

Anyons and the Exotic Particle on the Noncommutative Plane

Carrollian Wave Equations for Arbitrary Spin

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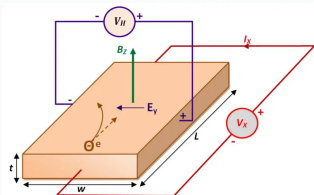
[arXiv:2608.29931](https://arxiv.org/abs/2608.29931)



Hall motion and guiding centres

The quantum Hall regime provides an experimentally realised setting in which an effective noncommutative geometry emerges.

Classical Hall measurement



Hall (1879); Landau (1930).

Image via Physics Stack Exchange.

Orbit and guiding centre

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\eta}.$$

- ▶ $\boldsymbol{\eta}$ is the cyclotron coordinate relative to the orbit centre.
- ▶ $\mathbf{R}$ locates that orbit on the sample.

For uniform fields the orbit centre carries the drift, $\dot{\mathbf{R}} = \mathbf{E} \times \mathbf{B} / B^2$.

Landau quantised the cyclotron motion $\rightarrow$ discrete levels $|n\rangle$. Non-commutative geometry emerges already in the Lowest Landau Level (LLL) $|0\rangle$.

Projection to the lowest Landau level

Landau quantisation turns the cyclotron sector into a harmonic oscillator,

$$H_B = \frac{\pi^2}{2M} = \hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right), \quad E_n = \hbar\omega_c \left(n + \frac{1}{2} \right).$$

Cyclotron sector

Projection $\Pi_{\text{LLL}} := |0\rangle\langle 0|$ fixes $n = 0$ and removes the cyclotron coordinate:

$$E_{\text{LLL}} = \frac{\hbar\omega_c}{2}, \quad \Pi_{\text{LLL}} \boldsymbol{\eta} \Pi_{\text{LLL}} = 0.$$

Guiding-centre sector

The projected position reduces to the orbit centre:

$$\Pi_{\text{LLL}} \mathbf{r} \Pi_{\text{LLL}} = \mathbf{R} \Pi_{\text{LLL}}, \quad [R_x, R_y] = \pm i \ell_B^2.$$

$$\bar{x}_i := \Pi_{\text{LLL}} x_i \Pi_{\text{LLL}}, \quad [\bar{x}_1, \bar{x}_2] = \pm i \ell_B^2,$$
$$\mathcal{A}_{\text{cell}} = 2\pi \ell_B^2 = \frac{\Phi_0}{|B|}, \quad N_\phi = \frac{A}{\mathcal{A}_{\text{cell}}}.$$

Landau (1930); Girvin–Jach (1984).

Coulomb interaction and fractionalisation

Inside the LLL, the kinetic energy is constant. The nontrivial many-body dynamics comes from the projected Coulomb interaction,

$$H_{\text{eff}} = \Pi_{\text{LLL}} \sum_{a < b} V_C(r_{ab}) \Pi_{\text{LLL}}.$$

Correlated ground state

At Laughlin filling

$$\nu = \frac{N_e}{N_\phi} = \frac{1}{m}, \quad m \text{ odd},$$

Coulomb correlations stabilise an incompressible quantum fluid.

Electrons remain fermions. The emergent quasiholes have fractional charge and spin.
Tsui et al. (1982); Laughlin (1983); Arovas–Schrieffer–Wilczek (1984).

Fractional quasiholes

Its elementary excitations carry

$$|q_{\text{qh}}| = \frac{e}{m}, \quad s_{\text{top}} = \frac{1}{2m} \pmod{1},$$
$$\vartheta_{\text{exch}} = 2\pi s_{\text{top}} = \frac{\pi}{m}.$$

From Hall anyons to Carroll wave equations

The physical benchmark

- ▶ In $2 + 1$ dimensions, quasiparticle exchanges form braids and can carry fractional phases.
Leinaas–Myrheim (1977); Wilczek (1982)
- ▶ The Hall problem links anyons, oscillator modules and noncommuting coordinates.
Horvathy–Plyushchay (2004)
- ▶ Relativistic one-anyon wave equations provide an independent representation-theoretic description of arbitrary spin.
Jackiw–Nair (1991); Plyushchay (1991); Cortés–Plyushchay (1994).

Since now anyons will be regarded as fundamental particles in $2+1$ dimensions.

Questions addressed here

Can the relativistic spin tower survive the Carroll ($c \rightarrow \infty$) limit?

Can its large-spin limit realise the exotic Carroll algebra and a noncommutative plane without a background magnetic field?

Relativistic starting point

The Cortés–Plyushchay system packages mass and spin into three first-order constraints:

$$V_\mu \psi = 0, \quad V_\mu = \alpha P_\mu - i\epsilon_{\mu\nu\lambda} P^\nu \mathcal{J}^\lambda + mc \mathcal{J}_\mu.$$

It implies the two Poincaré Casimir conditions,

$$(P^2 + m^2 c^2) \psi = 0, \quad (P \cdot \mathcal{J} - \alpha mc) \psi = 0.$$

Internal representation

$\mathcal{J}_\mu \in sp(2, \mathbb{R}) \cong so(2, 1)$ acts on a lowest-weight Lorentz module labelled by the real spin α .

Cortés and Plyushchay, J. Math. Phys. 35 (1994).

One tower, three particle sectors

Parameter	Lorentz module	Particle sector
$\alpha > 0$	unitary discrete series D_{α}^{+}	unitary anyons
$\alpha = -j = -\frac{1}{2}, -1, \dots$	finite quotient with $2j + 1$ states	bosons, fermions and higher spin
$-j < \alpha < -j + \frac{1}{2}$	half-bounded non-unitary module	interpolating anyons

At $\alpha = -j$, a singular vector appears at level $2j + 1$. Its descendants form a null invariant submodule, and the quotient terminates at $|2j\rangle$.

Bargmann (1947); Barut–Fronsdal (1965); Horvathy–Plyushchay–Valenzuela (2010).

A nontrivial fixed-energy Carroll limit

The scalar equation already determines the useful Carroll scaling:

$$\left(\partial_t^2 - c^2 \nabla^2 + m^2 c^4\right) \psi = 0.$$

Fixed mass

$c \rightarrow 0$ at fixed m gives

$$\partial_t^2 \psi = 0.$$

The rest-energy scale disappears.

Fixed rest energy

$c \rightarrow 0, m \rightarrow \infty$, with $E = mc^2$ fixed,

$$(\partial_t^2 + E^2) \psi = 0.$$

Two flat branches survive:

$$P_t := i\partial_t = \pm E$$

Lévy–Leblond (1965); Bergshoeff–Gomis–Longhi (2014); Marsot (2022).

This is the analog of the $c \rightarrow \infty$ Schrödinger equation.

The spin tower requires a grading

Write $\psi = \sum_{n \geq 0} \psi_n(x)|n\rangle$ and transform each level differently:

$$\Phi = S\psi, \quad S = c^{\mathbf{N}}, \quad \mathbf{N} = \mathcal{J}_0 - \alpha, \quad \phi_n = c^n \psi_n.$$

With $\mathcal{J}_{\pm} = \mathcal{J}_1 \pm i\mathcal{J}_2$, the ladder generators scale chirally,

$$S\mathcal{J}_+S^{-1} = c\mathcal{J}_+, \quad S\mathcal{J}_-S^{-1} = c^{-1}\mathcal{J}_-.$$

$$(\psi_0, \psi_1, \psi_2, \dots) \longmapsto (\psi_0, c\psi_1, c^2\psi_2, \dots).$$

The grading matters

It keeps every fixed level of the internal spin tower finite while preserving the ladder structure as $c \rightarrow 0$.

Horvathy–Plyushchay–Valenzuela (2010); Valenzuela, arXiv:2608.29931 (2026).

Carrollian anyon wave equations

The fixed- E contraction gives three first-order constraints,

$$\mathfrak{V}_a^C \Phi = 0, \quad a = 0, +, -.$$

With $P_{\pm} = P_1 \pm iP_2$ and $\mathbf{N} = \mathcal{J}_0 - \alpha$, their operators are

$$\mathfrak{V}_+^C = (\mathcal{J}_0 - \alpha)P_+ + (P_t - E)\mathcal{J}_+, \quad \mathfrak{V}_-^C = (P_t + E)\mathcal{J}_-, \quad (1)$$

$$\mathfrak{V}_0^C = -\alpha(P_t - E) + E\mathbf{N} - \frac{1}{2}P_+\mathcal{J}_-. \quad (2)$$

Vector square root

The three components retain the Lorentz-module ladder operators and encode a first-order square root of the scalar Carroll equation.

Cortés–Plyushchay (1994); Valenzuela, arXiv:2608.29931 (2026).

Constraint algebra and the Carroll equation

The nonvanishing commutators close on the constraints and one central operator:

$$\begin{aligned}[\mathfrak{V}_+^C, \mathfrak{V}_-^C] &= -P_+ \mathfrak{V}_-^C - 2\mathcal{J}_0 \mathcal{D}_C, \\[\mathfrak{V}_0^C, \mathfrak{V}_-^C] &= -E \mathfrak{V}_-^C, \\[\mathfrak{V}_0^C, \mathfrak{V}_+^C] &= P_+ \mathfrak{V}_0^C - P_t \mathfrak{V}_+^C + \mathcal{D}_C \mathcal{J}_+.\end{aligned}$$

On the common kernel of the three constraints,

$$[\mathcal{D}_C, \mathfrak{V}_a^C] = [\mathcal{D}_C, \mathcal{J}_a] = 0, \quad \mathcal{D}_C := (P_t - E)(P_t + E) = P_t^2 - E^2.$$

Hence the system determines the Carroll equation for every nontrivial spin module:

$$\mathcal{D}_C \Phi = 0.$$

Valenzuela, [arXiv:2608.29931](#) (2026).

Carrollian anyon equations in components

Taking matrix elements in the basis $|n\rangle$ gives the independent equations

$$\sqrt{n+2\alpha}(P_t - E)\phi_n + \sqrt{n+1}P_+\phi_{n+1} = 0, \quad (P_t + E)\phi_n = 0 \quad (n \geq 1).$$

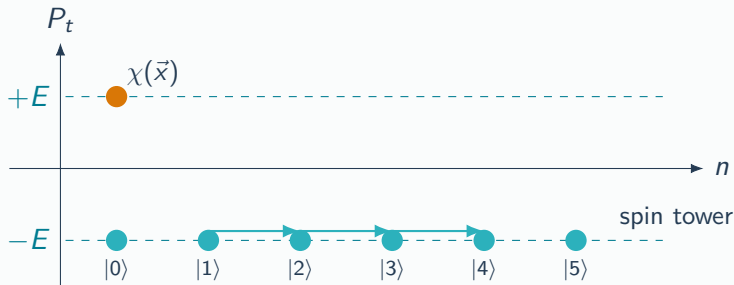
Square-root property

The second equation puts every $n \geq 1$ component on the $P_t = -E$ branch. Acting with $P_t + E$ on the first equation at $n = 0$ then gives $\mathcal{D}_C\phi_0 = 0$ for every nontrivial spin module.

Thus the component system makes the operator-algebra result explicit and is the Carroll counterpart of a Lévy–Leblond square root.

Lévy–Leblond (1967); Valenzuela, [arXiv:2608.29931](#) (2026).

Two energy branches, one asymmetric tower



- ▶ The positive and negative-energy datum occurs in the lowest component.
- ▶ The negative-energy datum smears on the full tower through P_+ action.
- ▶ Momentum labels states and recursion, but no longer changes energy.

Solutions and internal normalisability

For $p_+ \neq 0$, the negative-energy profiles obey

$$\varphi_n = \left(\frac{2E}{p_+}\right)^{n-1} \sqrt{\frac{(1+2\alpha)_{n-1}}{n!}} \varphi_1, \quad n \geq 1.$$

The ratio test gives

$$\left| \frac{\varphi_{n+1}}{\varphi_n} \right| \rightarrow \frac{2E}{|\vec{p}|}.$$

Infinite unitary tower

$\sum_n |\varphi_n|^2$ converges for

$$|\vec{p}| > 2E.$$

Critical circle

Among infinite towers, convergence at $|\vec{p}| = 2E$ survives only for

$$-\frac{1}{2} < \alpha < 0.$$

Horvathy–Plyushchay–Valenzuela (2010); Valenzuela, arXiv:2608.29931 (2026).

Carroll symmetry and arbitrary spin

The (Poincaré algebra) contracted boosts and rotation are

$$B_+ = x_+ P_t, \quad B_- = x_- P_t - i\mathcal{J}_-, \quad J = \epsilon_{ij} x_i P_j + \mathcal{J}_0.$$

Their nonvanishing kinematical brackets are

$$[B_i, P_j] = i\delta_{ij} P_t, \quad [J, P_i] = i\epsilon_{ij} P_j, \quad [J, B_i] = i\epsilon_{ij} B_j.$$

The two Carroll Casimirs have the on-shell values

$$\hat{\mathcal{C}}_1 = -P_t^2 = -E^2, \quad \hat{\mathcal{C}}_2 = P_t J - \epsilon_{ij} B_i P_j = E\alpha.$$

The dispersion is spin-independent, while the second Casimir retains arbitrary spin.

Marsot (2022); Valenzuela, arXiv:2608.29931 (2026).

Finite-spin Carroll fields

At $\alpha = -j$, the quotient terminates exactly at level $2j$.

j	components	Carroll system
$\frac{1}{2}$	2	Lévy–Leblond type
1	3	Deser–Jackiw–Templeton
$\frac{3}{2}$	4	Rarita–Schwinger
general j	$2j + 1$	finite higher-spin field

$P_t = +E$: independent datum at $n = 0$,

$P_t = -E$: independent datum at $n = 2j$.

Independent data occupy opposite ends of the finite tower.

Lévy–Leblond (1967); Deser–Jackiw–Templeton (1982); Rarita–Schwinger (1941).

The exotic Carroll scaling

A nonzero boost cocycle, in the limit $[\mathcal{J}_-, \mathcal{J}_+] = 2\mathcal{J}_0$, requires the internal spin contribution to grow as c^{-2} :

$$c \rightarrow 0, \quad \alpha \rightarrow +\infty, \quad \kappa_C = \alpha c^2 \text{ fixed}, \quad E = mc^2 \text{ fixed}.$$

$$\text{Lorentz spin } \alpha \xrightarrow[c \rightarrow 0]{\alpha c^2 \text{ fixed}} \text{central charge } \kappa_C.$$

Two different contractions

At fixed α the Carroll boosts commute. The correlated large-spin limit retains the finite boost cocycle κ_C .

Jackiw–Nair (2000); Duval–Horvathy (2000); Horvathy–Plyushchay (2002).

Lorentz tower to Heisenberg module

Define the contracted internal operators

$$b^{\pm} = \lim_{\alpha \rightarrow \infty} \frac{\mathcal{J}_{\pm}}{\sqrt{2\alpha}}, \quad \mathbf{N} = b^{+} b^{-}.$$

The limiting algebra and its action are exact:

$$[b^{-}, b^{+}] = \mathbf{1}, \quad b^{+}|n) = \sqrt{n+1}|n+1), \quad b^{-}|n) = \sqrt{n}|n-1).$$

Representation contraction

$sp(2) \rightarrow h(2)$, and D_{α}^{+} contracts to the oscillator Fock module on the algebraic span of fixed finite levels.

The tower remains infinite, but its algebraic organisation becomes exactly Heisenberg.

Exotic wave equation and coherent states

The limiting equations reduce to an energy condition and one polarisation:

$$(P_t - E)\Psi = 0, \quad \Lambda_- \Psi = 0, \quad \Lambda_- = \sqrt{2\kappa_C} P_- + 2E b^-.$$

For a momentum eigenstate,

$$b^- \Psi_{\vec{p}} = z(p_-) \Psi_{\vec{p}}, \quad z(p_-) = -\sqrt{\frac{\theta_C}{2}} p_-, \quad \theta_C = \frac{\kappa_C}{E^2}.$$

Hence

$$\Psi = \int d^2 p \mathcal{C}(\vec{p}) e^{-iEt + i\vec{p} \cdot \vec{x}} |z(p_-)).$$

Each plane wave carries the unique internal coherent state selected by its momentum.

Bargmann (1961); Horvathy–Plyushchay (2004).

A wave-equation realisation of the exotic Carroll algebra

The boosts contain the two oscillator components,

$$B_{\pm} = x_{\pm} P_t \pm i\sqrt{2\kappa_C} b^{\pm}.$$

The nonvanishing commutators are

$$\begin{aligned} [B_i, P_j] &= i\delta_{ij} P_t, & [B_i, B_j] &= -i\kappa_C \epsilon_{ij}, \\ [J, P_i] &= i\epsilon_{ij} P_j, & [J, B_i] &= i\epsilon_{ij} B_j. \end{aligned}$$

- ▶ P_t plays the central role analogous to the Bargmann mass.
- ▶ κ_C is the exotic boost charge generated by the scaled spin.
- ▶ The magnetic extension remains switched off: $[P_i, P_j] = 0$.

Marsot (2022); wave-equation realisation: Valenzuela, arXiv:2608.29931 (2026).

Spin becomes the exotic charge

When $[B_i, B_j] \neq 0$, the ordinary Carroll spin combination is no longer central:

$$[P_t J - \epsilon_{ij} B_i P_j, B_k] = i\kappa_C P_k.$$

The quadratic momentum term restores centrality,

$$\mathcal{C}_2^{\text{exo}} = P_t(P_t J - \epsilon_{ij} B_i P_j) + \frac{\kappa_C}{2} \vec{P}^2,$$

$$[\mathcal{C}_2^{\text{exo}}, P_i] = [\mathcal{C}_2^{\text{exo}}, B_i] = [\mathcal{C}_2^{\text{exo}}, J] = 0, \quad \mathcal{C}_2^{\text{exo}} \psi = 0.$$

Interpretation

The divergent relativistic spin α is transmuted into the finite exotic charge κ_C . No independent residual spin label remains in this contraction.

Observable coordinates on a noncommutative plane

On the physical energy shell $P_t = E$, define coordinates directly from the boosts:

$$\mathcal{X}_{\pm} = E^{-1} B_{\pm} = x_{\pm} \pm i\sqrt{2\theta_C} b^{\pm}, \quad \theta_C = \frac{\kappa_C}{E^2}.$$

They preserve the polarised physical subspace,

$$[\mathcal{X}_{\pm}, \Lambda_{-}] = 0, \quad [\mathcal{X}_i, P_j] = i\delta_{ij},$$

while their spatial components obey

$$[\mathcal{X}_1, \mathcal{X}_2] = -i\theta_C.$$

The exotic central charge becomes an observable area scale.

Duval–Horvathy (2000); Horvathy–Plyushchay (2002, 2004).

Landau and exotic Carroll geometries

Lowest Landau level	Exotic Carroll particle
guiding centres obey a Heisenberg algebra	Observable coordinates obey a Heisenberg algebra
guiding-centre states describe orbit centers	Observable coordinates describe the position of exotic particle position
ℓ_B^2 sets the noncommutative area	$\theta_C = \kappa_C/E^2$ sets the area

The analogy is algebraic and geometric; not an equivalence.

Girvin–Jach (1984); Horvathy (2002); Valenzuela, [arXiv:2608.29931](#) (2026).

Galilean and Carrollian descendants

	Galilean limit	Carroll limit
Scaling	$c \rightarrow \infty$, fixed m	$c \rightarrow 0$, fixed $E = mc^2$
Dynamics	$P_t = \vec{P}^2/2m$	$P_t = \pm E$
Energy branches	one branch survives	both branches survive
Momentum	controls kinetic energy	labels degenerate states
Norm ($\alpha > 0$)	$ \vec{p} < 2m$	$ \vec{p} > 2E$

Both limits use the same level grading. The Galilean phase selects one relativistic branch, while the phase-free Carroll frame retains the pair.

Horvathy–Plyushchay–Valenzuela (2010); Valenzuela, [arXiv:2608.29931](#) (2026).

Conclusions

- ▶ Landau-level projection provides a physical model of noncommutative coordinates and motivates the comparison.
- ▶ A level-dependent contraction yields first-order Carroll equations for arbitrary spin, including finite bosonic and fermionic quotients.
- ▶ The large-spin limit realises the exotic Carroll algebra on a coherent-state module.
- ▶ Observables coordinates span a noncommutative plane with $\theta_C = \kappa_C/E^2$.

In summary:

→ Carrollian wave equations for arbitrary spin for:

Anyons, Bosons and fermions.

→ A wave equation for a Carroll particle in the non-commutative plane.



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