

Corfu Summer Institute

*Workshop on Noncommutative and Generalised Geometries
and their Physical Applications
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Noncommutative Bianchi identities

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Plan:

-PART 1- Diff. geometry and Cartan calculus on a wide class of NC algebras

Connections

-NC Riemannian geometry:

Metrics

Existence and uniqueness of Levi-Civita connection

NC Gravity equations

[P.A. JGP 2026]

-PART 2- Bianchi identities

[P.A. arXiv 2608.24536]

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–PART 2- *Bianchi identities*

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Connections (and metrics) will be arbitrary fields: no bimodule connection condition (or metric centrality conditions). This is appropriate for dynamical fields like in NC gauge and gravity theories.

Data:

- (H, R) triangular Hopf algebra, or quantum group

We use the notation $R^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha \in H \otimes H$

Triangularity: $R^{-1} = R_{21}$.

- A an H -module algebra

A carries a representation of the quantum group H .

- *The product in A is R -commutative:*

$$ab = (\bar{R}^\alpha \triangleright b)(\bar{R}_\alpha \triangleright b) .$$

Examples:

$$[x, y] = i\hbar$$

$$xy = qyx$$

$$[x, y] = \frac{i}{\kappa} y$$

Examples:

- *All NC algebras arising as Drinfeld twist (2-cocycle) deformations of commutative algebras are of this kind: e.g. NC-torus; Connes-Landi spheres, Connes–Dubois-Violette NC manifolds....*
- *Any cotriangular Hopf algebra, for example Sweedler Hopf algebra H_4 .*

In the present study there is no assumption on the existence of derivations of the algebra, and no use of special coordinates. Indeed we use a global, coordinate independent, approach.

We retrieve the results in [Wess et al. 2005] on Moyal-Weyl quantum plane and in [Rosenberg '13] on T_θ^n by considering coordinates x^μ and partial derivatives ∂_μ .

$(H, \mathcal{R})$ a triangular Hopf algebra over $\mathbb{C}$ (or $\mathbb{C}[[\hbar]]$ power series in $\hbar$)

The H -module A -bimodule $\text{Der}_{\mathcal{R}}(A)$ in ${}^H_A\mathcal{M}_A^{\text{sym}}$ of braided derivations.

$u \in \text{Der}_{\mathcal{R}}(A) \subset \text{hom}_{\mathbb{C}}(A, A)$ with braided Leibniz rule

$$u(ab) = u(a)b + (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright u)(b) .$$

$au(b) := a u(b)$ and $ua := (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright u)$ are again braided derivations.

The braided commutator

$$[\ , \]_{\mathcal{R}} : \text{Der}_{\mathcal{R}}(A) \otimes \text{Der}_{\mathcal{R}}(A) \rightarrow \text{Der}_{\mathcal{R}}(A) , \quad u \otimes v \mapsto uv - (\bar{R}^\alpha \triangleright v)(\bar{R}_\alpha \triangleright u)$$

structures $\text{Der}_{\mathcal{R}}(A)$ as a Lie algebra,

$$[u, v]_{\mathcal{R}} = -[\bar{R}^\alpha \triangleright v, \bar{R}_\alpha \triangleright u]_{\mathcal{R}}$$

$$[u, [v, z]_{\mathcal{R}}]_{\mathcal{R}} = [[u, v]_{\mathcal{R}}, z]_{\mathcal{R}} + [\bar{R}^\alpha \triangleright v, [\bar{R}_\alpha \triangleright u, z]_{\mathcal{R}}]_{\mathcal{R}} .$$

1-forms are dual to vector fields

$$\Omega(A) := {}_A\text{hom}(\text{Der}(A), A)$$

(left A-linear maps). Pairing:

$$u \otimes_A \omega \mapsto \langle u, \omega \rangle$$

Exterior derivative

$$\langle u, da \rangle = u(a) ,$$

Contraction operator

$$i_u(\omega) = \langle u, \omega \rangle . \tag{1}$$

Generalize the pairing to the tensor algebra

$$\langle \nu \otimes_A u, \omega_1 \otimes_A \omega_2 \dots \omega_p \otimes_A v_1 \otimes_A \dots v_q \rangle = \langle \nu, \langle u_1, \omega_1 \rangle \omega_2 \dots \otimes_A v_1 \otimes_A v_q \rangle .$$

Exterior product

$$\omega \wedge \omega' := \omega \otimes_A \omega' - \bar{R}^\alpha \triangleright \omega' \otimes_A \bar{R}_\alpha \triangleright \omega , \tag{2}$$

is braided antisymmetric.

Lie derivative

$$\mathcal{L}_u(a) := u(a) , \quad \mathcal{L}_u(v) := [u, v] .$$

$\mathcal{L}$ is H -equivariant. Extended to the tensor algebra via:

$$\mathcal{L}_u(\nu \otimes_A \nu') = \mathcal{L}_u(\nu) \otimes_A \nu' + \bar{R}^\alpha \triangleright \nu \otimes_A \mathcal{L}_{\bar{R}_\alpha \triangleright u}(\nu')$$

and on contravariant tensor fields is canonically defined by duality,

$$\mathcal{L}_u \langle \nu, \theta \rangle = \langle \mathcal{L}_u \nu, \theta \rangle + \langle \bar{R}^\alpha \triangleright \nu, \mathcal{L}_{\bar{R}_\alpha \triangleright u} \theta \rangle \quad (3)$$

Theorem (*Braided Cartan calculus*) [T. Weber]

$$\begin{aligned} [\mathcal{L}_u, \mathcal{L}_v] &= \mathcal{L}_{[u, v]}, \quad [i_u, i_v] = 0, \\ [\mathcal{L}_u, i_v] &= i_{[u, v]}, \quad [i_u, d] = \mathcal{L}_u, \\ [\mathcal{L}_u, d] &= 0, \quad [d, d] = 0, \end{aligned}$$

where $[L, L'] = L \circ L' - (-1)^{|L||L'|} \bar{R}^\alpha(L') \circ \bar{R}_\alpha(L)$ is the graded braided commutator of $\mathbb{C}$ -linear maps L, L' on $\Omega^\bullet(A)$ of degree $|L|$ and $|L'|$.

Connections and Cartan equation

Def. A right connection on a module Γ in ${}^H_A \mathcal{M}_A^{\text{sym}}$ (e.g. $\text{Der}(A)$ or $\Omega(A)$) is a $\mathbb{C}$ -linear map

$$\nabla : \Gamma \rightarrow \Gamma \otimes_A \Omega \quad (4)$$

which satisfies the Leibniz rule, for all $s \in \Gamma$, $a \in A$,

$$\nabla(sa) = \nabla(s)a + s \otimes_A da . \quad (5)$$

A left connection on Γ is a $\mathbb{C}$ -linear map

$$\nabla : \Gamma \rightarrow \Omega \otimes_A \Gamma \quad (6)$$

which satisfies the Leibniz rule,

$$\nabla(as) = da \otimes_A s + a \nabla(s) . \quad (7)$$

A right connection ∇ on Γ is also a braided left connection [P.A, Schenkel '14] (similarly a left connection ∇ on Γ is also a braided right connection):

$$\nabla(as) = (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright \nabla)(s) + \bar{R}^\alpha \triangleright s \otimes_A \bar{R}_\alpha \triangleright da$$

If ∇ is H -equivariant we recover the notion of connection on central A -bimodules studied in [Dubois-Violette, Michor].

Given connections $\nabla \in \text{Con}_A(\Gamma)$, $\hat{\nabla} \in \text{Con}_A(\hat{\Gamma})$ on the modules Γ and $\hat{\Gamma}$ in ${}^H_A \mathcal{M}_A^{\text{sym}}$ we consider the connection $\nabla \oplus_{\mathcal{R}} \hat{\nabla} \in {}_A \text{Con}(\Gamma \otimes_A \hat{\Gamma})$ **well defined** by

$$\begin{aligned} \nabla \oplus_{\mathcal{R}} \hat{\nabla} : \Gamma \otimes_A \hat{\Gamma} &\longrightarrow \Gamma \otimes_A \hat{\Gamma} \otimes_A \Omega_{\mathcal{R}}(A) \\ s \otimes_A \hat{s} &\longmapsto \tau_{23} \circ (\nabla(s) \otimes_A \hat{s}) + (\bar{R}^\alpha \triangleright s) \otimes_A (\bar{R}_\beta \triangleright \hat{\nabla})(\hat{s}) \end{aligned}$$

where we have used the braiding isomorphism

$$\tau_{23} : \Gamma \otimes_A \Omega_{\mathcal{R}}(A) \otimes_A \hat{\Gamma} \rightarrow \Gamma \otimes_A \hat{\Gamma} \otimes \Omega_{\mathcal{R}}(A).$$

(a similar formula holds for left connections).

Extend ∇ to

$$d_{\nabla} : \Gamma \otimes_A \Omega^{\bullet}(A) \longrightarrow \Gamma \otimes_A \Omega^{\bullet+1}(A) ,$$

by

$$d_{\nabla}(s \otimes_A \theta) = \nabla(s) \otimes_A \theta + s \otimes_A d\theta ,$$

Curvature

The curvature of $\nabla : \Gamma \rightarrow \Omega(A) \otimes_A \Gamma$ is

$$d_{\nabla}^2 = d_{\nabla} \circ d_{\nabla} .$$

It is a left $\Omega_{\mathcal{R}}^{\bullet}(A)$ -linear map, $\Omega_{\mathcal{R}}^{\bullet}(A) \otimes_A \Gamma \rightarrow \Omega_{\mathcal{R}}^{\bullet+2} \otimes_A \Gamma$

Torsion Let $I \in \Omega(A) \otimes_A \text{Der}(A)$ give the identity map $\theta \rightarrow \langle I, \theta \rangle = \theta$

The torsion of ∇ is

$$d_{\nabla}(I) \in \Omega^2(A) \otimes_A \text{Der}(A)$$

Def. Connection along vector field is

$$\nabla_u := i_u \circ \nabla$$

It is the composition of ∇ acting from the *right* and i_u acting from the *left*.
More in general:

$$d_{\nabla_u} := i_u \circ d_{\nabla} + d_{\nabla} \circ i_u , \quad (9)$$

Lemma Braided Leibniz rule, for d_{∇_u}

$$d_{\nabla_u}(\theta \wedge \sigma) = \mathcal{L}_u(\theta) \wedge \sigma + (\bar{R}^\alpha \triangleright \theta) \wedge d_{\nabla_{\bar{R}_\alpha \triangleright u}}(\sigma) .$$

Theorem Braided Cartan relation for d_{∇_u}

$$d_{\nabla_u} i_v - i_{\bar{R}^\alpha \triangleright v} d_{\nabla_{\bar{R}_\alpha \triangleright u}} = i_{[u,v]} .$$

All other expression of curvature and torsion are equivalent due to the above Cartan relation.

Def. $R_{\nabla}(u, v, z) := (\nabla_u \circ \nabla_v - \nabla_{\bar{R}^\alpha \triangleright v} \circ \nabla_{\bar{R}_\alpha \triangleright u} - \nabla_{[u, v]})(z) .$

$$T_{\nabla}(u, v) := \nabla_u v - \nabla_{\bar{R}^\alpha \triangleright v} \bar{R}_\alpha \triangleright u - [u, v] .$$

Proposition

$$R_{\nabla}(u, v, s) = -\langle u \otimes_A v, d_{\nabla}^2 s \rangle$$

$$T_{\nabla}(u, v) = \langle u \otimes_A v, d_{\nabla}(I) \rangle$$

Riemannian geometry

Let $g \in \Omega(A) \otimes_A \Omega(A)$.

Def. g is braided symmetric if invariant under the braiding realized by $\mathcal{R}$.

Example: $\omega \otimes \omega' + (\bar{R}^\alpha \triangleright \omega') \otimes (\bar{R}_\alpha \triangleright \omega)$ is braided symmetric.

Given $g \in \Omega(A) \otimes_A \Omega(A)$ we write $g = g^a \otimes_A g_a$ and define

$$g^b : \mathfrak{X}(A) \rightarrow \Omega(A), \quad v \mapsto g^b(v) = \langle v, g^a \rangle g_a$$

Def. A pseudo-Riemannian metric on $\mathfrak{X}(A)$ is a braided symmetric element with g^b that is an isomorphism.

Let $g \in \Omega(A) \otimes_A \Omega(A)$ be a pseudo-Riemannian metric. A connection $\nabla \in \text{Con}_A(\Omega(A))$ is metric compatible if it satisfies $\nabla(g) = 0$. A connection $\nabla \in {}_A\text{Con}(\mathfrak{X}(A))$ is metric compatible if its dual $\nabla \in \text{Con}_A(\Omega(A))$ is metric compatible. It follows

$$d\langle v \otimes_A z, g \rangle = \langle \nabla(v \otimes_A z), g \rangle$$

A Levi-Civita connection is a metric compatible and torsion free connection.

Existence and uniqueness of Levi-Civita connection is proven, similarly to the classical case, via a **braided Koszul formula**.

For all $u, v, z \in \text{Der}(A)$, (braiding omitted)

$$\begin{aligned}\mathcal{L}_u \langle v \otimes_A z, g \rangle &= \langle \nabla_u (v \otimes_A z), g \rangle \\ &= \langle z \otimes_A \nabla_v u, g \rangle + \langle [u, v] \otimes_A z, g \rangle + \langle v \otimes_A \nabla_u z, g \rangle\end{aligned}$$

Summing $\mathcal{L}_u \langle v \otimes_A z, g \rangle - \mathcal{L}_z \langle u \otimes_A v, g \rangle + \mathcal{L}_v \langle z \otimes_A u, g \rangle$ (braiding omitted) we obtain

$$\begin{aligned}2 \langle {}^\alpha v \otimes_A \nabla_{\alpha u} z, g \rangle &= \mathcal{L}_u \langle v \otimes_A z, g \rangle - \mathcal{L}_{\alpha v} \langle \alpha u \otimes_A z, g \rangle + \mathcal{L}_{\alpha \beta z} \langle \alpha u \otimes_A \beta v, g \rangle \\ &\quad - \langle [u, v] \otimes_A z, g \rangle + \langle u \otimes_A [v, z], g \rangle + \langle [u, {}^\beta z] \otimes_A \beta v, g \rangle .\end{aligned}$$

were ${}^\alpha v := \bar{R}^\alpha \triangleright v$ and ${}_\alpha u := \bar{R}_\alpha \triangleright u$. Now, since u, v, z are arbitrary, the pairing is nondegenerate and the metric is also nondegenerate, knowledge of the l.h.s. uniquely defines the Levi-Civita connection.

Conclusions

- Given a wide class of algebras A :
all those admitting an action of a triangular Hopf algebra, including all those obtained from Drinfeld twist (2-cocycle deform.) of commutative manifolds
- Given an arbitrary braided symmetric metric g on A

We have shown existence and uniqueness of the Levi-Civita connection ∇ .

This gives Einstein equations on A .

Ricci tensor (trace of Riemann tensor):

$$Ric(u, v) = \langle \omega^i, R_{\nabla}(e_i, u, v) \rangle' .$$

Einstein equations in vacuum

$$Ric(u, v) = \lambda \langle u \otimes_A u, g \rangle , \quad (\lambda \in \mathbb{k}) .$$

Bianchi identities

- Chern-Weil theory of characteristic classes
- integrability condition for the Yang-Mills equations
- contracted Bianchi identity in GR $\Rightarrow$ covariant conservation of the energy-momentum tensor.

Comparing different approaches to NC gauge theories:

- Twisted gauge theories [ADMSW LMP 06]
- Braided L^∞ -approach [DCGRS LMP 2021]
- Principal bundle approach [ALP JNG 25]

NC Bianchi identity in the context of projective modules appear in Connes foundational paper IHES 85.

More explicit treatment in [Landi LNP 1997].

Recent work in [Beggs,Majid 2018] and their book [BM2020] with also first Bianchi identity.

We recover the above results and provide a global coefficient expression on arbitrary finitely generated projective modules;

We present equivalent expressions in terms of R_{∇} and T_{∇} for algebras A that are R -commutative.

Bianchi id.

Commut. case

$$R = dA + A \wedge A$$

③

(& local)

$$DR = dR + A \wedge R - R \wedge A = 0$$

②

NC & global

$$\mathbb{D}: \Gamma \rightarrow \Omega \otimes \Gamma \quad d_{\mathbb{D}}: \Omega^i \otimes \Gamma \rightarrow \Omega^{i+1} \otimes \Gamma$$

$$R = d_{\mathbb{D}}^2: \Gamma \rightarrow \Omega^2 \otimes \Gamma$$

Bianchi $d_{\mathbb{D}} \circ A - R \circ d_{\mathbb{D}} = 0$ trivially by associativity

. set $\mathbb{D}(e_i) = A_i^j e_j$ and recover 1 & 2 (e_i) basis of Γ

NC case caveat:

$$\nabla^{\text{Hom}} \varphi := \nabla' \circ \varphi - \varphi \circ \nabla \in {}_A \text{Hom}(P, \mathcal{R} \otimes_A P') \simeq \mathcal{R} \otimes_A {}_A \text{Hom}(P, P')$$

but ∇^{Hom} is not a connection:

$$\nabla^{\text{Hom}}(a\varphi) \neq da \otimes \varphi + a \otimes \nabla^{\text{Hom}} \varphi$$

only if ∇ is H -equivariant (bimodule connection) ∇^{Hom} is a conn.

Bianchi is however

$$\underline{d_{\nabla} \circ R - R \circ d_{\nabla} = 0}$$

In terms of $R(u, v) : \mathcal{P} \rightarrow \mathcal{P}$ $\forall u, v \in \mathcal{D}_2(A)$

$$s \mapsto R(u, v)s = R(u, v, s) = \langle u \otimes v, d_{\mathcal{D}}^2 s \rangle$$

Bianchi

$$\nabla_u \circ R(v, z) - R(u, v) \circ \nabla_z + R(u, [v, z]) + \text{b.c.p.}(u, v, z) = 0$$

or

$$\nabla_u \circ R(v, z) - R(\alpha v, \beta z) \circ \nabla_{\rho \alpha u} + R(u, [v, z]) + \text{b.c.p.}(u, v, z) = 0$$

Proof Apply $i_u \circ i_v \circ i_z$ to $d_W \circ A - R \circ d_W = 0$

and use Cartan: $i_z d_W = d_{W_z} - d_W i_z$

$$i_v d_{W_z} = d_{W_z} \circ i_{\alpha^v} + i_{[v, z]}$$

Equivalent expression of Bianchi using any

$$\nabla: \mathcal{K} \rightarrow \Omega \otimes \mathcal{K}.$$

$$\chi \equiv \text{Der}(A).$$

Lemma:

$$\lambda \circ (\nabla_n(v \otimes z) + u \otimes [v, z]) + \text{b.c.p.}(u, v, z) = -u \wedge T[v, z] + \text{b.c.p.}(u, v, z)$$

Bianchi:

$$\nabla_n(R(v \otimes z \otimes s)) - R(\nabla_n(v \otimes z \otimes s)) - R(u, (\nabla T)[v, z], s) + \text{b.c.p.}(u, v, z) = 0.$$

In particular, if $\nabla: \mathcal{K} \rightarrow \Omega \otimes \mathcal{K}$, has vanishing torsion

$$\nabla_n(R(v \otimes z \otimes s)) - R(\nabla_n(v \otimes z \otimes s)) + \text{b.c.p.}(u, v, z) = 0.$$

1st Bianchi: Let $I \in \Omega^2 \otimes \mathcal{X}$ give the identity map $\mathcal{X} \rightarrow \mathcal{X}$
 $u \mapsto \langle u, I \rangle = u$
 Torsion is $d_{\nabla} I \in \Omega^3 \otimes \mathcal{X}$

Bianchi: $d_{\nabla}(d_{\nabla} I) = d_{\nabla}^2 I$ trivial associativity property in $\Omega^3 \otimes \mathcal{X}$

Then apply $i_u \circ i_v \circ i_z$,

use $T(v, z) = -i_v i_z d_{\nabla} I$

and obtain:

$$R(u \otimes v \otimes z) - (\nabla_u \circ T - T \circ \nabla_u)(v \otimes z) + T(u \otimes T(v, z)) + \text{b.c.p.}(u, v, z) = 0$$

□