

Gauged Poisson (and Courant) sigma models and their geometry

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§1. Introduction

To do

Gauging of AKSZ sigma models (Special Batalin-Vilkovisky formalism)
Alexandrov, Kontsevich, Schwartz, Zaboronsky '97

Purposes • Equivariant generalizations of geometry in physics

cf. Equivariant BV formalism, weak Q-manifolds

Bonechi-Cattaneo-Qiu-Zabzine '20, Grigoriev-Rudinsky '25

Carow-Watamura-NI-Kaneko-Watamura '19

cf. Realizations of momentum maps by Q-manifolds and AKSZ sigma models
Zucchini '08, Bonechi-Cabrera-Zabzine '12

Gauged sigma models geometry in target space + symmetry

pre-(multi)symplectic (and metric) + Lie groups

(Gauged nonlinear sigma models) Hull-Spence '91

pre-(multi)symplectic (and metric) + Lie algebroids(groupoids)

Chatzistavrakidis-Deser-Jonke '16, Chatzistavrakidis-Deser-Jonke-Strobl '16

Poisson + Lie algebroids(algebras)?

Courant algebroids + Lie algebroids (Courant algebroids)?

Higher groupoids + another higher groupoids?

Applications of AKSZ sigma models

- 2D Poisson sigma models NI-Izawa '93, NI '94, Schaller-Strobl '94

JT gravity, 2D dilation gravity

Teitelboim '83, Jackiw '85,

Chirco-Vacchiano-Vitale '25, Girelli-Pollack-Riello '26,,,

Kontsevich formulas in deformation quantizations, noncommutative gauge theory via NS B-field

Kontsevich '97, Cattaneo-Felder '99

- 3D Courant sigma models

NI '03, Roytenberg '07

Cherns-Simons theory, T-duality, generalized geometry

- higher dimensions

Brane current algebras

NI-Xu '13, Arvanitakis '21

Plan of Talk

- Poisson sigma models
- Gauging of PSM
- GPSM with boundary

§2. Poisson sigma model

Σ : 2D manifold

$A_{\mu i}(\sigma)$: gauge field ($\mu = 1, 2$), $X^i(\sigma)$: scalar field
 $\pi^{ij}(X) = -\pi^{ji}(X)$.

$$S_{cl} = \int_{\Sigma} d^2\sigma \left[\epsilon^{\mu\nu} A_{\mu i} \partial_{\nu} X^i + \frac{1}{2} \epsilon^{\mu\nu} \pi^{ij}(X) A_{\mu i} A_{\nu j} \right]$$

The theory is gauge invariant under the following gauge transformations,

$$\delta X^i = -\pi^{ij}(X) \epsilon_j \quad \delta A_{\mu i} = \partial_{\mu} \epsilon_i + \frac{1}{2} \frac{\partial \pi^{jk}(X)}{\partial X^i} A_{\mu j} \epsilon_k$$

if π^{ij} is a Poisson tensor, i.e.,

$$\frac{\partial \pi^{ij}}{\partial X^m} \pi^{mk} + \frac{\partial \pi^{jk}}{\partial X^m} \pi^{mi} + \frac{\partial \pi^{ki}}{\partial X^m} \pi^{mj} = 0 \iff \delta S = 0 \quad (1)$$

(1) is the Jacobi identity of the Poisson bracket

$$\{F(X), G(X)\}_{PB} \equiv \frac{1}{2} \pi^{ij}(X) \frac{\partial F}{\partial X^i} \frac{\partial G}{\partial X^j}$$

Batalin-Vilkovisky formalism

The BV action functional

$$\begin{aligned} S = \int_{\Sigma} d^2\sigma \left[\epsilon^{\mu\nu} A_{\mu i} \partial_{\nu} X^i + \frac{1}{2} \epsilon^{\mu\nu} \pi^{ij}(X) A_{\mu i} A_{\nu j} \right. \\ \left. - X^{*i} \pi^{ij} c_j + A^{*\nu i} \left(\partial_{\nu} c_i + \frac{\partial \pi^{jk}}{\partial X^i} A_{\nu j} c_k \right) + \frac{1}{2} \frac{\partial \pi^{jk}}{\partial X^i} c^{*i} c_j c_k \right. \\ \left. - \frac{1}{4} \frac{\partial^2 \pi^{ij}}{\partial X^k \partial X^l} \epsilon_{\mu\nu} A_i^{*\mu} A_j^{*\nu} c_k c_l \right], \end{aligned}$$

satisfies the BV master equation, $\{S, S\} = 0$, where $\{-, -\}$ is the BV (anti)bracket.

§3. AKSZ formulation

Supefield formalism

Introduce a (vector) supercoordinate θ^μ ($\mu = 1, 2$), where degree=ghost number: $|\theta| = 1$

Define **superfield** by

$$\begin{aligned} \mathbf{X}^i(\sigma, \theta) &\equiv \textcolor{red}{X}^i + \theta^\mu A_\mu^{+i} + \frac{1}{2} \theta^\mu \theta^\nu c_{\mu\nu}^{+i}, & |\mathbf{X}| &= 0, \\ \mathbf{Z}_i(\sigma, \theta) &\equiv -c_i + \theta^\mu \textcolor{red}{A}_{\mu i} + \frac{1}{2} \theta^\mu \theta^\nu X_{\mu\nu}^+, & |\mathbf{Z}| &= 1, \end{aligned}$$

BV action

$$S = \int_{T[1]\Sigma} d^2\sigma d^2\theta \left(\mathbf{Z}_i d\mathbf{X}^i + \frac{1}{2} \pi^{ij}(\mathbf{X}) \mathbf{Z}_i \mathbf{Z}_j \right)$$

Here $d \equiv \theta^\mu \partial_\mu$.

The BV bracket is

$$\{F, G\} \equiv \int_{T[1]\Sigma} d^2\sigma d^2\theta \left(F \frac{\overleftarrow{\partial}}{\partial \mathbf{X}^i} \frac{\overrightarrow{\partial}}{\partial \mathbf{Z}_i} G - F \frac{\overleftarrow{\partial}}{\partial \mathbf{Z}_i} \frac{\overrightarrow{\partial}}{\partial \mathbf{X}^i} G \right)$$

A **graded Poisson bracket** of degree 1.

Classical master equation to homological condition

$$S_0 = \int_{T[1]\Sigma} d^2\sigma d^2\theta \mathbf{Z}_i d\mathbf{X}^i, S_\pi = \int_{T[1]\Sigma} d^2\sigma d^2\theta \frac{1}{2}\pi^{ij}(\mathbf{X}) \mathbf{Z}_i \mathbf{Z}_j,$$

$\{S, S\} = \{S_0, S_0\} + 2\{S_0, S_\pi\} + \{S_\pi, S_\pi\}$. Trivially, $\{S_0, S_0\} = \{S_0, S_\pi\} = 0$. Therefore,

Theorem 1. $\{S, S\} = 0 \iff \{S_\pi, S_\pi\} = 0$.

Note: The target space quantity corresponding to S_π is denoted by $\Theta_\pi = \frac{1}{2}\pi^{ij}(x)z_iz_j$.

§4. Lie algebroid gauging

Lie algebroids

A *Lie algebroid* $(E, [\cdot, \cdot], \rho)$ is a vector bundle $E \rightarrow M$ together with a bundle morphism (an anchor map) $\rho: E \rightarrow TM$ as well as a Lie bracket $[\cdot, \cdot]$, satisfying the Leibniz rule

$$[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)f \cdot e_2,$$

for every $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$.

Example 1. [Lie algebra] *Let a manifold M be one point $M = \{pt\}$. Then a Lie algebroid is a Lie algebra $\mathfrak{g}$.*

Example 2. [Vector fields] *If $E = TM$ and $\rho = \text{id}$, then the bracket $[-, -]$ is a normal Lie bracket of vector fields and $(TM, \text{id}, [-, -])$ is a Lie algebroid.*

QP-manifolds of Lie algebroids

1. The target space $T^*[1]M$ is generalized to $T^*[1]E[1]$.

$T^*[1]E[1]$: (x^i, a^a, z_i, y_a) of degree $(0, 1, 1, 0)$

2. The graded symplectic form on $T^*[1]E[1]$:

$$\omega_{grad} = \delta x^i \wedge \delta z_i - \delta a^a \wedge \delta y_a.$$

3. The homological function for a Lie algebroid is

$$\Theta_E = \rho_a^i(x) z_i a^a + \frac{1}{2} C_{ab}^c(x) a^a a^b y_c,$$

where for the basis e_a , $\rho(e_a) = \rho_a^i(x) \partial_i$ and $[e_a, e_b] = C_{ab}^c(x) e_c$.

Lie algebroid gauging

Homological functions

$$\begin{aligned}\Theta_\pi &= \frac{1}{2}\pi^{ij}(x)z_iz_j, \\ \Theta_E &= \rho_a^i(x)z_ia^a + \frac{1}{2}C_{ab}^c(x)a^aa^by_c,\end{aligned}$$

$\{\Theta_\pi, \Theta_\pi\} = 0$ iff π is a Poisson structure.

$\{\Theta_E, \Theta_E\} = 0$ iff E is a Lie algebroid.

A candidate gauged homological function is $\Theta = \Theta_\pi + \Theta_E$.

Problems

- 1, Θ_π (and Θ) is not globally well-defined.
- 2, In general, $\{\Theta_\pi, \Theta_E\} \neq 0$, thus, $\{\Theta, \Theta\} \neq 0$ for $\Theta = \Theta_\pi + \Theta_E$.
Thus, $\{S, S\} \neq 0$!

Coordinate transformations under diffeomorphisms and transition functions on the graded manifold,

$$x'^i = x'^i(x), \quad a'^a = M_b^a(x)a^b, \quad y'_a = M_a^b(x)y_b,$$

$$z'_i = \frac{\partial x^j}{\partial x'^i} \left(z_j + M_a^b(x) \partial_i M_b^c(x) a^a y_c \right),$$

Connections, torsions and curvatures

A connection $\nabla : \Gamma(E) \rightarrow \Gamma(E \otimes T^*M)$ on a vector bundle E .
 $\omega_b^a = \omega_{ai}^b(x)dx^i$: a connection 1-form.

The *curvature*, $R \in \Omega^2(M, E \otimes E^*)$, the *E-torsion*, $T \in \Gamma(E \otimes \wedge^2 E^*)$, and the *basic curvature* ${}^E S \in \Omega^1(M, \wedge^2 E^* \otimes E)$ are

$$R(v, v') := [\nabla_v, \nabla_{v'}] - \nabla_{[v, v']},$$

$$T(e, e') := \nabla_{\rho(e)}e' - \nabla_{\rho(e')}e - [e, e'],$$

$${}^E S(e, e') := [e, \nabla e'] - [e', \nabla e] - \nabla[e, e'] - \nabla_{\rho(\nabla e)}e' + \nabla_{\rho(\nabla e')}e$$

for $v, v' \in \mathfrak{X}(M)$ and $e, e' \in \Gamma(E)$.

Covariantized homological functions The global covariantized coordinate of z_i on $T^*[1]E[1]$ is $z_i^\nabla := z_i + \omega_{ai}^b a^a y_b$.

$$\begin{aligned}\Theta_\pi^\nabla &= \frac{1}{2}\pi^{ij}(x)z_i^\nabla z_j^\nabla \neq \Theta_\pi, \\ \Theta_E^\nabla &= \rho_a^i(x)z_i^\nabla a^a - \frac{1}{2}T_{ab}^c(x)a^a a^b y_c = \Theta_E.\end{aligned}$$

These satisfies

$$\begin{aligned}\{\Theta_\pi^\nabla, \Theta_\pi^\nabla\} &= \pi^{ij}\pi^{kl}R_{ija}^b a^a y_b z_k^\nabla z_l^\nabla, \\ \{\Theta_\pi^\nabla, \Theta_E^\nabla\} &= \frac{1}{2}{}^E\nabla_a \pi^{ij} z_i^\nabla z_j^\nabla a^a - \frac{1}{2}\pi^{ijE}S_{jab}^c z_i^\nabla a^a a^b y_c, \\ \{\Theta_E^\nabla, \Theta_E^\nabla\} &= 0,\end{aligned}$$

where ${}^E\nabla_e v = [\rho(e), v] + \rho(\nabla_v e)$ is the basic E -connection on TM for a vector field v and $e \in \Gamma(E)$.

Theorem 2. [NI '25] $\Theta^\nabla := \Theta_\pi^\nabla + \Theta_E^\nabla$. $\{\Theta^\nabla, \Theta^\nabla\} = 0$ if and only if

$${}^E\nabla\pi = 0, \quad (\pi^\sharp)^2 R = 0, \quad \pi^\sharp {}^E S = 0.$$

Gauged Poisson sigma models

Superfields (Local coordinates) corresponding to local coordinates (x^i, z_i, a^a, y_a) of degree $(0, 1, 1, 0)$ are

$$\begin{aligned} \mathbf{X}^i : T[1]\Sigma &\rightarrow M, & \mathbf{Z}_i &\in \Gamma(T[1]\Sigma, \mathbf{X}^*T^*[1]M), \\ \mathbf{A}^a &\in \Gamma(T[1]\Sigma, \mathbf{X}^*E[1]), & \mathbf{Y}_a &\in \Gamma(T[1]\Sigma, \mathbf{X}^*E^*). \end{aligned}$$

The BV action functional (the homological function) is

$$\begin{aligned} S = \int_{T[1]\Sigma} d\lambda_2 & \left[\mathbf{Z}_i^\nabla d\mathbf{X}^i - \mathbf{Y}_a D\mathbf{A}^a \right. \\ & \left. + \frac{1}{2}\pi^{ij}(\mathbf{X}) \mathbf{Z}_i^\nabla \mathbf{Z}_j^\nabla + \rho_a^i(\mathbf{X}) \mathbf{Z}_i^\nabla \mathbf{A}^a - \frac{1}{2}T_{ab}^c(\mathbf{X}) \mathbf{A}^a \mathbf{A}^b \mathbf{Y}_c \right], \end{aligned}$$

where $D\mathbf{A}^a = d\mathbf{A}^a - \omega_{bi}^a(\mathbf{X}) \mathbf{A}^b d\mathbf{X}^i$.

Proposition 1. $\{S, S\} = 0$ is satisfied iff conditions in Theorem 2 hold.

- The target space structure of a gauged AKSZ sigma model is formulated by equations,

$$\{S_0 + S_{gauge}, S_0 + S_{gauge}\} = \mathcal{F}$$

or

$$(Q_0 + Q_{gauge})^2 = \iota_{\mathcal{F}} \text{ with a 'curvature' } \mathcal{F}.$$

§5. GPSM with boundary

Let $\partial\Sigma \neq \emptyset$.

We assume the boundary condition, $\mathbf{Z}_i = 0$, and the boundary term,

$$S_{\partial\Sigma} = \int_{T[1]\partial\Sigma} d\lambda_1 \mu_a(\mathbf{X}) \mathbf{A}^a,$$

where $\mu = \mu_a(x)e^a \in \Gamma(E^*)$. The total BV action is

$$\begin{aligned} S_b^\nabla = & \int_{T[1]\Sigma} d\lambda_2 \left[\mathbf{Z}_i^\nabla d\mathbf{X}^i - \mathbf{Y}_a D\mathbf{A}^a + \frac{1}{2}\pi^{ij}(\mathbf{X}) \mathbf{Z}_i^\nabla \mathbf{Z}_j^\nabla \right. \\ & \left. + \rho_a^i(\mathbf{X}) \mathbf{Z}_i^\nabla \mathbf{A}^a - \frac{1}{2}T_{ab}^c(\mathbf{X}) \mathbf{A}^a \mathbf{A}^b \mathbf{Y}_c \right] + \int_{T[1]\partial\Sigma} d\lambda_1 \mu_a(\mathbf{X}) \mathbf{A}^a, \end{aligned}$$

must satisfies $\{S_b^\nabla, S_b^\nabla\} = 0$. This equation gives consistent boundary conditions.

Proposition 2. *Assume Proposition 2, i.e.*

${}^E\nabla\pi = 0, (\pi^\sharp)^2R = 0$ and $\pi^\sharp{}^ES = 0$.

Then, $\{S_b^\nabla, S_b^\nabla\} = 0$ if

1. *$X = \text{constant map on } \partial\Sigma$,*
2. *$\rho(e) = -\pi^\sharp(\nabla\mu)(e)$,*
3. *${}^E\mathrm{d}\mu(e_1, e_2) = -\pi((\nabla\mu)(e_1), (\nabla\mu)(e_2))$,*

where $e, e_1, e_2 \in \Gamma(E)$. ${}^E\mathrm{d}$ is the E -differential of the Lie algebroid

E .

Conditions 1, 2, 3 are equivalent to *Hamiltonian Lie algebroids over Poisson manifolds*.

Blohmann-Ronchi-Weinstein '23, (Blohmann-Weinstein '18)

Example

If $E = M \times \mathfrak{g}$ is an action Lie algebroid with a Lie algebra $\mathfrak{g}$ and $\nabla = d$, μ is an *infinitesimally equivariant momentum map*.

Theorem 3. [NI '25] *Under the condition in Theorem 2, ${}^E\nabla\pi = 0$, $(\pi^\sharp)^2R = 0$, $\pi^\sharp{}^ES = 0$, the GPSM with boundary satisfies the classical master equation $\{S_b^\nabla, S_b^\nabla\} = 0$ if the target space E is a Hamiltonian Lie algebroid over a Poisson manifold.*

§6. 3 dimensional case

NI '26

QP-manifolds of degree 2

Courant sigma models with Lie algebroid or Courant algebroid gauging. Connections, E -torsions and basic curvatures in Courant algebroids appear.

The *curvature*, $R \in \Omega^2(M, E \otimes E^*)$, the E -*torsion*, (or the Gualtieri torsion,) $T \in \Gamma(\wedge^3 E^*)$ and the *basic curvature* ${}^E S \in \Omega^1(M, \wedge^3 E^*)$ are

$$R(v_1, v_2) := [\nabla_{v_1}, \nabla_{v_2}] - \nabla_{[v_1, v_2]},$$

$$\begin{aligned}
T(e_1, e_2, e_3) &:= \langle {}^E\nabla_{e_1}e_2 - {}^E\nabla_{e_2}e_1 - [e_1, e_2]_E, e_3 \rangle \\
&\quad + \frac{1}{2}(\langle {}^E\nabla_{e_3}e_1, e_2 \rangle - \langle {}^E\nabla_{e_3}e_2, e_1 \rangle), \\
{}^E S(e_1, e_2, e_3)v &:= \langle \nabla_v[e_1, e_2]_E - [\nabla_v e_1, e_2]_E - [e_1, \nabla_v e_2]_E \\
&\quad - \nabla_{{}^E\nabla_{e_2}v}e_1 + \nabla_{{}^E\nabla_{e_1}v}e_2, e_3 \rangle + \frac{1}{2} \left(\langle \nabla_{{}^E\nabla_{e_3}v}e_1, e_2 \rangle - \langle \nabla_{{}^E\nabla_{e_3}v}e_2, e_1 \rangle \right),
\end{aligned}$$

for $e_1, e_2, e_3 \in \Gamma(E)$ and $v, v_1, v_2 \in \mathfrak{X}(M)$.

Theorem 4. [NI '26] *The BV action functional S satisfies the classical master equation $\{S, S\} = 0$ if and only if $\nabla\rho = 0$, $R = {}^E S = 0$, and ${}^E\nabla H = 0$.*

§. Summary

- Gauged Poisson and Courant sigma models have been proposed.
- The structure is an *equivariant double complex*,

$$(Q_0 + Q_1)^2 = \iota_{\mathcal{F}} \text{ with a 'curvature' } \mathcal{F}.$$

Torsions and curvatures in various algebroids appear.

- Consistency requires compatibility of two QP-manifold structures.
- Gauged AKSZ sigma models with boundary require structures of generalizations of *momentum maps*.

Thank you for your attention!