

Quantum Structure of Spacetime and Its Imprint on Hawking Radiation via Entropy

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From Statistical Mechanics to Quantum Gravity

- For a system in thermal equilibrium,

$$Z(\beta) = \text{Tr } e^{-\beta H}, \quad \beta = \frac{1}{k_B T}.$$

- The trace represents a weighted sum over microscopic states:

$$Z(\beta) = \sum_n e^{-\beta E_n}.$$

- Thermodynamic quantities follow from Z :

$$F = -\frac{1}{\beta} \ln Z, \quad S = k_B (1 - \beta \partial_\beta) \ln Z.$$

- In particular, for Ω equally accessible states,

$$S = k_B \ln \Omega.$$

What microscopic states underlie black-hole entropy?

Semiclassical Approach: Gibbons–Hawking: Commun. Math. Phys., 1977)

- For a fixed differentiable manifold M , consider the Euclidean gravitational partition function:

$$Z_E(\beta) \sim \int \mathcal{D}g \, e^{-I_E[g]/\hbar}.$$

- In the semiclassical treatment considered here, the path integral is evaluated within a fixed manifold/topology sector:

$$\mathcal{M}, \text{ topology} = \text{fixed}.$$

- The metric is the dynamical field being integrated over:

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}.$$

- In the semiclassical approximation, the dominant contribution comes from a classical Euclidean saddle g_0 :

$$Z_E(\beta) \approx e^{-I_E[g_0]/\hbar}.$$

The Euclidean gravitational action is

$$I_E = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{h} K.$$

The second term is the Gibbons–Hawking–York boundary term.

Semiclassical Approach: Gibbons–Hawking (II)

- Choose the Euclidean Schwarzschild solution as the saddle:

$$g_0 = g_{\text{Schw}}^{(E)}.$$

- For the Euclidean Schwarzschild saddle, regularity at the horizon requires

$$\beta = \frac{2\pi}{\kappa}, \quad T_H = \frac{1}{\beta} = \frac{\kappa}{2\pi}.$$

- Evaluating the Euclidean action on the Schwarzschild saddle gives

$$I_E[g_0] = \frac{\beta M}{2}.$$

- Using the thermodynamic relation

$$S = \beta E - I_E, \quad E = M,$$

gives

$$S_{BH} = \frac{\beta M}{2} = \frac{A}{4G\hbar}.$$

The semiclassical theory reproduces the black-hole area law.

Beyond the Semiclassical Saddle: Quantum Matter Corrections

$$g_{\mu\nu} = g_{\mu\nu}^{(0)}, \quad \Phi = \text{quantum}$$

The matter-field partition function is

$$Z_{\Phi}[g^{(0)}] = \int \mathcal{D}\Phi \, e^{-I_E[g^{(0)}, \Phi]/\hbar}$$

Integrating out Φ defines the effective action:

$$e^{-\Gamma[g^{(0)}]/\hbar} = \int \mathcal{D}\Phi \, e^{-I_E[g^{(0)}, \Phi]/\hbar}$$

$$I_E^{\text{eff}} = I_E^{\text{cl}} + \Gamma$$

Quantum matter fluctuations $\rightarrow$ one-loop effective action $\rightarrow$ entropy correction

Beyond the Semiclassical Result

$$S_{BH} = \frac{A}{4G\hbar}$$



Classical Schwarzschild geometry

$$g_{\mu\nu} = g_{\mu\nu}^{(0)}$$

+

Quantum matter fluctuations

$$\Phi = \text{quantum}$$



$$S = S_{BH} + \Delta S_{\text{quantum}}$$

Limitations of the Semiclassical Description

- The calculation is performed on a **fixed classical spacetime manifold**.
- The gravitational path integral integrates over metric configurations, but the underlying manifold and topology remain classical:

$$M = \text{fixed}, \quad \mathcal{D}g \text{ integrated.}$$

- The semiclassical approximation is controlled by fluctuations around a classical saddle:

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}.$$

- It does not provide a microscopic description of the **spacetime degrees of freedom** responsible for the black-hole entropy.
- At the Planck scale, the assumptions of a smooth classical spacetime and arbitrarily precise localization are expected to break down.

What replaces the classical notion of spacetime in the UV?

Why Go Beyond Smooth Manifold?

- Key question:

Should we quantize only the metric, or the entire structure of spacetime?

Events vs Coordinates in General Relativity

How do we know we live in a spacetime?

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- In General Relativity, spacetime is modeled as a smooth manifold.
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- What is physically real are **events**:
 - localized physical processes (interactions, detector clicks).

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Key message

Physics is about events, not points of a manifold.

Operational Construction of Spacetime Coordinates

You localize an event by:

- sending a light signal at time t_1 ,
- receiving it back at time t_2 .

You then define:

$$x = \frac{c(t_2 - t_1)}{2}, \quad t = \frac{t_1 + t_2}{2}.$$

Important

The quantities (x, t) are *not measured objects*. They are *constructed* from measurement outcomes (t_1, t_2) .

From Events to Manifold (Smolin2011)

- Events are observer-independent physical occurrences.
- Coordinate assignments are observer- and convention-dependent.
- A sufficiently dense set of localized events allows one to:
 - define neighborhoods,
 - introduce continuity,
 - reconstruct a differentiable manifold.

Key statement

The spacetime manifold is reconstructed from events; it is not assumed *a priori*.

What Changes at Short Distances?

- Event localization requires energy.
- At high energies, gravity backreacts.
- Localization cannot be arbitrarily sharp.

Lead-in

Quantum gravity makes the emergent nature of spacetime unavoidable.

Operational Motivation: Why Spacetime Cannot Be Sharply Localized (Doplicher1994)

To localize an event in a region of size Δx , one needs high momentum

$$\Delta p \sim \frac{\hbar}{\Delta x}, \quad E \sim c \Delta p \sim \frac{\hbar c}{\Delta x}.$$

This energy backreacts gravitationally. Its Schwarzschild radius is

$$r_s = \frac{2GM}{c^2} = \frac{2GE}{c^4}.$$

Avoid gravitational collapse: require $r_s \lesssim \Delta x$, hence

$$\frac{2G}{c^4} \frac{\hbar c}{\Delta x} \lesssim \Delta x \implies (\Delta x)^2 \gtrsim \frac{2G\hbar}{c^3} \implies \Delta x \gtrsim \sqrt{2} \ell_P.$$

Physical meaning

Below the Planck length the operational notion of a sharply localized spacetime point breaks down.

Quantum Spacetime: Noncommutative Geometry (Connes 1994)

Spacetime becomes “pixelated” $\rightarrow$ coordinates should not commute.

$$[\hat{x}^a, \hat{x}^b] = i \theta^{ab}(\hat{x}) \implies \Delta x^a \Delta x^b \geq \frac{1}{2} |\theta^{ab}|$$

This is **pre-geometric**—there is no notion of points, distances, or classical geometry.

- **Problem: How can a classical geometric structure (manifold) emerge from this algebra?**

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 - **Spectral approach (Connes):** Geometry reconstructed from algebra + Hilbert space + Dirac operator
 - **Effective approach (this work):** Seiberg–Witten map $\Rightarrow$ deformed fields and metric

Conceptual Shift

Semiclassical Gravity	Quantum Spacetime
Classical manifold	Quantum structure
Metric fluctuates	Structure fluctuates
Thermodynamic description	Microscopic description

Basic Idea of Noncommutative Geometry

- At very short distances, spacetime coordinates are taken to be noncommuting:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\tilde{\Theta}^{\mu\nu}.$$

- Consequently, ordinary multiplication is replaced by the Moyal–Weyl star product:

$$f \star g = fg + \frac{i}{2}\tilde{\Theta}^{\mu\nu}(\partial_\mu f)(\partial_\nu g) + \mathcal{O}(\tilde{\Theta}^2).$$

- Here $\tilde{\Theta}^{\mu\nu}$ sets the scale and orientation of the noncommutativity.

Idea

Noncommutativity modifies the way fields are multiplied, leading to short-distance corrections.

Seiberg–Witten Map

- In a noncommutative gauge theory, the NC gauge field $\hat{A}_\mu$ is related to an ordinary gauge field A_μ .
- The Seiberg–Witten map preserves gauge equivalence:

$$\hat{A}(A, \tilde{\Theta}) + \hat{\delta}_{\tilde{\lambda}} \hat{A}(A, \tilde{\Theta}) = \hat{A}(A + \delta_\lambda A, \tilde{\Theta}).$$

- For an Abelian gauge field,

$$\delta_\lambda A_\mu = \partial_\mu \lambda, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

- To first order in the NC parameter,

$$\hat{A}_\mu = A_\mu - \frac{g}{2} \tilde{\Theta}^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) + \mathcal{O}(\tilde{\Theta}^2)$$

g is a coupling/normalization convention

Gravitational Identification of the Coupling

$$g = \frac{m_p}{\hbar c}$$

Using

$$m_p = \sqrt{\frac{\hbar c}{G}}, \quad l_p = \sqrt{\frac{G \hbar}{c^3}},$$

we obtain

$$g = \frac{1}{l_p c^2}$$

$\tilde{\Theta}^{\alpha\beta} \longrightarrow$ NC correction involving $g\tilde{\Theta}^{\alpha\beta}$

$$\Theta \equiv \frac{\tilde{\Theta}}{l_p}, \quad [\tilde{\Theta}] = L^2, \quad [\Theta] = L$$

Strategy: From Schwarzschild to a NC-Corrected Geometry

1. Identify a gravitational variable

Starting from the Schwarzschild geometry, its weak-field limit gives

$$g_{tt} \simeq - \left(1 + \frac{2\Phi_N}{c^2} \right), \quad \Phi_N(r) = -\frac{GM}{r}.$$

$$\boxed{g_{\mu\nu}^{\text{Schw}} \xrightarrow{\text{weak field}} \Phi_N}$$

2. Introduce the NC deformation and reconstruct the geometry

We use Φ_N as the gravitational variable in the Abelian/GEM description:

$$\Phi_N \xrightarrow{\text{SW}} \hat{\Phi}_N \longrightarrow T_{\mu\nu}^{\text{eff}} \xrightarrow{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}^{\text{eff}}} g_{\mu\nu}^{\text{eff}}.$$

The Newtonian potential is an intermediate variable, not the final gravitational

Why the Newtonian Potential?

- The SW map is formulated in terms of gauge potentials.
- In the weak-field limit of GR, the gravitational field is described by the Newtonian/GEM potentials.
- For a static, non-rotating source, the relevant potential is

$$\Phi_N(r) = -\frac{GM}{r}.$$

- We therefore use Φ_N as the gravitational variable to which the SW deformation is applied.

$$\boxed{\Phi_N \xrightarrow{\text{SW}} \hat{\Phi}_N}$$

The subsequent construction restores the relativistic geometry.

From Schwarzschild Geometry to the Newtonian Potential

- In the weak-field limit,

$$g_{00} \simeq - \left(1 + \frac{2\Phi_N}{c^2} \right).$$

- For a point mass,

$$\Phi_N = -\frac{GM}{r}.$$

- Thus Φ_N captures the leading gravitoelectric effect of the source.
- **Strategy:** use the weak-field gravitational potential as the starting variable for the noncommutative deformation.

Linearized Gravity: Trace-Reversed Perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

$$h \equiv \eta^{\mu\nu} h_{\mu\nu}$$

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$$

$$\partial^\mu \bar{h}_{\mu\nu} = 0 \quad \Longrightarrow \quad \square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Trace reversal makes the linearized Einstein equations take a simple wave-equation form.

Weak-Field Gravity and Gravitoelectromagnetism: Mashhoon, arXiv:gr-qc/0311030v2

For a slowly moving source,

$$T_{00} = \rho c^2, \quad T_{0i} = c j_i.$$

The relevant components of $\bar{h}_{\mu\nu}$ are written as

$$\bar{h}_{00} = \frac{4\Phi}{c^2}, \quad \bar{h}_{0i} = -\frac{2A_g{}_i}{c^2}$$

$$\bar{h}_{\mu\nu} \longrightarrow (\Phi, \mathbf{A}_g)$$

Static, nonrotating source

$$\mathbf{A}_g = 0, \quad \Phi = \Phi_N = -\frac{GM}{r}$$

linearized GR $\longrightarrow$ GEM potential representation

Gauge Freedom from Infinitesimal Diffeomorphisms

- Consider an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} - \epsilon^{\mu}(x)$$

where ϵ^{μ} is an infinitesimal vector field.

- To linear order,

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}$$

- For the trace-reversed perturbation,

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} - \eta_{\mu\nu}\partial_{\alpha}\epsilon^{\alpha}$$

- The transverse gauge is preserved if

$$\square\epsilon^{\mu} = 0.$$

Here ϵ^{μ} represents an infinitesimal coordinate/gauge transformation, not a new physical field.

Mashhoon, Sec. 1.2, Eqs. (1.12)–(1.13).

Gauge Freedom of the GEM Potentials

- Mashhoon's post-Newtonian ordering,

$$\epsilon^0 = O(c^{-3}), \quad \epsilon^i = O(c^{-4}),$$

together with

$$\square \epsilon^\mu = 0,$$

leads to the GEM gauge function

$$\Psi = \frac{c^2}{4} \epsilon^0, \quad \square \Psi = 0.$$

- The GEM potentials transform as

$$\Phi' = \Phi - \frac{1}{c} \partial_t \Psi$$

$$\frac{\mathbf{A}'_g}{2} = \frac{\mathbf{A}_g}{2} + \nabla \Psi$$

Noncommutative Correction to the Potential

- Apply the first-order Seiberg–Witten map to

$$A_\mu^G = \left(\frac{\Phi_N}{c}, 0, 0, 0 \right).$$

- This generates a noncommutative correction:

$$\hat{\Phi}_N = -\frac{GM}{r} + \frac{\Theta G^2 M^2}{c^2 r^3}.$$

- With

$$\tilde{\Theta} = \Theta l_p^2, \quad l_p = \sqrt{\frac{\hbar G}{c^3}}.$$

- The correction scales as

$$\boxed{\delta\Phi_N \sim \frac{1}{r^3}}.$$

Interpretation

Noncommutativity produces a short-distance correction to the gravitational potential.

From Potential to Effective Density

- Interpret the corrected potential through the Newtonian Poisson equation:

$$\nabla^2 \hat{\Phi}_N = 4\pi G \hat{\rho}.$$

- The NC correction

$$\delta\Phi_N \sim r^{-3}$$

gives

$$\boxed{\hat{\rho} \sim r^{-5}}.$$

- Thus the NC correction can be represented as an effective gravitational source.

Important

This is an effective source generated by the modified geometry; no additional physical matter is introduced.

Construction of the Effective Energy-Momentum Tensor

- The NC-corrected potential defines an effective energy density,

$$\hat{\rho}(r) = \frac{1}{4\pi G} \nabla^2 \hat{\Phi}_N(r).$$

- For a static, spherically symmetric source, introduce an effective anisotropic energy-momentum tensor:

$$T^{\text{eff} \mu}_{\nu} = \text{diag}(-\hat{\rho}, p_r, p_\theta, p_\theta).$$

- We seek a Schwarzschild-type geometry,

$$ds^2 = -f(r)c^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2.$$

Determining the Effective Pressures

- For the Schwarzschild-type metric,

$$G^t_t = G^r_r.$$

- Using the Einstein equations,

$$G^\mu_\nu = \frac{8\pi G}{c^4} T^{\text{eff} \mu}_\nu,$$

we obtain

$$T^{\text{eff} t}_t = T^{\text{eff} r}_r.$$

- Since

$$T^{\text{eff} t}_t = -\hat{\rho}, \quad T^{\text{eff} r}_r = p_r,$$

it follows that

$$\boxed{p_r = -\hat{\rho}}.$$

- Energy-momentum conservation,

$$\nabla_\mu T^{\text{eff} \mu\nu} = 0,$$

then gives $p_\theta = -\hat{\rho} - \frac{r}{2}\partial_r\hat{\rho}$.

From Effective Energy-Momentum to Geometry

- The effective source enters the Einstein equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}^{\text{eff}}.$$

- For the static, spherically symmetric geometry,

$$ds^2 = -f(r)c^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$

the tt Einstein equation gives

$$\frac{d}{dr}[r(1-f)] = \frac{8\pi G}{c^2} r^2 \hat{\rho}(r).$$

- Introduce the enclosed effective mass:

$$m(r) = 4\pi \int_0^r dr' r'^2 \hat{\rho}(r').$$

- Hence,

$$f(r) = 1 - \frac{2Gm(r)}{c^2 r}.$$

Modified Schwarzschild Geometry

- Substituting the NC-corrected density into the mass function,

$$m(r) = 4\pi \int dr r^2 \hat{\rho}(r),$$

gives

$$m(r) = M - \frac{3GM^2\Theta}{c^2 r^2}.$$

- Therefore, the metric function becomes

$$f(r) = 1 - \frac{2Gm(r)}{c^2 r}.$$

- To first order in the NC parameter,

$$f(r) = 1 - \frac{2GM}{c^2 r} + \frac{6G^2 M^2 \Theta}{c^4 r^3}.$$

- In the commutative limit,

$$\Theta \rightarrow 0 \quad \Rightarrow \quad f(r) \rightarrow 1 - \frac{2GM}{c^2 r}.$$

Horizon Structure of the NC Geometry

$$f(r) = 1 - \frac{2m}{r} + \frac{6m^2\Theta}{r^3} + \mathcal{O}(\Theta^2)$$

$$f(r_h) = 0 \quad \Rightarrow \quad r_{\pm} = m \pm \sqrt{m^2 - 3m\Theta}$$

$m > 3\Theta$: two horizons $\Rightarrow$ NC black hole

$m = 3\Theta$: extremal geometry

$m < 3\Theta$: no horizon

For the following analysis: $m > 3\Theta$

outer horizon:

$$r_+ = m + \sqrt{m^2 - 3m\Theta}$$

NC-Corrected Black-Hole Horizon

$$r_+ = m + \sqrt{m^2 - 3m\Theta}$$

For

$$\frac{\Theta}{m} \ll 1,$$

$$r_+ \simeq 2m - \frac{3}{2}\Theta$$

$$\Theta \rightarrow 0$$

$$\Rightarrow$$

$$r_+ \rightarrow 2m$$

NC deformation shifts the Schwarzschild horizon

Effective Reconstruction

$$\hat{\Phi}_N \longrightarrow \rho_{\text{eff}} \longrightarrow T_{\mu\nu}^{\text{eff}} \longrightarrow g_{\mu\nu}^{\text{eff}}$$

$$\hat{\Phi}_N \xrightarrow{\nabla^2 \hat{\Phi}_N = 4\pi G \rho_{\text{eff}}} \rho_{\text{eff}}$$

static spherical symmetry

$$\underbrace{\rho_{\text{eff}} + ds^2 = -f(r)c^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \longrightarrow T_{\mu\nu}^{\text{eff}}}_{\nabla_\mu T_{\text{eff}}^{\mu\nu} = 0 \text{ assumptions}}$$

$$T_{\mu\nu}^{\text{eff}} \xrightarrow{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}^{\text{eff}}} g_{\mu\nu}^{\text{eff}}$$

Quantum Field on the NC-Corrected Geometry

$$g_{\mu\nu}^{\text{eff}}$$



$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) = 0$$

$$\xi = 0$$

massless minimally coupled scalar



$$(t, r) \text{ sector}$$

WKB Description of the Probe Field

For a rapidly varying semiclassical mode, write

$$\Phi(r, t) = \exp \left[-\frac{i}{\hbar} S(r, t) \right]$$

$S(r, t)$: phase/action of the propagating scalar mode

Substitution into the Klein–Gordon equation gives

$$(\partial_t S)^2 - f^2(r)(\partial_r S)^2 - i\hbar \mathcal{Q}[S] = 0$$

$\mathcal{Q}[S]$: terms containing second derivatives of S

Semiclassical Hamilton–Jacobi Equation

Taking the leading order in $\hbar$,

$$\hbar \rightarrow 0$$

the quantum Hamilton–Jacobi equation reduces to

$$(\partial_t S_0)^2 - f^2(r)(\partial_r S_0)^2 = 0$$

S_0 : semiclassical action of the massless mode

$$\partial_t S_0 = \pm f(r) \partial_r S_0$$

$+ / -$: outgoing / ingoing trajectories

Beyond the Semiclassical Approximation

The full action is expanded as

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \dots$$

- S_0 : semiclassical contribution
- $\hbar S_1$: first quantum correction
- $\hbar^2 S_2$: second quantum correction
- $\vdots$

Substitution generates a hierarchy:

$$\mathcal{O}(\hbar^0), \mathcal{O}(\hbar^1), \mathcal{O}(\hbar^2), \dots$$



$$S_0, S_1, S_2, \dots$$

Structure of the WKB Tower

At each order,

$$\boxed{\partial_t S_i = \pm f(r) \partial_r S_i} \quad (i = 0, 1, 2, \dots)$$

Each quantum order obeys the same characteristic equation

Therefore,

$$\boxed{S_i(r, t) \propto S_0(r, t)}$$

Higher-order actions have the same functional dependence as S_0

Quantum-Corrected WKB Action

Since

$$S_i \propto S_0,$$

the full action can be written as

$$S = (1 + \gamma_1 \hbar + \gamma_2 \hbar^2 + \cdots) S_0.$$

For a 4D Schwarzschild-type background,

$$\gamma_i = \frac{\beta_i}{M^{2i}}$$

with β_i dimensionless.

Hence

$$S = \left[1 + \sum_{i=1}^{\infty} \beta_i \frac{\hbar^i}{M^{2i}} \right] S_0$$

Leading Quantum Correction

Keeping only the first quantum correction,

$$S = \left(1 + \beta_1 \frac{\hbar}{M^2} + \mathcal{O}(\hbar^2) \right) S_0$$

$$S_1 = \frac{\beta_1}{M^2} S_0$$

β_1 : unknown dimensionless quantum coefficient

$$\text{Im } S = \left(1 + \beta_1 \frac{\hbar}{M^2} + \cdots \right) \text{Im } S_0$$

Tunnelling Probability

For the outgoing mode,

$$P_{\text{out}} = |\phi_{\text{out}}|^2$$

with

$$P_{\text{out}} = \exp \left[-\frac{4\omega}{\hbar} \left(1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right) \text{Im} \int \frac{dr}{f(r)} \right]$$

Here $f(r) = g(r)$ for our effective Schwarzschild-type geometry.

$$\text{The horizon pole} \implies \text{Im} \int \frac{dr}{f(r)} \neq 0$$

Corrected Hawking Temperature

Detailed balance:

$$P_{\text{out}} = e^{-\omega \beta_{\text{corr}}} P_{\text{in}}$$

Comparing with the tunnelling probability,

$$P_{\text{out}} \sim \exp \left[-\omega \beta_H \left(1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right) \right],$$

we identify

$$\beta_{\text{corr}} = \beta_H \left(1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right)$$

where

$$\beta_H = \frac{1}{k_B T_H}$$

Quantum-Corrected Entropy

Use the corrected inverse temperature in the first law:

$$dS_{\text{BH}} = \beta_{\text{corr}} dM$$

$$\beta_{\text{corr}} = \frac{8\pi M}{\hbar} \left(1 + \beta_1 \frac{\hbar}{M^2} + \beta_2 \frac{\hbar^2}{M^4} + \cdots \right)$$

Therefore,

$$S_{\text{BH}} = \int \beta_{\text{corr}} dM$$

The Problem: Determining β_1

$$\beta_1 ?$$

WKB

Determines the structure of the quantum correction, but not its coefficient.



Need independent quantum information



Renormalized one-loop contribution to its stress-tensor trace

The Problem: Determining β_1

- From WKB analysis:

$$S = \left(1 + \beta_1 \frac{\hbar}{M^2} \right) S_0$$

- β_1 encodes quantum corrections
- **Issue:**

WKB determines the structure of corrections, but not the coefficient β_1

- **Question:** How do we determine β_1 ?

Strategy: Fixing the One-Loop Coefficient

Consider a constant metric rescaling

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = k g_{\mu\nu}, \quad k = 1 + \delta k$$

Under this transformation, the Schwarzschild mass parameter scales as

$$\bar{M} = k^{-1} M \simeq (1 - \delta k) M$$

At $\mathcal{O}(\hbar)$,

$$S_1 = \frac{\beta_1}{M^2} S_0$$

Key idea

Relate the scale variation of S_1 to the trace of the quantum energy-momentum tensor.

Quantum Effective Action

Start with a quantum matter field on a classical background:

$$S[g_{\mu\nu}, \phi]$$

$g_{\mu\nu}$: classical background, ϕ : quantum field

Integrating out the matter field,

$$Z[g] = \int \mathcal{D}\phi e^{iS[g, \phi]} \equiv e^{iI[g]}$$

$I[g]$ = effective action including quantum fluctuations

The energy-momentum tensor is obtained from its metric variation:

$$\delta I = \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}$$

Scale Variation and the Trace of $T_{\mu\nu}$

Under

$$g_{\mu\nu} \rightarrow k g_{\mu\nu}, \quad k = 1 + \delta k,$$

we have

$$\delta g_{\mu\nu} = g_{\mu\nu} \delta k$$

Therefore,

$$\begin{aligned} \delta I &= \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} \\ &= \frac{\delta k}{2} \int d^4x \sqrt{-g} T^{\mu\nu} g_{\mu\nu} \end{aligned}$$

$$T^{\mu\nu} g_{\mu\nu} = T^\mu{}_\mu$$

Hence

One-Loop WKB Correction

The tunnelling mode is described by a Hamilton–Jacobi/WKB action

$$S = S_0 + \hbar S_1 + \cdots$$

Dimensional analysis gives

$$S_1 = \frac{\beta_1}{M^2} S_0$$

Important distinction

S : one-particle WKB action

$I[g]$: quantum effective action

At $\mathcal{O}(\hbar)$, the WKB correction is related to the one-loop contribution.

Connecting the WKB Correction to the Trace

QFT variation = WKB variation

From the scale variation of the quantum effective action,

$$\text{Im} \int d^4x \sqrt{-g} \langle T^\mu{}_\mu \rangle = 2 \frac{\delta \text{Im} S_1^{\text{out}}}{\delta k}$$

outgoing contribution satisfies

$$\frac{\delta \text{Im} S_1^{\text{out}}}{\delta k} = \frac{\beta_1}{M^2} \text{Im} S_0^{\text{out}}$$

Therefore,

$$\text{Im} \int d^4x \sqrt{-g} \langle T^\mu{}_\mu \rangle = \frac{2\beta_1}{M^2} \text{Im} S_0^{\text{out}}$$

Determining β_1

For Schwarzschild,

$$\text{Im} S_0^{\text{out}} = -4\pi\omega M$$

Hence

$$\text{Im} \int d^4x \sqrt{-g} \langle T^\mu{}_\mu \rangle = -\frac{8\pi\omega}{M} \beta_1$$

For a stable black hole,

$$\omega = M$$

and therefore

$$\beta_1 = -\frac{1}{8\pi} \text{Im} \int d^4x \sqrt{-g} \langle T^\mu{}_\mu \rangle$$

$$\text{WKB coefficient} \longleftrightarrow \text{quantum trace}$$

Quantum Trace and Trace Anomaly

For a conformally coupled massless field,

$$T^\mu{}_\mu|_{\text{classical}} = 0$$

but quantum renormalization gives

$$\langle T^\mu{}_\mu \rangle_{\text{ren}} \neq 0$$

Trace anomaly

For a minimally coupled scalar,

$$\xi = 0, \quad T^\mu{}_\mu|_{\text{classical}} \neq 0$$

but it also has a quantum contribution

$$\langle T^\mu{}_\mu \rangle_{1\text{-loop}} \neq 0$$

Determining β_1 from the Quantum Trace

The WKB expansion gives

$$S = S_0 + \hbar S_1 + \mathcal{O}(\hbar^2), \quad S_1 = \frac{\beta_1}{M^2} S_0$$

The quantum effective action has the expansion

$$\Gamma[g] = I_{\text{cl}}[g] + \hbar \Gamma_{1\text{-loop}}[g] + \mathcal{O}(\hbar^2)$$

$\Downarrow$

$$\langle T^\mu{}_\mu \rangle_{\text{ren}} = T^\mu{}_\mu|_{\text{cl}} + \hbar \langle T^\mu{}_\mu \rangle_{1\text{-loop}} + \cdots$$

$$S_1 \longrightarrow \beta_1 \longleftarrow \langle T^\mu{}_\mu \rangle_{1\text{-loop}}$$

Banerjee–Majhi: the $\mathcal{O}(\hbar)$ WKB correction is related to the one-loop quantum trace.

Curvature and Trace Anomaly (NC)

- For the NC-corrected Schwarzschild-type metric

$$f(r) = 1 - \frac{2M}{r} + \frac{6M^2\Theta}{r^3},$$

the curvature invariants to $\mathcal{O}(\Theta)$ are

$$R_{\mu\nu\rho\sigma}^2 = \frac{48M^2}{r^6} - \frac{960M^3\Theta}{r^8},$$

$$R = -\frac{12M^2\Theta}{r^5}, \quad R_{\mu\nu}R^{\mu\nu} = \mathcal{O}(\Theta^2).$$

- The Laplacian term is

$$\square R = \frac{600M^3\Theta}{r^8} - \frac{240M^2\Theta}{r^7}.$$

- Hence, using

$$\langle T^\mu{}_\mu \rangle_{1\text{-loop}} = \frac{1}{2880\pi^2} (R_{\mu\nu\rho\sigma}^2 - R_{\mu\nu}R^{\mu\nu} + \square R),$$

Curvature and Trace Anomaly (NC)

we obtain

$$\langle T^\mu{}_\mu \rangle_{1\text{-loop}} = \frac{M^2}{60\pi^2 r^6} - \Theta \left(\frac{M^3}{8\pi^2 r^8} + \frac{M^2}{12\pi^2 r^7} \right)$$

Evaluation of β_1^{NC}

The trace-anomaly relation gives

$$\beta_1 = -\frac{1}{8\pi} \text{Im} \int d^4x \sqrt{-g} \langle T^\mu{}_\mu \rangle_{1\text{-loop}}.$$

Separate the commutative and NC contributions:

$$\beta_1(M) = \frac{1}{360\pi} + \Theta \hat{\beta}_1^{\text{NC}}(M)$$

where

$$\hat{\beta}_1^{\text{NC}} = -\frac{1}{8\pi} \text{Im} \int d^4x \sqrt{-g} \left[-\frac{M^3}{8\pi^2 r^8} - \frac{M^2}{12\pi^2 r^7} \right].$$

Using

$$t \rightarrow -i\tau, \quad \int d\tau = \beta_H = 8\pi M, \quad \int d\Omega = 4\pi,$$

$$\hat{\beta}_1^{\text{NC}} = 4\pi M \int_{2M}^{\infty} dr \left(\frac{M^3}{8\pi^2 r^6} + \frac{M^2}{12\pi^2 r^5} \right)$$

Evaluation of $\widehat{\beta}_1^{\text{NC}}$

Using

$$\int_{2M}^{\infty} \frac{dr}{r^6} = \frac{1}{5(2M)^5}, \quad \int_{2M}^{\infty} \frac{dr}{r^5} = \frac{1}{4(2M)^4},$$

we obtain

$$\widehat{\beta}_1^{\text{NC}} = 4\pi M \left[\frac{M^3}{8\pi^2} \frac{1}{5(2M)^5} + \frac{M^2}{12\pi^2} \frac{1}{4(2M)^4} \right].$$

Therefore,

$$\widehat{\beta}_1^{\text{NC}} = \frac{1}{120\pi M}$$

and hence

$$\beta_1(M) = \frac{1}{360\pi} + \frac{\Theta}{120\pi M}$$

$$\widehat{\beta}_1^{\text{NC}} \propto M^{-1} \Rightarrow \text{NC correction depends on the horizon scale.}$$

Quantum-Corrected Entropy

Use the corrected inverse temperature in the first law:

$$dS_{\text{BH}} = \beta_{\text{corr}} dM$$

$$\beta_{\text{corr}} = \frac{8\pi M}{\hbar} \left[1 + \beta_1(M) \frac{\hbar}{M^2} + \beta_2 \frac{\hbar^2}{M^4} + \dots \right]$$

Therefore,

$$S_{\text{BH}} = \int \beta_{\text{corr}} dM$$

For a constant β_1 ,

$$S_{\text{BH}} = \frac{4\pi M^2}{\hbar} + 8\pi\beta_1 \ln M - \frac{4\pi\hbar\beta_2}{M^2} + \dots$$

NC effects enter through the modified $\beta_1(M)$.

Quantum-Corrected Black-Hole Entropy

$$S_{\text{BH}} = \underbrace{\frac{A}{4\hbar}}_{\text{semiclassical}} + \underbrace{\frac{1}{90} \ln\left(\frac{A}{A_0}\right)}_{\text{quantum matter}} - \underbrace{\frac{\Theta}{15M}}_{\text{NC spacetime}} + \underbrace{\mathcal{O}(A^{-1})}_{\text{higher corrections}}$$

A_0 : arbitrary reference area (e.g. $A_0 \sim \ell_P^2$).

$$S_{\text{BH}} = S_{\text{semiclassical}} + S_{\text{quantum matter}} + S_{\text{NC}} + \dots$$

Physical Interpretation: Quantum Structure and Entropy

- **Noncommutative spacetime modifies the black-hole entropy:**

$$S_{\text{BH}} = \frac{A}{4\hbar} + \frac{1}{90} \ln\left(\frac{A}{A_0}\right) - \frac{\Theta}{15M} + \dots$$

- The NC contribution

$$\delta S_{\text{NC}} = -\frac{\Theta}{15M}$$

represents a **UV imprint of spacetime noncommutativity** on black-hole thermodynamics.

- The correction is not simply a modification of the classical geometry: it changes the **thermodynamic state** associated with the black hole.
- Hence, the quantum structure of spacetime can leave an **indirect imprint on a macroscopic observable**:

UV structure $\longrightarrow$ black-hole entropy

Physical Interpretation: Entropy and Microscopic States

The Parikh–Wilczek relation can be interpreted as

$$\Gamma \sim e^{S_{\text{after}} - S_{\text{before}}} = \frac{\Omega_{\text{after}}}{\Omega_{\text{before}}}$$

For our NC correction,

$$\frac{(\Omega_{\text{after}}/\Omega_{\text{before}})_{\text{NC}}}{(\Omega_{\text{after}}/\Omega_{\text{before}})_{\text{comm}}} = \exp \left[-\frac{\Theta \omega}{15M(M - \omega)} \right]$$

- The NC deformation therefore changes the **relative statistical weight** of the initial and final black-hole macrostates.
- For $\Theta > 0$,

$$\Gamma_{\text{NC}} < \Gamma_{\text{comm}}$$

for the same M and ω within our approximation.

- Thus, the quantum structure of spacetime can leave an **indirect imprint on Hawking radiation** through the modified entropy difference.

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