

Conditions on minimal twist for the Standard Model and the σ field

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Workshop on Noncommutative and Generalised Geometries and their Physical
Applications, Corfu, 19 September 2026



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Spectral triple [1]

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of

- A unital, involutive algebra $\mathcal{A}$
- A separable Hilbert space $\mathcal{H}$ with a faithful representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$
- A self-adjoint Dirac operator $D : \mathcal{H} \rightarrow \mathcal{H}$ where $[D, \pi(a)]$ is bounded for all $a \in \mathcal{A}$

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- Grading operator $\Gamma : \mathcal{H} \rightarrow \mathcal{H}$ (+ extra conditions)
- Antilinear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$ which implements the involution of the algebra $J\pi(a)J^{-1} = \pi(a)^\dagger = \pi(a^*)$
such that $J^2 = \epsilon \mathbb{I}$, $JD = \epsilon' DJ$, $J\Gamma = \epsilon'' \Gamma J$
where $\epsilon, \epsilon', \epsilon'' \in \{-1, 1\}$

Moreover, the following conditions are required

- order zero condition $[\pi(a), J\pi(b^*)J^{-1}] = 0$
- first-order condition $[[D, \pi(a)], J\pi(b^*)J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}$

The spectral triple of the Standard Model [1]

The spectral triple of the Standard Model is the almost-commutative spectral triple which is the product of the canonical triple with the finite-dimensional internal triple

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_{SM}, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \not{D} \otimes \mathbb{I}_F + \gamma^5 \otimes D_F$$

where

$$\mathcal{A}_{SM} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_F = \mathbb{C}^{32n} \text{ } n \text{ is number of generation,}$$

$$D_F = D_Y + D_M$$

containing the Yukawa couplings of fermions and the Majorana mass of the neutrino

where

$$D_Y = \begin{pmatrix} D_0 & \\ & D_0^\dagger \end{pmatrix}, \quad D_M = \begin{pmatrix} 0 & D_R \\ D_R^\dagger & 0 \end{pmatrix}.$$

and

$$J = J_{\mathcal{M}} \otimes J_F, \quad \Gamma = \gamma^5 \otimes \Gamma_F$$

The spectral triple of the Standard Model [2]

The bosonic degrees of freedom are obtained by a fluctuation of the Dirac operator, that is, $D \rightarrow D' = D + A + JAJ^{-1}$ where

$$A = \sum_i \pi(a_i)[D, \pi(b_i)], \quad a_i, b_i \in \mathcal{A},$$

is self-adjoint.

The bosonic spectral action can be obtained from the asymptotic expansion $\Lambda \rightarrow \infty$ of the spectral action

$$\text{Tr}f\left(\frac{D'}{\Lambda}\right).$$

Dirac operator

$$D = \not{D} \otimes \mathbb{I}_F + \gamma^5 \otimes D_F$$

(f being smooth approximation of the characteristic function of $[0, 1]$)

The spectral triple of the Standard Model [3]

Problem

The computation of the Higgs mass gives 170 GeV while the experiment shows 125 GeV.

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This problem can be cured by a new scalar field σ which couples to the Higgs in a suitable way. (Elias-Miro, Espinosa, Giudice, Lee, Strumia 2012)

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This problem can be cured by a new scalar field σ which couples to the Higgs in a suitable way. (Elias-Miro, Espinosa, Giudice, Lee, Strumia 2012)
This field is obtained by turning into a field the neutrino Majorana mass (Chamseddine, Connes 2012)

$$k_R \rightarrow k_R \sigma.$$

which can be obtained from the fluctuation of the off-diagonal part of D_F but

$$[\gamma^5 \otimes D_M, \pi(a)] = 0, \quad \forall a \in \mathcal{A}.$$

The spectral triple of the Standard Model [4]

$$D + A + JAJ^{-1}$$
$$A = \sum_i \pi(a_i)[D, \pi(b_i)]$$

Attempts of solution

- adding new fermions (Stephan 2009)
- relaxing the first-order condition (Chamseddine, Connes, Suijlekom 2013)
- modifying axiom regarding the real structure (Brzezinski, Ciccoli, Dabrowski, Sitarz 2016)
- twisting the original noncommutative geometry

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Twisted spectral triple [1]

The twisted spectral triple is $(\mathcal{A}, \mathcal{H}, D), \rho$ but now substitute $[D, \pi(a)]$ with

$$[D, \pi(a)]_{\rho} := D\pi(a) - \pi(\rho(a))D$$

where ρ is an automorphism of $\mathcal{A}$.

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Minimal twist (Devastato, Lizzi, Martinetti 2014)

The simplest way to construct a twisted spectral triple with the same Hilbert space and Dirac operator is to double the algebra and make them act independently on the left and right components of spinors.

$$\mathcal{A} = (C^{\infty}(\mathcal{M}) \otimes \mathcal{A}_{SM}) \oplus (C^{\infty}(\mathcal{M}) \otimes \mathcal{A}_{SM})$$

with a **flip**

$$\rho((a, a')) := (a', a)$$

Twisted spectral triple [2]

Define the representation by

$$\pi((a, a')) := \pi_+(a) + \pi_-(a') = \frac{\mathbb{I} + T}{2} \pi_0(a) + \frac{\mathbb{I} - T}{2} \pi_0(a')$$

where $\pi_0(a) = \pi_{\mathcal{M}}(f) \otimes \pi_F(a_F)$ and $T = \gamma^5 \otimes T_F$ with

$$\begin{array}{ll} T = T^\dagger & T_F = T_F^\dagger \\ T^2 = \mathbb{I} & \implies T_F^2 = \mathbb{I} \\ [T, \pi_0(a)] = 0 & [T_F, \pi_F(a_F)] = 0 \end{array}$$

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The conditions become

- order zero condition $[\pi((a, a')), J\pi((b^*, b'^*))J^{-1}] = 0$
- twisted first-order condition $[[D, \pi((a, a'))]_{\rho}, J\pi((b^*, b'^*))J^{-1}]_{\rho^o} = 0$

Twisted spectral triple [3]

(Filaci, Martinetti 2021)

Proposition 1

The minimal twist satisfies the order zero condition if and only if

$$[T_F, J_F T_F J_F^{-1}] = 0 \quad \text{and} \quad [T_F, J_F \pi_F(a_F) J_F^{-1}] = 0$$

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Proposition 2

Consider the minimal twist such that the order zero condition holds. Then the first-order condition holds if and only if

$$\{\{D_F, T_F\}, J_F T_F J_F^{-1}\} = 0 \quad \text{and} \quad [\{D_F, T_F\}, J_F \pi_F(a_F) J_F^{-1}] = 0$$

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Lemma 1

If the representation is faithful, then T_F cannot be written as a representation of the algebra. In general, $T_F \in \mathcal{U}(\mathcal{B}(\mathcal{H})) \setminus \mathcal{U}(\pi_F(a_F))$.

Conditions on the twisting operator [1]

Lemma 1

If the representation is faithful, then T_F cannot be written as a representation of the algebra. In general, $T_F \in \mathcal{U}(\mathcal{B}(\mathcal{H})) \setminus \mathcal{U}(\pi_F(a_F))$.

Lemma 2 (Filaci, Martinetti 2023)

If $\{T_F, D_M\} = 0$, then the fluctuation of Dirac operator does not generate an extra scalar field.

Therefore, in order to get an extra scalar field, $\{T_F, D_M\} \neq 0$.

Conditions on the twisting operator [2]

Lemma 3

If the order-zero and twisted first-order conditions hold then T_F is diagonal.

Conditions on the twisting operator [2]

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If the order-zero and twisted first-order conditions hold then T_F is diagonal.

Lemma 4

If T_F is diagonal and the twisted first-order condition holds, then the twisted fluctuation does not generate an extra scalar field.

Conditions on the twisting operator [2]

Lemma 3

If the order-zero and twisted first-order conditions hold then T_F is diagonal.

Lemma 4

If T_F is diagonal and the twisted first-order condition holds, then the twisted fluctuation does not generate an extra scalar field.

Theorem (UMI)(Ubonsri, Martinetti, Iseppi 2026)

For the spectral triple

$(C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \not{D} \otimes \mathbb{I} + \gamma^5 \otimes D_F; J_{\mathcal{M}} \otimes J_F)$ defined as in the original spectral triple for Standard Model and $T = \gamma^5 \otimes T_F$ where T_F satisfies order-zero condition, together with a minimal twist by a flip procedure, the twisted first-order condition and an extra scalar field cannot be obtained at the same time.

Conditions on the twisting operator [3]

$$T_F = T_F^\dagger$$

$$T_F^2 = \mathbb{I}$$

$$[T_F, \pi_F(a_F)] = 0$$

$$[T_F, J_F T_F J_F^{-1}] = 0$$

$$[T_F, J_F \pi_F(a_F) J_F^{-1}] = 0$$

With the original
 $\pi_F(a_F), H_F, D_F, J_F$.

$$\begin{aligned} \{\{D_F, T_F\}, J_F T_F J_F^{-1}\} &= 0 \\ \{\{D_F, T_F\}, J_F \pi_F(a_F) J_F^{-1}\} &= 0 \end{aligned} \implies \{T_F, D_M\} = 0$$

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- Minimal twist of the spectral triple of Standard Model provides an extra scalar field.
- In order to obtain an extra scalar field, one has to relax the twisted first-order condition. **The twisted spectral triple without the first-order condition was already discussed in (Martinetti, Zanchettin 2021).**
- The minimal twist procedure is still interesting because it is related to a change of signature from Euclidean to Lorentzian.

Thank you for your attention.

Support slides

If T_F anticommutes with D_M , then

$$(\gamma^5 \otimes D_M)\pi_+(a) = \pi_-(a)(\gamma^5 \otimes D_M) + \frac{1}{2}(\mathbb{I} - T)[\gamma^5 \otimes D_M, \pi_0(a)]$$

$$(\gamma^5 \otimes D_M)\pi_-(a) = \pi_+(a)(\gamma^5 \otimes D_M) + \frac{1}{2}(\mathbb{I} + T)[\gamma^5 \otimes D_M, \pi_0(a)]$$

So,

$$\begin{aligned} [\gamma^5 \otimes D_M, \pi((a, a'))]_\rho &= (\gamma^5 \otimes D_M)(\pi_+(a) + \pi_-(a')) \\ &\quad - (\pi_+(a') + \pi_-(a))(\gamma^5 \otimes D_M) \\ &= [\gamma^5 \otimes D_M, \pi_0(a)] + [\gamma^5 \otimes D_M, \pi_0(a')] = 0 \end{aligned}$$

Thus, in order to get an extra scalar field, $\{T_F, D_M\} \neq 0$.

In tensorial form, it is convenient to label the $32n$ degrees of freedom of $\mathcal{H}_F$ by a multi-index $CI\alpha$ defined as follow

- $C = 0, 1$ is for a particle ($C = 0$) or antiparticle ($C = 1$)
- $I = 0, 1, 2, 3$ is the leptocolor index where $I = 0$ means lepton and $I = 1, 2, 3$ means quark
- $\alpha = \dot{1}, \dot{2}, 1, 2$ is the flavor index
 - $\dot{1} = \nu_R, \quad \dot{2} = e_R, \quad 1 = \nu_L, \quad 2 = e_L$ for leptons ($I = 0$)
 - $\dot{1} = u_R, \quad \dot{2} = d_R, \quad 1 = u_L, \quad 2 = d_L$ for quarks ($I = 1, 2, 3$)

Therefore, the representation of the algebra $(c, q, m) \in \mathcal{A}_{SM}$ can be written as

$$\pi(a) = \begin{pmatrix} Q & \\ & M \end{pmatrix}_c^D$$

where

$$Q = Q_{I\alpha}^{J\beta} = \delta_I^J Q_\alpha^\beta, \quad M = M_{I\alpha}^{J\beta} = \delta_\alpha^\beta M_I^J$$

which

$$Q_\alpha^\beta = \begin{pmatrix} c & & \\ & \bar{c} & \\ & & q \end{pmatrix}_\alpha^\beta, \quad M_I^J = \begin{pmatrix} c & & \\ & & \\ & & m \end{pmatrix}_I^J.$$

The real structure is

$$J_F = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix}_C^D.$$

And the Dirac operator is

$$D_F = D_Y + D_M$$

where

$$D_Y = \begin{pmatrix} D_0 & \\ & D_0^\dagger \end{pmatrix}_C^D, \quad D_M = \begin{pmatrix} 0 & D_R \\ D_R^\dagger & 0 \end{pmatrix}_C^D.$$

Gauge transformation

$$\rho(U)D_A U^{-1} = D_{A^u}$$

where $U = uJuJ^{-1}$ and $\rho(U) = \rho(u)J\rho(u)J^{-1}$.

The latter does not preserve self-adjointness of D_A . However, it is self-adjoint with respect to the inner product induced by the twist

$$\langle \psi, O\phi \rangle_\rho = \langle \rho(O)^\dagger \psi, \phi \rangle_\rho$$

If $\rho(O) = ROR^\dagger$ for some unitary operator R , then

$$\langle \psi, \phi \rangle_\rho = \langle \psi, R\phi \rangle$$

We need $\rho(\pi((a, a'))) = \pi(\rho((a, a')))$, so $R = \gamma^0$.

Therefore,

$$\rho(\gamma^0) = \gamma^0, \quad \rho(\gamma^a) = -\gamma^a, \quad \forall a = 1, 2, 3.$$

Compare to Wick rotation

$$W(\gamma^0) = \gamma^0, \quad W(\gamma^a) = i\gamma^a, \quad \forall a = 1, 2, 3.$$

Thus, in some sense $\rho = W^2$.