606.03856, to fill out Vasily’s paper to cumulants.)

Conclusion

I rather recklessly embarked on a paper entitled “Multiscale Loop Vertex Expansion for Cumulants, the ϕ_2^4 Model” (arXiv:2608.27188).

For this, I have befitting greatly of three papers:

- Magnen and me, Constructive ϕ^4 Field Theory without Tears, Annals Henri Poincaré, 9 (2008)
- Gurau and me, The Multiscale Loop Vertex Expansion, Annals Henri Poincaré, 16 (2015)
- Zhituo and me, Corrected Loop Vertex Expansion for ϕ_2^4 Theory, Journal of Mathematical Physics, 56 (6) (2015)

Conclusion

I would like to also fill out various results who only deals about free energy to cumulants, to extend to Hermitian, symmetric, symplectic matrices, etc...

For the more distant future, maybe I should write a book formed by conjectures that have resisted in good old constructive quantum field theory?

Thanks for your attention!