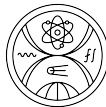


Spectral Geometry in Noncommutative Models

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Why revisit the fuzzy onion?

fuzzy onion $\longrightarrow$ Dirac operator

- We want to describe fermions on the fuzzy onion.
- The radial derivative was introduced by hand as a finite difference between neighbouring fuzzy spheres.
- The polarization-tensor basis works well in practice, but we would like to understand its geometric origin.

Can these ingredients arise naturally from the geometry?

Why start upstairs?

Odd-dimensional spaces do not have a canonical non-degenerate Poisson structure.

$\mathbb{C}^2 \simeq \mathbb{R}^4$ does have the oscillator quantization

$$\{z_\alpha, \bar{z}_\beta\} \sim -i\delta_{\alpha\beta} \quad \rightsquigarrow \quad [a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}.$$

So the strategy is: quantize the clean space, then reduce.

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Fuzzy onion: a quick reminder

$$\psi = \left(\begin{array}{c} (.) \\ \begin{pmatrix} \cdot & \cdot \end{pmatrix} \\ \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \\ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \end{array} \right)$$


Comparing neighbouring fuzzy spheres

On the n -th fuzzy sphere we expand

$$\psi^{(n)} = \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} \psi_{\ell m}^{(n)} Y_{\ell m}^{(n)},$$

where $Y_{\ell m}^{(n)}$ are the polarization tensors (the fuzzy spherical harmonics).

To move the same angular configuration to the next shell, we keep its coefficients and only change the matrix representation:

$$\text{Up}_n(\psi^{(n)}) = \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} \psi_{\ell m}^{(n)} Y_{\ell m}^{(n+1)}.$$

Going down, the highest angular momentum no longer fits:

$$\text{Down}_n(\psi^{(n)}) = \sum_{\ell=0}^{n-2} \sum_{m=-\ell}^{\ell} \psi_{\ell m}^{(n)} Y_{\ell m}^{(n-1)}.$$

The radial derivative was then put in by hand

Both neighbouring fields can now be transported to the n -th shell:

$$\psi^{(n+1)} \xrightarrow{\text{Down}_{n+1}} M_n(\mathbb{C}), \quad \psi^{(n-1)} \xrightarrow{\text{Up}_{n-1}} M_n(\mathbb{C}).$$

This allows us to define the central radial difference

$$(\partial_r \psi)^{(n)} = \frac{\text{Down}_{n+1}(\psi^{(n+1)}) - \text{Up}_{n-1}(\psi^{(n-1)})}{2\lambda}.$$

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From $\mathbb{C}^2$ to $\mathbb{R}^3$

Start with

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 \setminus \{0\}, \quad x_i = z^\dagger \sigma_i z.$$

The coordinates x_i are invariant under the common phase,

$$z \longmapsto e^{i\alpha} z, \quad x_i \longmapsto x_i,$$

so the phase parametrizes the Hopf fibre.

Using Euler angles,

$$z_1 = \rho \cos \frac{\theta}{2} e^{\frac{i}{2}(\varphi + \chi)}, \quad z_2 = \rho \sin \frac{\theta}{2} e^{\frac{i}{2}(-\varphi + \chi)},$$

and therefore

$$\vec{x} = \rho^2 (\sin \theta \cos \varphi, -\sin \theta \sin \varphi, \cos \theta).$$

$$r = \rho^2, \quad \chi = \text{fibre coordinate}.$$

Quantizing the $\mathbb{R}^3_\lambda$ coordinates

Quantization upstairs is standard:

$$z_\alpha \longrightarrow \sqrt{\lambda} a_\alpha, \quad \bar{z}_\alpha \longrightarrow \sqrt{\lambda} a_\alpha^\dagger, \quad [a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}.$$

The Hopf coordinates become

$$x_i = \lambda a^\dagger \sigma_i a,$$

and satisfy

$$[x_i, x_j] = 2i\lambda \varepsilon_{ijk} x_k.$$

The total occupation number

$$N = a_1^\dagger a_1 + a_2^\dagger a_2$$

labels the fuzzy spheres, and we choose

$$r = \lambda(N+1), \quad r^2 = \vec{x}^2 + \lambda^2.$$

ordering: The shift in the definition of r is a convention we have to choose.

Functions on $\mathbb{R}_\lambda^3$

The oscillator Fock space decomposes into fixed- N sectors,

$$\mathcal{H}_F = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad \dim \mathcal{H}_n = n + 1.$$

A function downstairs should be invariant along the Hopf fibre, so we define

$$A_\lambda = \{\Phi \in \text{End}(\mathcal{H}_F) \mid [N, \Phi] = 0\}.$$

Since $[N, \Phi] = 0$, the operator Φ preserves every fixed- N sector:

$$\Phi = \bigoplus_{n=0}^{\infty} \Phi_n, \quad \Phi_n \in \text{End}(\mathcal{H}_n).$$

Therefore

$$A_\lambda \simeq \bigoplus_{n=0}^{\infty} \text{End}(\mathcal{H}_n) \simeq \bigoplus_{n=0}^{\infty} M_{n+1}(\mathbb{C}).$$

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The Hopf-adapted geometry

$$ds_4^2 = d\rho^2 + \frac{\rho^2}{4} [d\theta^2 + \sin^2 \theta d\varphi^2 + (d\chi + \cos \theta d\varphi)^2].$$

$$e^0 = d\rho, \quad e^1 = \frac{\rho}{2} d\theta, \quad e^2 = \frac{\rho}{2} \sin \theta d\varphi,$$

$$e^3 = \frac{\rho}{2} (d\chi + \cos \theta d\varphi).$$

The e^3 direction contains the Hopf fibre coordinate χ .

Spinors upstairs and downstairs

Upstairs the Dirac operator acts on

$$V_4 = \mathbb{C}^4 = V^+ \oplus V^-.$$

Downstairs we want

$$V_3 = \mathbb{C}^2.$$

We identify the downstairs spinor space with the negative Weyl sector:

$$j_- : V_3 \rightarrow V^-, \quad P_- : V^- \rightarrow V_3.$$

Thus a 3D spinor is first viewed as a Weyl spinor upstairs.

Why the fibre gamma?

The four-dimensional Dirac operator acts on the full spinor space,

$$D_4 : V_4 \longrightarrow V_4.$$

Since it anticommutes with chirality,

$$\Psi^- \in V^- \quad \Longrightarrow \quad D_4 \Psi^- \in V^+.$$

The fibre gamma also changes chirality,

$$i\gamma^3 : V^+ \longrightarrow V^-.$$

Therefore

$$V^- \xrightarrow{D_4} V^+ \xrightarrow{i\gamma^3} V^-.$$

On V^- the combinations

$$\tau^j = i\gamma^3 \gamma^j \Big|_{V^-}, \quad j = 0, 1, 2,$$

give the effective three-dimensional Clifford action.

The reduced Dirac operator

We identify the 3D spinor space with the negative Weyl sector,

$$j_- : V_3 \longrightarrow V^-, \quad P_- : V^- \longrightarrow V_3.$$

The reduction is

$$\psi \xrightarrow{j_-} \Psi^- \xrightarrow{D_4} \Psi^+ \xrightarrow{i\gamma^3} \tilde{\Psi}^- \xrightarrow{P_-} D_{3,\text{red}}\psi.$$

Thus

$$D_{3,\text{red}} = P_- i\gamma^3 D_4 j_-.$$

For fibre-independent spinors,

$$D_{3,\text{red}} = \tau^0 \left(\partial_\rho + \frac{1}{\rho} \right) + \frac{2}{\rho} \tau^1 \left(\partial_\theta + \frac{1}{2} \cot \theta \right) + \frac{2}{\rho \sin \theta} \tau^2 \partial_\varphi.$$

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Quantize the upstairs Dirac

The clean object is still the Dirac operator upstairs on $\mathbb{C}_\lambda^2$:

$$\partial_{z_\alpha} \rightsquigarrow -\frac{1}{\sqrt{\lambda}} [a_\alpha^\dagger, \cdot], \quad \partial_{\bar{z}_\alpha} \rightsquigarrow \frac{1}{\sqrt{\lambda}} [a_\alpha, \cdot].$$

This gives

$$D_4^{\text{osc}} = \frac{1}{\sqrt{\lambda}} \sum_{\alpha=1}^2 \left(-\Gamma_\alpha [a_\alpha^\dagger, \cdot] + \Gamma_{\bar{\alpha}} [a_\alpha, \cdot] \right).$$

This is the direct oscillator analogue of the four-dimensional Dirac operator.

Fibre grading of the oscillator algebra

Recall that N generates the Hopf-fibre rotation. We introduce

$$E_k = \{\Phi \mid [N, \Phi] = k\Phi\}.$$

The ordinary functions on $\mathbb{R}^3_\lambda$ are the neutral sector,

$$A_\lambda = E_0.$$

The basic oscillator operations shift this grading:

$$[a_\alpha^\dagger, \Phi] \in E_{k+1}, \quad [a_\alpha, \Phi] \in E_{k-1}, \quad \Phi \in E_k.$$

So $[a_\alpha^\dagger, \cdot]$ and $[a_\alpha, \cdot]$ move us between fibre-charge sectors.

What are the Γ_α ?

The complex Clifford matrices are

$$\begin{aligned}\Gamma_1 &= \gamma^0 + i\gamma^1, & \Gamma_{\bar{1}} &= \gamma^0 - i\gamma^1, \\ \Gamma_2 &= \gamma^2 + i\gamma^3, & \Gamma_{\bar{2}} &= \gamma^2 - i\gamma^3.\end{aligned}$$

In D_4^{osc} the two ingredients act on different factors:

$$\Phi \otimes \nu \longmapsto [a_\alpha^\dagger, \Phi] \otimes \Gamma_\alpha \nu.$$

The Γ matrices act only on the spinor index, so they do not change the oscillator grading E_k .

They do, however, change chirality:

$$\Gamma_\alpha, \Gamma_{\bar{\alpha}} : V^- \longrightarrow V^+.$$

The neutral space is not closed

Start with a spinor whose coefficient is an ordinary function on $\mathbb{R}^3_\lambda$,

$$\Phi \otimes v^- \in E_0 \otimes V^-.$$

The two pieces of D_4^{osc} produce

$$[a_\alpha^\dagger, \Phi] \otimes \Gamma_\alpha v^- \in E_{+1} \otimes V^+,$$

and

$$[a_\alpha, \Phi] \otimes \Gamma_{\bar{\alpha}} v^- \in E_{-1} \otimes V^+.$$

Therefore

$$D_4^{\text{osc}} : E_0 \otimes V^- \longrightarrow (E_{+1} \oplus E_{-1}) \otimes V^+.$$

Instead of projecting these terms away, let us understand what $E_{\pm 1}$ actually mean.

Relation to Kováčik–Prešnajder monopoles

In the oscillator description of the $\mathbb{R}^3_\lambda$ monopoles, one considers states with a fixed imbalance between creation and annihilation operators,

$$\kappa = \#(a^\dagger) - \#(a).$$

Equivalently,

$$[N, \Psi_\kappa] = \kappa \Psi_\kappa$$

up to the convention for the sign of the charge.

This is precisely the same grading as

$$E_k = \{\Phi \mid [N, \Phi] = k\Phi\}.$$

The sectors appearing in the Dirac construction are the same type of sectors already known from $\mathbb{R}^3_\lambda$ monopoles.

Charged intermediate spinors

$$S_\lambda^- = E_0 \otimes V^-.$$

But

$$D_4^{\text{osc}} : S_\lambda^- \longrightarrow (E_{+1} \oplus E_{-1}) \otimes V^+.$$

The full spinor remains Hopf-equivariant; we suppress the spinorial details.

The opposite charge shifts are

$$E_{+1} \xrightarrow{a_\alpha} E_0, \quad E_{-1} \xrightarrow{a_\alpha^\dagger} E_0.$$

Therefore

$$V_\lambda = \sum_{\alpha=1}^2 \left(a_\alpha \Gamma_{\bar{\alpha}} - a_\alpha^\dagger \Gamma_\alpha \right)$$

is the natural oscillator part of the vertical Clifford map.

The noncommutative fibre Clifford map

Including the radial normalization, we take

$$G_\lambda = \frac{i}{2}(N+1)^{-1/4} V_\lambda (N+1)^{-1/4}.$$

It maps the charged positive-Weyl sector back to the neutral one,

$$S_\lambda^+ \xrightarrow{G_\lambda} S_\lambda^-.$$

Therefore the classical pattern survives:

$$S_\lambda^- \xrightarrow{D_4^{\text{osc}}} S_\lambda^+ \xrightarrow{G_\lambda} S_\lambda^-.$$

ordering: The symmetric placement of $(N+1)^{-1/4}$ is an ordering choice.

The reduced noncommutative operator

The whole construction mirrors the classical reduction:

$$\psi_\lambda \xrightarrow{j_\lambda} \Psi_\lambda^- \xrightarrow{D_4^{\text{osc}}} \Psi_\lambda^+ \xrightarrow{G_\lambda} \tilde{\Psi}_\lambda^- \xrightarrow{P_\lambda} D_\lambda^{(0)} \psi_\lambda.$$

Hence

$$D_\lambda^{(0)} = P_\lambda G_\lambda D_4^{\text{osc}} j_\lambda.$$

The monopole sectors appear only in the intermediate Weyl sector; the reduced operator acts again on neutral 3D spinors.

Where choices remain

The Hopf construction fixes much of the structure, but not everything.

$$r = \lambda(N + 1) \quad \textbf{ordering:choice of radial convention}$$

$$G_\lambda = \frac{i}{2}(N + 1)^{-1/4} V_\lambda (N + 1)^{-1/4} \quad \textbf{ordering:ordering choice}$$

$$j_\lambda, P_\lambda \quad \textbf{choice:identification of the reduced spinor space}$$

The choices have not disappeared, but we can now see exactly where they enter.

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What has been achieved

Starting from the four-dimensional geometry we obtained

D_4^{osc} directly from oscillator quantization on $\mathbb{C}_\lambda^2$,

a natural equivariant spinor structure

$$S_\lambda^- \xrightarrow{D_4^{\text{osc}}} S_\lambda^+,$$

where the intermediate charged components are precisely the monopole-type sectors,

and a quantization of the vertical Clifford map

$$S_\lambda^+ \xrightarrow{\gamma_{\text{vert},\lambda}} S_\lambda^-.$$

Together,

$$D_\lambda^{(0)} = P_\lambda \gamma_{\text{vert},\lambda} D_4^{\text{osc}} j_\lambda.$$

What is still missing?

The operator

$$D_{\lambda}^{(0)}$$

is a geometrically motivated candidate, not yet a complete spectral triple.

- choose the Hilbert-space completion and representation of A_{λ} ,
- specify the operator domain,
- establish symmetry / self-adjointness,
- check the commutators

$$[D_{\lambda}, a],$$

- establish the appropriate compactness and spectral properties,
- understand the finite fuzzy-onion truncation and the induced radial coupling.

Takeaway

The Hopf route does not remove every choice.
It tells us where the structure comes from.

$$D_4 \longrightarrow D_4^{\text{osc}} \longrightarrow \text{charged / monopole sectors} \longrightarrow \gamma_{\text{vert},\lambda} \longrightarrow D_\lambda^{(0)}.$$