

Locally Compact Quantum Groups and Para-Hermitian Symmetric Spaces

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Corfu 2026

Fourier theory on a locally compact group

Let G be a locally compact group and $f \in L^2(G)$. Formally,

$$f(g) = \int_{\widehat{G}} \mathrm{Tr}(\widehat{f}(\pi)^* \pi(g)) d[\pi],$$

where

$$\widehat{f}(\pi) = \pi(f) = \int_G f(g) \pi(g) dg.$$

Examples.

- $G = U(1)$: ordinary Fourier series, with $\widehat{U(1)} \simeq \mathbb{Z}$.
- $G = SU(2)$: Peter–Weyl decomposition:

$$f(g) = \sum_{j \in \frac{1}{2}\mathbb{N}_0} \sqrt{2j+1} \, \mathrm{Tr}(\widehat{f}(j)^* \pi_j(g)).$$

From compact to locally compact quantum groups

- Woronowicz's compact quantum groups provide a quantum analogue of Peter–Weyl theory.
- Kustermans–Vaes develop the theory of locally compact quantum groups.
- For $SU_q(2)$, the Haar state leads to a q -deformed spectral/Fourier theory.

Guiding question

What replaces the compact picture in the non-compact setting, for instance for deformations related to the κ -Poincaré group?

A possible framework is provided by locally compact quantum groups and cocycle deformation.

Multiplicative unitaries and coproducts

Let $M \subset B(H)$ be a von Neumann algebra and let

$$W \in \mathcal{U}(H \otimes H)$$

be a multiplicative unitary (satisfying the pentagon equation). Define

$$\Delta(x) = W(1 \otimes x)W^*, \quad x \in M.$$

Then (M, Δ) is the von Neumann algebraic core of a locally compact quantum group.

Group example. For a locally compact group G ,

$$H = L^2(G), \quad M = L^\infty(G),$$

and, in the convention used in the notes,

$$(W\phi)(x, y) = \phi(x, xy).$$

The dual locally compact quantum group

The dual von Neumann algebra is generated by slices of W :

$$\widehat{M} = \{(\omega \otimes \text{id})(W) ; \omega \in B(H)_*\}''.$$

The dual multiplicative unitary is

$$\widehat{W} = \Sigma W^* \Sigma,$$

where $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$.

For a locally compact group,

$$\widehat{L^\infty(G)} = \text{VN}(G) = \lambda(G)'' \subset B(L^2(G)).$$

Cocycle deformation (De Commer)

If Ω is a unitary dual 2-cocycle, then the cocycle-twisted coproduct

$$\widehat{\Delta}_\Omega(x) = \Omega \widehat{\Delta}(x) \Omega^*$$

defines a new locally compact quantum group.

Examples and deformation mechanisms

Examples appearing in the manuscript include:

- compact quantum groups;
- Stachura's locally compact quantum group version of the Jordanian twist;
- the Baaj–Skandalis matched-pair construction;
- cocycle constructions of Bieliavsky–Gayral–Neshveyev–Tuset.

A basic semidirect-product situation is

$$G = Q \ltimes V,$$

where Q is locally compact, V is locally compact abelian, and there is $\xi_0 \in \widehat{V}$ such that

$$Q \longrightarrow \widehat{V}, \quad q \longmapsto q \cdot \xi_0$$

is a measurable equivalence onto an open orbit (up to null sets).

From a quantization map to a dual cocycle

Proposition (scheme)

Let (H, π) be a square-integrable unitary representation of G . Assume that

$$\text{Op} : L^2(G) \longrightarrow \text{HS}(H)$$

is unitary and G -equivariant. If $W^*(G)$ is a type I factor, then Op determines a unitary dual 2-cocycle.

Heuristically, define a star product by transporting operator composition:

$$\text{Op}(f_1 \star f_2) = \text{Op}(f_1) \text{Op}(f_2).$$

Thus

$$f_1 \star f_2 = \text{Op}^{-1}(\text{Op}(f_1) \text{Op}(f_2)).$$

Kernel description of the star product

Assume the product admits a Schwartz-kernel description:

$$(f_1 \star f_2)(g) = \int_{G \times G} K(x, y) f_1(gx) f_2(gy) dx dy.$$

The corresponding dual cocycle is formally

$$\Omega = \int_{G \times G} K(x, y) \lambda_x \otimes \lambda_y dx dy.$$

This makes explicit the passage

equivariant quantization $\implies$ associative product $\implies$ dual cocycle.

Para-Hermitian symmetric spaces

A para-Hermitian symmetric space is denoted

$$(M, \omega, s, J),$$

with:

- (M, ω) symplectic;
- $J \in \text{End}(TM)$ satisfying $J_x^2 = \text{id}$;
- the eigenspaces $T_x^\pm M$ of J_x are Lagrangian;
- $s : M \times M \rightarrow M$, $(x, y) \mapsto s_x(y)$, is smooth and

$$s_x^2 = \text{id}_M, \quad s_x(x) = x \text{ (isolated fixed point),}$$

$$s_x s_y s_x = s_{s_x(y)}, \quad s_x^* \omega = \omega, \quad s_x^* J = J.$$

The affine group $GL_2(\mathbb{R}) \ltimes \mathbb{R}^2$

Let

$$G = GL_2(\mathbb{R}) \ltimes \mathbb{R}^2.$$

Proposition

There exists an irreducible para-Hermitian symmetric space (M, ω, s, J) such that:

- 1 G embeds into $\text{Aut}(M, \omega, s, J) \times PGL_2(\mathbb{R})$;
- 2 the induced action of G on

$$M \times PGL_2(\mathbb{R})/GL_1(\mathbb{R})$$

is strictly transitive on a dense open orbit.

The symmetric space occurring here is

$$M = SL_3(\mathbb{R})/S(GL_1(\mathbb{R}) \times GL_2(\mathbb{R})).$$

Inönü–Wigner contraction

Proposition

There exists an Inönü–Wigner contraction

$$M \longrightarrow M_0 := T^*V \simeq \mathbb{R}^4$$

and a Lie-group injection of the form

$$G \longrightarrow (\mathrm{Sp}(T^*V) \ltimes H(T^*V)) \times \mathrm{PGL}_2(\mathbb{R}),$$

such that G acts simply transitively on its dense open orbit in

$$\mathbb{R}^4 \times \mathrm{AdS}_2.$$

This contracted picture is the bridge to explicit representation-theoretic quantization.

Induced representations

Let $H(T^*V)$ be the Heisenberg group associated with T^*V . The manuscript considers induced representations of the form

$$\mathrm{Ind}_Z^{H(T^*V)}(e^{i\hbar\zeta}) \simeq (L^2(V), \pi_{\hbar})$$

and

$$\mathrm{Ind}_{GL_1(\mathbb{R})}^{PGL_2(\mathbb{R})}(\nu) \simeq (L^2(S^1), \pi_{\nu}^+).$$

Their tensor product gives a unitary irreducible representation of G (by restriction):

$$(L^2(S^1 \times V), U_{\nu, \hbar}), \quad U_{\nu, \hbar} = \pi_{\nu}^+ \otimes \pi_{\hbar}.$$

A Lagrangian matched pair

Introduce the subgroups

$$Q = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\} \ltimes \mathbb{R}^2.$$

Then multiplication gives a measurable equivalence

$$Q \times N \longrightarrow G, \quad (q, n) \longmapsto qn.$$

Frobenius structure. Let

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and choose $\xi_0 := E^* + e_2^* \in \mathfrak{n}^*$. It is a character of $\mathfrak{n}$ and its pull-back

$$\sigma := \text{pr}_{\mathfrak{n}}^*(\xi_0) \in \mathfrak{g}^*$$

has non-degenerate Chevalley coboundary

$$\delta\sigma = \omega_0.$$

Symplectic open embeddings and the horocycle picture

Lemma

The dense open embeddings

$$G \hookrightarrow M \times \text{AdS}_2, \quad G \hookrightarrow M_0 \times \text{AdS}_2 = T^*V \times \text{AdS}_2$$

are symplectic when G is equipped with the left-invariant form $\tilde{\omega}_0$.

Moreover, via the horocycle picture,

$$L^2(S^1 \times V) \simeq L^2(N).$$

Hence the representation of G above may be written as

$$(L^2(N), U_{\nu, \hbar} = \pi_{\nu}^+ \otimes \pi_{\hbar}).$$

Mackey theory and the BKS intertwiner

Mackey induction gives

$$\mathrm{Ind}_N^G(e^{i\xi_0}) \simeq (L^2(Q), U_{\xi_0}).$$

Proposition

- ① G has a dense open coadjoint orbit;
- ② the representations $U_{\nu, \hbar}$ and U_{ξ_0} are square-integrable.

Corollary

There is a canonical Blattner–Kostant–Sternberg unitary intertwiner

$$\mathcal{F}_{\nu, \hbar, \xi_0} : L^2(N) \longrightarrow L^2(Q).$$

A G -equivariant quantization map

Define the unitary

$$K : L^2(G) \longrightarrow L^2(Q \times Q)$$

by

$$K := W_Q \circ (\mathrm{id}_Q \otimes \mathcal{F}_{\nu, \hbar, \xi_0}).$$

Then K is G -equivariant between the left regular representation of G and

$$U_{\xi_0} \otimes U_{\xi_0}^*.$$

Using

$$L^2(Q \times Q) \simeq \mathrm{HS}(L^2(Q)),$$

we obtain the quantization map

$$\mathrm{Op} : L^2(G) \longrightarrow \mathrm{HS}(L^2(Q)), \quad \mathrm{Op}(f) = K(f),$$

which is unitary and G -equivariant.

Type I property and the resulting LCQG

Proposition

For the affine group

$$GL_n(\mathbb{R}) \ltimes \mathbb{R}^n,$$

the group von Neumann algebra $W^*(G)$ is a type I factor.

Combining this with the previous quantization scheme gives:

Corollary

The BKS intertwiner $\mathcal{F}_{\nu, \hbar, \xi_0}$ canonically induces a unitary dual 2-cocycle on

$$W^*\left(GL_2(\mathbb{R}) \ltimes \mathbb{R}^2\right),$$

and hence a locally compact quantum group obtained by cocycle deformation.

- S. L. Woronowicz, compact matrix pseudogroups / compact quantum groups.
- J. Kustermans and S. Vaes, locally compact quantum groups in the von Neumann algebraic setting.
- K. De Commer, cocycle twisting of locally compact quantum groups.
- S. Baaj and G. Skandalis, multiplicative unitaries and matched pairs.
- P. Stachura, locally compact quantum group deformation related to the Jordanian twist.
- P. Bieliavsky, V. Gayral, S. Neshveyev and L. Tuset, cocycle deformation / quantization constructions.