

KITAEV'S QUANTUM DOUBLE MODEL AND STRINGS IN MAGNETIC FIELDS

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Workshop on Noncommutative and Generalised Geometries
and their Physical Applications

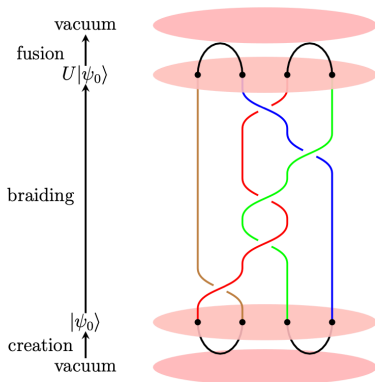
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Topological quantum computing

Quantum computing: use the power of the superposition principle

$$\sum_x c_x |x\rangle \xrightarrow{U_f} \sum_x c_x |f(x)\rangle$$

Problems (decoherence, error correction) that can be solved using robustness of topological states



- anyons with labels $a, b, c..$ in dimension $2 + 1$
- non abelian representation of the braid group
- computation = braiding
- measurement = fusing
 $ab = \sum_c N_{ab}^c c$

picture from lectures by [Cui 18](#)
see also [Preskill 04](#) and [Simon 17](#)

Kitaev's quantum double model

Model based on an oriented ribbon graph $\Gamma = (E, V, F)$ with Hilbert space $L^2(G^E)$ and Hamiltonian (Kitaev 94)

$$H = \sum_v \mu_v (1 - P_v) + \sum_f \nu_f (1 - Q_f) \quad \mu_v, \nu_f \geq 0$$

- P_v projects onto gauge invariant states at vertex v

$$P_v \Psi(\{g_e\}) = \frac{1}{|G|} \sum_{g \in G} \Psi(\{g \triangleright_{e \ni \partial v} g_e\})$$

$g \triangleright_{e \ni \partial v} g_e = g^{-1} g_e$ (e leaving v) or $g_e g$ (e entering v)

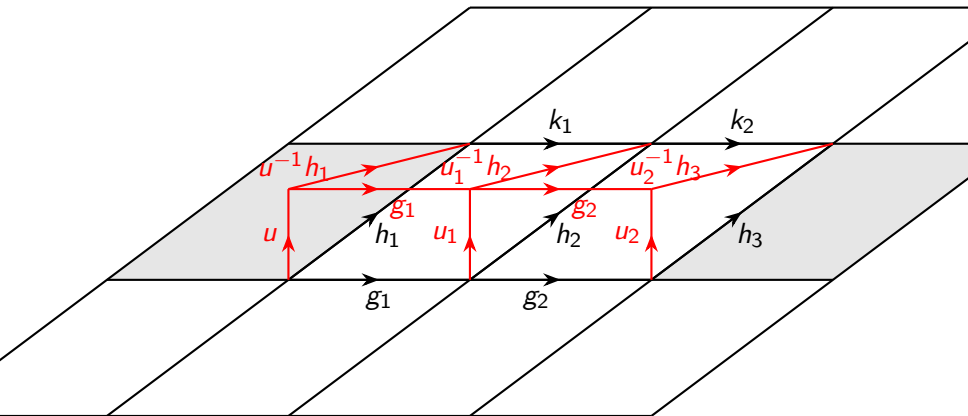
- Q_f implements the flatness constraint around the face f

$$Q_f \Psi(\{g_e\}) = \delta\left(\prod_{e \in \partial f} g_e^{\pm}\right) \Psi(\{g_e\})$$

P_v, Q_f mutually commuting projectors $\Rightarrow$ ground state obeying $P_v |\Psi_0\rangle = 0$ and $Q_f |\Psi_0\rangle = 0$ for all v and f

Excitations

Elementary excitation created by ribbon operators $\mathcal{L}_g^u(v_1, f_1 | v_2, f_2)$



Lift by u parallel transported $u_1 = g_1^{-1}ug_1$, $u_2 = (g_1g_2)^{-1}ug_1g_2$ to ensure flatness (except on boundary) and multiply by $\delta_{g_1g_2,g}F(u)$, invariant under homotopy with fixed boundary

Drinfel'd quantum double of a finite group

Ribbon operators belong to $D[G]$ (quasi-triangular Hopf algebra)
with basis $w \square_g$

- product (concatenation of ribbons)

$$w \square_g v \square_h = \delta_{w, g^{-1}vg} w \square_{gh}$$

- coproduct allowing tensor product of representations
(superposition of ribbons)

$$\Delta(w \square_g) = \sum_{uv=w} u \square_g \otimes v \square_g$$

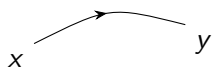
- R -matrix (braiding, $\sigma = (\text{flip}) \circ R$)

$$\Delta^{\text{op}} = R \Delta R^{-1} \quad \text{with} \quad R = \sum_{u,v} u \square_e \otimes v \square_u$$

Suitable framework for topological quantum computing
([Kitaev 94](#), see also the lectures [Cui 18](#) and [Lo et al 25](#))

Particle in a magnetic field

Particle on X in a magnetic field B propagating from x to y


$$\exp i \int A \quad \text{with } dA = B$$

if G acts on X and B invariant, A only invariant up to gauge transformation: $d(g^*A - A) = g^*B - B = 0 \Rightarrow g^*A - A = d\phi_g$
Propagation commutes with magnetic translations

$$T_g \psi(x) = \exp \{ -i\phi_g(x) \} \psi(xg)$$

Magnetic translation define a projective representation of G

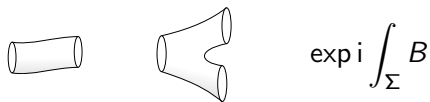
$$T_g T_h = \omega_{g,h} T_{gh}$$

with $\omega_{g,h} = \exp -i\{g^*\phi_h - \phi_{gh} + \phi_g\}$ a constant 2-cocycle
 $\delta\omega_{g,h,k} = \omega_{h,k} \omega_{gh,k}^{-1} \omega_{h,gk} \omega_{g,h}^{-1} = 1$ ensuring associativity

$$(T_g T_h) T_k = T_g (T_h T_k)$$

Strings in an invariant magnetic field

For string in a magnetic field (3-form), with 2-form potential B ,
 $C = dB$



If C is invariant, B is not necessarily invariant but there exist
 1-form A_g and 0-form $\Phi_{g,h}$, defined up to gauge transformations

$$C = dB \quad B \rightarrow B + d\Lambda$$

$$g^*B - B = dA_g \quad A_g \rightarrow A_g + g^*\Lambda - \Lambda + d\Theta_g$$

$$g^*A_h - A_{gh} + A_g = d\Phi_{g,h} \quad \Phi_{g,h} \rightarrow \Phi_{g,h} + g^*\Theta_h - \Theta_{gh} + \Theta_g$$

leading to a constant 3-cocycle (lack of associativity)

$$\omega_{g,h,k} = \exp -i \{ g^*\Phi_{h,k} - \Phi_{gh,k} + \Phi_{g,hk} - \Phi_{g,h} \}$$

Holds for X finite with n -forms antisymmetric functions on X^{n+1}

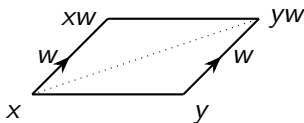
$$d\Omega_n(x_0, \dots, x_n) = \sum_i (-1)^i \Omega_n(x_0, \dots, \check{x}_i, \dots, x_n)$$

$$g^*\Omega_n(x_0, \dots, x_n) = \Omega_n(x_0g, \dots, x_ng)$$

Magnetic translation of twisted sectors

Magnetic amplitude for a finite string twisted sector (x, xw) propagating to (y, yw) with $x, y \in X$ and winding $w \in G$

$$\exp i\{B(x, y, yw) + B(yw, xw, x) + A_w(x, y)\}$$



Propagation commutes with T_g^w (winding $g^{-1}wg \rightarrow w$)

$$T_g \psi(x, xw) = \exp -iA_g(x, xw) \frac{\exp i\phi_{w,g}(x)}{\exp i\phi_{g,g^{-1}wg}(x)} \psi(xg, xwg)$$

obeying the associative multiplication law

$$T_g^w T_h^v = \delta_{g^{-1}vg, w} \frac{\omega_{w,g,h} \omega_{g,h,(gh)^{-1}wgh}}{\omega_{g,g^{-1}wg,h}} T_{gh}^w$$

Multiplication in the twisted Drinfel'd double $D_\omega[G]$ (Dikgraaf, Pasquier, Roche 91)

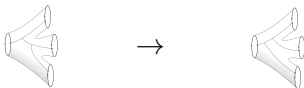
Quasi-Hopf structure from strings

Compatibility with string amplitudes (Jureit, Krajewski 06)

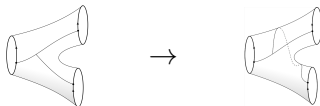
- Coproduct from decay $\mathcal{H}_{uv} \rightarrow \mathcal{H}_u \otimes \mathcal{H}_v$

$$\Delta T_g^w = \sum_{u,v \in G} \delta_{w,uv} \frac{\omega_{u,v,g} \omega_{g^{-1}ug, g^{-1}vg, g}}{\omega_{u, g^{-1}vg, g}} T_g^u \otimes T_g^v$$

with Drinfel'd associator $\Omega = \sum_{u,v,w} \omega_{u,v,w} T_e^u \otimes T_e^v \otimes T_e^w$
 $(\mathcal{H}_u \otimes \mathcal{H}_v) \otimes \mathcal{H}_w \rightarrow \mathcal{H}_u \otimes (\mathcal{H}_v \otimes \mathcal{H}_w)$



- Braiding $\sigma : \mathcal{H}_u \otimes \mathcal{H}_v \rightarrow \mathcal{H}_{uvu^{-1}} \otimes \mathcal{H}_u$

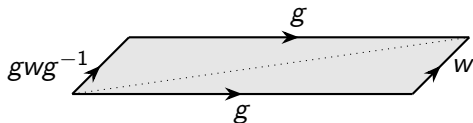


- Antipode from orientation reversing

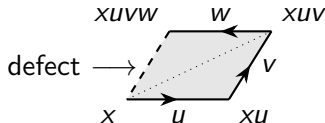
Hilbert space and Hamiltonian

Hilbert space $\mathcal{H} = L^2(G^{2|E|} \times X^{|V|})$ associated to an oriented ribbon graph $\Gamma = (E, V, F)$ with basis $|\{g_e, w_e, x_v\}\rangle$

- Every edge is a triangulated rectangle with gauge variable g_e , incoming $g_e^{-1}w_e g_e$ and outgoing w_e windings



- Every vertex with n incident edges is a triangulated polygon with $n + 1$ sides, a corner carrying $x_v \in X$, next n sides attached to windings $w_1, \dots, w_n$ and $x_v w_1 \cdots w_i$ at corner i , with defect $w_1, \dots, w_n$ on last edge



Projection onto magnetic amplitude

For a vertex v with defect $w_v = w_1 \cdots w_n$, as an incoming string

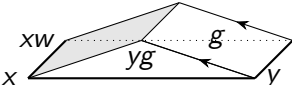
$$\Pi_v \Psi(\{g_e, w_e, x_v\}) = \frac{1}{|X|} \sum_{y \in X} \exp i C(\text{tetrahedra on } y, x_v, \{g_e, w_e\}_{v \ni e})$$

$$\exp i \{B(x_v, y, yw_v) + B(yw_v, x_v w_v, x_v) + A_{w_v}(x_v, y)\} \Psi(\{g_e, w_e, x_{v' \neq v}, y\})$$

Mutually commuting projectors such that $\prod_{v \in V} \Pi_v$ projects on magnetic amplitudes on the Γ : $\Psi_{\text{string}}(\{g_e, w_e, y_v\}) \times F(\{g_e, w_e\})$

- triangle in a polygonal vertex $x \triangleleft_{xu}^{xuv}$

$$\exp i \Phi_{u,v}(x) \times \exp i B(x, xu, xuv)$$

- edge 

$$\exp i \{B(x, yg, ygw) + B(ygw, xw, x) + A_w(x, yg)\}$$

$$\times \exp -i A_g(y, yw) \times \frac{\exp i \phi_{w,g}(y)}{\exp i \phi_{g,g^{-1}wg}(y)}$$

Spin network expression

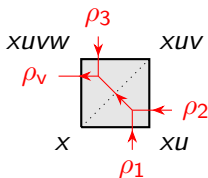
Irreducible representations of $D_\omega[G]$ are $\rho = (C, r)$ with C a conjugation class and r a projective representation of any centralizer of $w \in C$ and obey

$$\sum_b \mathcal{D}_{a,b}^\rho \left(\square_g^w \right) \mathcal{D}_{b,c}^\rho \left(\square_h^v \right) = \frac{\delta_{gvg^{-1},w} \omega_{w,g,h} \omega_{g,h,(gh)^{-1}wgh}}{\mathcal{D}_{a,c}^\rho \left(\square_{gh}^w \right) \omega_{g,g^{-1}wg,h}}$$

They define an orthonormal basis of $L^2(G^2)$ (Peter-Weyl theorem)

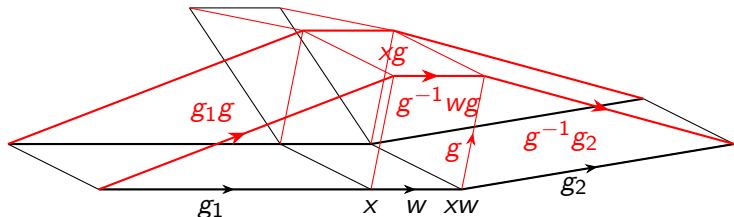
$$\frac{1}{|G|} \sum_{g,w} \sqrt{d_\rho} \mathcal{D}_{a,b}^\rho(g, w) \sqrt{d_{\rho'}} \overline{\mathcal{D}}_{c,d}^{\rho'}(g, w) = \delta_{\rho,\rho'} \delta_{a,c} \delta_{b,d}$$

Spin network basis of $L^2(G^{2|E|} \times X^{|V|})$ by contracting with intertwiners $\mathcal{I}_{v,\rho_1,\dots,\rho_n,\rho_v}$, with ρ_v associated to the total winding



Finite group gauge transformations at vertices

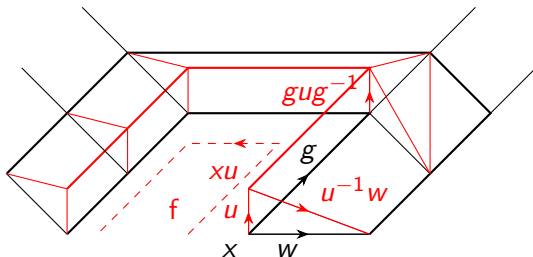
$$\begin{aligned}
 U_{v,g} \Psi(\{g_e, w_e, x_{v'}\}) &= \Psi(\overbrace{\{g \triangleright_v g_e, g \triangleright_v w_e, g \triangleright_v x_{v'}\}}^{\text{action on variable at vertex } v}) \\
 &\times \underbrace{\exp i A_g(x_{v'}, x_{v'} w) \times \frac{\exp i \phi_{g, g^{-1} w g}(x_v)}{\exp i \phi_{w, g}(x_v)}}_{\text{magnetic translation of string}}
 \end{aligned}$$



If $\Pi_v \Psi = \Psi$ for all v , then $U_{v,g} \Psi = (\prod \omega) \Psi$ with $\prod \omega \in U(1)$ corresponding to products, coproducts and antipodes of elements in $D_\omega[D]$ that can be compensated in internal space $\{g_e, w_e\}$

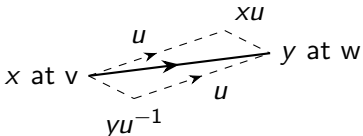
String insertions on faces

$h_f = \prod_{e,v \in f} (g_e^\pm \text{ or } w_v^\pm)$ holonomy including defect at vertices
 $u \rightarrow u_1 \rightarrow \dots$ parallel transported around the face by inner automorphism with $h_f u = u h_f$ (closing condition)



$$\begin{aligned}
 D_{f,\rho} \Psi(\{g_e, w_e, x_v\}) &= \sum_{\substack{u \in G \\ h_f u = u h_f}} \Psi(\overbrace{\{g_e, u \triangleright_f w_e, u \triangleright_f x_v\}}^{\text{action on variable around } f}) \\
 &\times \underbrace{\prod \exp i C}_{\text{magnetic flux on tube}} \times \underbrace{\Psi_{\text{string}}(\{g_e, w_v, u, x_v\})}_{\text{string on lateral side}} \times \underbrace{\text{Tr} \prod_{g_i} \mathcal{D}(u_i \square)}_{\text{representation } \rho}
 \end{aligned}$$

Simple $D = 1$ model

Single edge oriented from v to w 

$$\Pi_v \Psi(x, y, u) = \frac{1}{|X|} \sum_{x'} \exp i \left\{ -B(x, x', yu^{-1}) + A'(x, x') \right\} \Psi(x', y, u)$$

$$\Pi_w \Psi(x, y, u) = \frac{1}{|X|} \sum_{y'} \exp i \left\{ B(y, y', xu) - A'(y, y') \right\} \Psi(x, y', u)$$

projected on $(\mathcal{D}^\rho$ projective irrep of G with ω , basis of $L^2(G))$

$$\Psi_{ij}^\rho(x, y, g) = \exp i \left\{ A(xu, y) + \Phi_u(x) \right\} \mathcal{D}_{ij}^\rho(u)$$

Under a gauge transformation (invariant states given by characters $\text{Tr } D^\rho(g)$ on closed graphs $x = y, g = h$)

$$\begin{aligned} U_{v,g} U_{w,h} \Psi_{ij}^\rho(x, y, u) &= \Psi_{ij}^\rho(xg, yh, g^{-1}uh) \exp i \left\{ \Phi_g(x) - \Phi_h(y) \right\} \\ &= \sum_{k,l} \mathcal{D}_{lj}(h) \frac{1}{\omega_{g,g^{-1}}} \mathcal{D}_{ik}(g^{-1}) \Psi_{kl}^\rho(x, y, u) \end{aligned}$$

Hamiltonian and symmetries

The Hamiltonian is a sum of mutually commuting projectors

$$H = \sum_v \lambda_v (1 - \Pi_v) + \sum_v \mu_v (1 - P_v) + \sum_f \nu_f (1 - Q_f)$$

with λ_v, μ_v, ν_f positive coupling constants and

$$P_v = \frac{\delta(w_v)}{|G|} \sum_{g \in G} U_{v,g} \quad Q_f = \frac{d_\rho}{|G|} \sum_{\rho} V_{f,\rho}$$

It commutes with $U_{v,g}$ (unitary symmetry) and $V_{f,\rho}$ (non invertible symmetry), obeying the algebra ($N_{\rho,\rho'}^{\rho''}$ fusion coefficients)

$$U_{v,g} U_{v,h} = \frac{\omega_{g,g^{-1}w_v g,h}}{\omega_{w_v,g,h} \omega_{g,h,(gh)^{-1}w_v gh}} U_{v,gh} \quad V_{f,\rho} V_{f,\rho'} = \sum_{\rho''} N_{\rho,\rho'}^{\rho''} V_{f,\rho''}$$

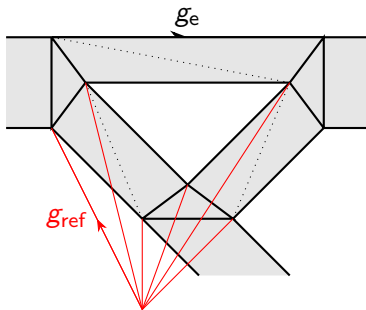
- Π_v projects onto magnetic amplitudes
- P_v projects onto gauge invariant states
- Q_f imposes flatness and translates w

Ground state

Unique ground state Ψ_0 if Γ can be drawn on a genus 0 surface Σ

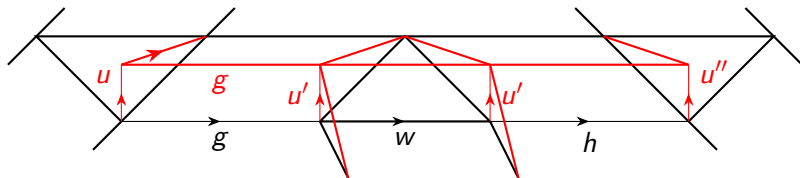
$$\Psi_0(\{x_v, g_e, w_e\}) = \Psi_{\text{string}}(\{x_v, g_e, w_e\}) \times \frac{\Psi_{\text{DW}}(\{g_e, w_e = e\})}{\Psi_{\text{DW}}(\{g_e, w_e\})}$$

$\Psi_{\text{DW}}(\{g_e, w_e\})$ is the partition function of the finite group G Chern-Simons theory with 3-cocycle ω (Dijkgraaf Witten 90) on a triangulated 3-sphere that bound Σ , where each tetrahedron (in red) is associated to $\omega_{g,h,k}$ with $g, h, k \sim g_{\text{ref}}, g_e, w_e$



Excitations as open strings

Elementary excitations are open strings carrying a representation of $D_\omega[G]$ at both ends, created by a ribbon operator



- parallel transport created winding u using gauge fields (not windings) $u \rightarrow u' = g^{-1}ug \rightarrow u'' = (gh)^{-1}u gh$
- insert string magnetic tube $\exp -iC$, string wave function Ψ_{string} and representation $\mathcal{D}_{ij}^\rho$ on lateral face
- translate transversal gauge fields to ensure flatness

Homotopy invariance ensured if no defect is encountered by moving the ribbon

Ribbon operators can be concatenated, superposed and braided