

Action principle for κ -Minkowski noncommutative U(1) gauge theory

from Lie-Poisson electrodynamics

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Outline:

- * Lie-Poisson electrodynamics (LPE): definition and overview
- * Classical action for LPE and Lagrangian field equations
- * Application to the κ -Minkowski case
- * Comparison with previous results

Lie-Poisson electrodynamics: definition

- The starting point is a d -dimensional space-time $\mathcal{M} \simeq \mathbb{R}^d$, equipped with a given Poisson bracket,

$$\{x^\mu, x^\nu\} = \Theta^{\mu\nu}(x) \quad \mu, \nu = 0, \dots, d-1,$$

- Poisson electrodynamics (PE) is a non-Abelian deformation of usual electrodynamics, where the infinitesimal gauge transformations close the **Poisson gauge algebra**,

$$[\delta_f, \delta_g]A_\mu = \delta_{\{f,g\}}A_\mu, \quad \lim_{\Theta \rightarrow 0} \delta_f A_\mu = \partial_\mu f.$$

- PE can be seen as a semiclassical limit of the noncommutative $U(1)$ gauge theory [Kupriyanov, Vitale 2020] and [Kupriyanov, Szabo 2022],

$$[\delta_f, \delta_g]A_\mu = \delta_{-i[f,g]_\star}A_\mu, \quad [f,g]_\star = f \star g - g \star f, \quad [f,g]_\star \simeq i\{f,g\}.$$

Thus, we refer to our Poisson bracket as *noncommutativity*.

Lie-Poisson electrodynamics: definition

- Consider a class of Poisson bivectors, which are linear in coordinates,

$$\Theta^{\mu\nu}(x) = \mathcal{C}_\xi^{\mu\nu} x^\xi.$$

The constants $\mathcal{C}_\xi^{\mu\nu}$ are the structure constants of a given d -dimensional Lie algebra $\mathfrak{g}$. Various noncommutativities which have attracted significant attention in the literature are of the Lie-algebraic type:

$$\{x^\mu, x^\nu\} = \mathcal{C}_\lambda^{\mu\nu} x^\lambda.$$

- For example, at $d = 4$ (I present non-zero Poisson brackets only):

$$\begin{aligned} \{x^j, x^k\} &= \ell \varepsilon^{ijk} x^k, & (\text{the } \mathfrak{su}(2) \text{ noncommutativity}) \\ \{x^3, x^1\} &= -\ell x^2, \quad \{x^3, x^2\} = -\ell x^1, & (\text{the angular noncommutativity}) \\ \{x^0, x^j\} &= \underbrace{\ell}_{\frac{1}{\kappa}} x^j. & (\text{the } \kappa\text{-Minkowski noncommutativity}) \end{aligned}$$

- The corresponding Poisson electrodynamics we call the **Lie-Poisson electrodynamics** (LPE). Today I shall discuss the LPE only.

Overview of the construction: what we are looking for

- First of all, for a given Poisson bivector $\Theta^{\mu\nu}$ we would like to *define* the deformed gauge transformations.

$$\delta_f A_\mu =? \quad [\delta_f, \delta_g] A_\mu = \delta_{\{f,g\}} A_\mu, \quad \lim_{\Theta \rightarrow 0} \delta_f A_\mu = \partial_\mu f.$$

- We would also like to define the gauge-covariant field strength.

$$\mathcal{F}_{\mu\nu} =? \quad \delta_f \mathcal{F}_{\mu\nu} = \{\mathcal{F}_{\mu\nu}, f\}, \quad \lim_{\Theta \rightarrow 0} \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu =: F_{\mu\nu}.$$

- Finally, we are looking for the deformed classical action, which respects the Poisson gauge symmetry and which reduces to the standard Maxwell action in the commutative limit.

$$S[A] =? \quad \delta_f S[A] = 0, \quad \lim_{\Theta \rightarrow 0} S[A] = -\frac{1}{4} \int_{\mathcal{M}} dx F_{\mu\nu} F_{\alpha\beta} \eta^{\mu\alpha} \eta^{\nu\beta}.$$

Overview of the construction: master equations

- The key technical elements of the construction are two $d \times d$ field-dependent matrices, $\gamma(A)$ and $\rho(A)$, which solve the master equations

$$\begin{cases} \gamma_\mu^\nu(A) \frac{\partial \gamma_\lambda^\xi(A)}{\partial A_\mu} - \gamma_\mu^\xi(A) \frac{\partial \gamma_\lambda^\nu(A)}{\partial A_\mu} = C_\mu^{\nu\xi} \gamma_\lambda^\mu(A) \\ \gamma_\lambda^\nu(A) \frac{\partial \rho_\xi^\mu(A)}{\partial A_\lambda} + \rho_\xi^\lambda(A) \frac{\partial \gamma_\lambda^\nu(A)}{\partial A_\mu} = 0 \end{cases},$$

- and tend to the identity matrix in the commutative limit,

$$\lim_{\Theta \rightarrow 0} \gamma = \mathbb{I}, \quad \lim_{\Theta \rightarrow 0} \rho = \mathbb{I}.$$

- For a given Poisson bivector $\Theta^{\mu\nu} = C_\xi^{\mu\nu} x^\xi$, defining the noncommutativity, the main constituents of the Poisson gauge formalism, viz. the deformed gauge transformations $\delta_f A$, the deformed field strength $\mathcal{F}$ and the deformed gauge-covariant derivative $\mathcal{D}$, are completely determined by the matrices.

Overview of the construction: basic elements

The required deformed gauge transformations $\delta_f A$, deformed field strength $\mathcal{F}$ and the deformed gauge-covariant derivative $\mathcal{D}$ were constructed in [Kupriyanov, Vitale 2020] and [Kupriyanov 2021]

Definition		Property
$\delta_f A_\mu = \gamma_\mu^\nu \partial_\nu f + \{A_\mu, f\}$		$[\delta_f, \delta_g] A_\mu = \delta_{\{f, g\}} A_\mu$
$\mathcal{F}_{\mu\nu}(x) = \rho_\mu^\alpha \rho_\nu^\beta (\gamma_\alpha^\xi \partial_\xi A_\beta - \gamma_\beta^\xi \partial_\xi A_\alpha + \{A_\alpha, A_\beta\})$		$\delta_f \mathcal{F}_{\mu\nu} = \{\mathcal{F}_{\mu\nu}, f\}$
$\mathcal{D}_\mu \psi(x) = \rho_\mu^\alpha (\gamma_\alpha^\lambda \partial_\lambda \psi + \{A_\alpha, \psi\})$		$\delta_f (\mathcal{D}_\mu \psi) = \{\mathcal{D}_\mu \psi, f\}$

In the last line ψ is an arbitrary field which transforms in a covariant manner under gauge transformations: $\delta_f \psi = \{\psi, f\}$.

Overview of the construction: universal LPE

- In [Kupriyanov, Szabo 2022] and [Kupriyanov, Kurkov, Vitale 2022] special solutions of the two master equations were obtained in terms of matrix-valued functions.

- Introducing the matrix,

$$[\hat{A}]_{\mu}^{\beta} = -i\mathcal{C}_{\mu}^{\alpha\beta} A_{\alpha},$$

we can represent these universal solutions as follows

$$\begin{cases} \gamma(A) = G(\hat{A}), \\ \rho(A) = (\gamma(A) - i\hat{A})^{-1} \end{cases}, \quad \text{with} \quad G(p) := \frac{i p}{2} + \frac{p}{2} \cot \frac{p}{2}$$

- These solutions are valid for *arbitrary* structure constants $\mathcal{C}_{\mu}^{\alpha\beta}$. The corresponding Lie-Poisson gauge theory will be referred to as the “universal LPE”.

Overview of the construction: universal-equivalent LPE

- For any invertible field redefinition

$$A \rightarrow \tilde{A}(A), \quad \text{such that} \quad \lim_{\Theta \rightarrow 0} \tilde{A}(A) = A,$$

the quantities

$$\tilde{\gamma}_{\nu}^{\mu}(\tilde{A}) = \left(\gamma_{\xi}^{\mu}(A) \cdot \frac{\partial \tilde{A}_{\nu}}{\partial A_{\xi}} \right) \Big|_{A=A(\tilde{A})}, \quad \tilde{\rho}_{\nu}^{\mu}(\tilde{A}) = \left(\frac{\partial A_{\xi}}{\partial \tilde{A}_{\mu}} \cdot \rho_{\nu}^{\xi}(A) \right) \Big|_{A=A(\tilde{A})},$$

are again the solutions of the master equations. The matrices $\tilde{\gamma}$ and $\tilde{\rho}$ define one more Poisson model for the same Poisson bivector Θ , which we call the “universal-equivalent”.

- The infinitesimal gauge transformation of the new model reads:

$$\tilde{\delta}_f \tilde{A}_{\mu} = \tilde{\gamma}_{\mu}^{\nu}(\tilde{A}) \partial_{\nu} f + \{ \tilde{A}_{\mu}, f \}.$$

Upon the field redefinition the gauge orbits are mapped onto the gauge orbits,

$$\tilde{A}(A + \delta_f A) = \tilde{A}(A) + \tilde{\delta}_f \tilde{A}(A).$$

LPE: gauge (non)invariance of the action

- The Lagrangian density, obeying the gauge-covariance condition $\delta_f \mathcal{L} = \{\mathcal{L}, f\}$, can be easily constructed:

$$\mathcal{L} = -\frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}_{\alpha\beta} \eta^{\mu\alpha} \eta^{\nu\beta}.$$

However, the corresponding classical action

$$S_{\mathfrak{g}}[A] = \int_{\mathcal{M}} dx \mathcal{L},$$

is gauge-invariant for unimodular algebras $\mathfrak{g}$ only, $\mathcal{C}_{\mu}^{\mu\nu} = 0$.

- The gauge covariance of the Lagrangian density does not imply gauge invariance of the action:

$$\begin{aligned} \delta_f S_{\mathfrak{g}} &= \int_{\mathcal{M}} dx \{\mathcal{L}, f\} \\ &= \int_{\mathcal{M}} dx \left(\underbrace{-\partial_{\nu} (\partial_{\mu} f \cdot \Theta^{\mu\nu} \mathcal{L})}_{\text{nice t.d.}} - \underbrace{\partial_{\mu} f \cdot \mathcal{C}_{\nu}^{\nu\mu} \cdot \mathcal{L}}_{\text{problematic!}} \right) \neq 0. \end{aligned}$$

- This property is nothing but the semiclassical version of the non-cyclicity of the corresponding $\star$ -product,

$$\int_{\mathcal{M}} dx [\cdot, \cdot]_{\star} \neq 0.$$

LPE: from gauge-covariance to gauge-invariance

- In my recent article, J.Phys.A 59 (2026) 14, 145402, I proved that any quantity Q , transforming in a gauge-covariant manner $\delta_f Q = \{Q, f\}$, can be converted into a gauge-invariant object (up to a total derivative) by means of an integrating factor.
- A suitable integrating factor, valid for all gauge-covariant objects Q , can be chosen as follows:

$$M_A(x) := \left(\det [\gamma(A(x)) \rho(A(x))] \right)^{-1}.$$

- Then, the expression

$$\check{Q}(x) := M_A(x) Q(x)$$

is gauge-invariant up to a total derivative:

$$\delta_f \check{Q}(x) = \partial_\nu (x^\xi C_\xi^{\nu\sigma} \check{Q}(x) \partial_\sigma f).$$

LPE: gauge-invariant action

- By choosing the Lagrangian density

$$\check{\mathcal{L}}(x) = M_A(x) \underbrace{\left(-\frac{1}{4} \mathcal{F}_{\mu\nu}(x) \mathcal{F}^{\mu\nu}(x) \right)}_{\text{gauge-covariant}},$$

we obtain a gauge-invariant action

$$S_{\mathbf{g}}[A] = \int_{\mathcal{M}} dx \check{\mathcal{L}}(x).$$

- Being an integral of a total derivative, the corresponding variation of the action vanishes, provided the gauge field $A(x)$ decays sufficiently fast at infinity:

$$\delta_f S_{\mathbf{g}}[A] = \int_{\mathcal{M}} dx \partial_\nu \left(x^\xi \mathcal{C}_\xi^{\nu\sigma} \check{\mathcal{L}}(x) \partial_\sigma f \right) = 0.$$

- The above classical action has the correct commutative limit:

$$\lim_{\mathcal{C} \rightarrow 0} S_{\mathbf{g}}[A] = \int_{\mathcal{M}} dx \left(-\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \right).$$

LPE: Euler-Lagrange field equations

- The gauge-invariance of the action implies that the corresponding Euler-Lagrange equations

$$\mathcal{E}_{EL}^\mu(x) = 0, \quad \mathcal{E}_{EL}^\mu(x) := \frac{\delta S_g[A]}{\delta A_\mu(x)},$$

are not independent but obey the Noether identity:

$$\partial_\nu (\gamma_\mu^\nu(A) \mathcal{E}_{EL}^\mu(x)) + \{A_\mu, \mathcal{E}_{EL}^\mu(x)\} + \mathcal{C}_\nu^{\xi\nu} \partial_\xi A_\mu \mathcal{E}_{EL}^\mu(x) = 0.$$

- In [J.Phys.A 59 \(2026\) 14, 145402](#) I have shown that these Euler-Lagrange equations admit the manifestly gauge-covariant form $\mathcal{E}_G^\mu(x) = 0$ with

$$\mathcal{E}_G^\mu(x) := \mathcal{D}_\xi \mathcal{F}^{\xi\mu} + \frac{1}{2} \mathcal{F}_{\lambda\omega} \mathcal{C}_\nu^{\lambda\omega} \mathcal{F}^{\mu\nu} - \mathcal{F}_{\lambda\omega} \mathcal{C}_\nu^{\mu\omega} \mathcal{F}^{\lambda\nu} - \frac{1}{4} (\mathcal{C}_\nu^{\nu\mu} \mathcal{F}_{\lambda\omega} \mathcal{F}^{\lambda\omega} + 4 \mathcal{C}_\nu^{\nu\lambda} \mathcal{F}_{\lambda\omega} \mathcal{F}^{\omega\mu}).$$

- Under the gauge transformations of A ,

$$\delta_f \mathcal{E}_G^\mu = \{\mathcal{E}_G^\mu, f\}.$$

LPE: κ -Minkowski case

- In the four-dimensional κ -Minkowski case:

$$\{x^\mu, x^\nu\} = \kappa^{-1} (\delta_0^\mu x^\nu - \delta_0^\nu x^\mu),$$

and

$$\mathcal{C}_\sigma^{\sigma\nu} = -3\delta_0^\nu \kappa^{-1} \neq 0.$$

- The matrices γ and ρ can be chosen as follows:

$$\gamma(A) = \begin{pmatrix} 1 & -\frac{A_1}{\kappa} & -\frac{A_2}{\kappa} & -\frac{A_3}{\kappa} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\frac{A_0}{\kappa}} & 0 & 0 \\ 0 & 0 & e^{\frac{A_0}{\kappa}} & 0 \\ 0 & 0 & 0 & e^{\frac{A_0}{\kappa}} \end{pmatrix},$$

so

$$M_A(x) = \exp(-3\kappa^{-1}A_0(x)).$$

- So the corresponding action becomes

$$S_g[A] = \int_{\mathcal{M}} dx \exp(-3\kappa^{-1}A_0(x)) \left(-\frac{1}{4} \mathcal{F}_{\mu\nu}(x) \mathcal{F}^{\mu\nu}(x) \right).$$

LPE: comparison with previous results

- At the leading order in κ^{-1} , my integrating factor

$$M_A(x) = 1 - 3\kappa^{-1}A_0(x) + \mathcal{O}(\kappa^{-2})$$

perfectly agrees with the field-dependent volume factor proposed in JHEP 09 (2005), 068 by M. Dimitrijevic et al. within the perturbative first-order approach to noncommutative electrodynamics on the κ -Minkowski space-time.

- In the article JHEP 11 (2023) 200 written together with Kupriyanov and Vitale, on purely general grounds, we proposed a one-parameter family of deformed Maxwell equations for the κ -Minkowski LPE.

$$\mathcal{E}_G^\mu = 0, \quad \mathcal{E}_G^\mu = \mathcal{D}_\xi \mathcal{F}^{\xi\mu} + \frac{1}{2} \mathcal{F}_{\lambda\omega} \mathcal{C}_\nu^{\lambda\omega} \mathcal{F}^{\mu\nu} - \mathcal{F}_{\lambda\omega} \mathcal{C}_\nu^{\mu\omega} \mathcal{F}^{\lambda\nu} + \alpha (\mathcal{C}_\nu^{\nu\mu} \mathcal{F}_{\lambda\omega} \mathcal{F}^{\lambda\omega} + 4 \mathcal{C}_\nu^{\nu\lambda} \mathcal{F}_{\lambda\omega} \mathcal{F}^{\omega\mu}), \quad \alpha \in \mathbb{R}.$$

The left-hand sides of these equations obey the gauge identity

$$\mathcal{D}_\mu \mathcal{E}_G^\mu = \frac{1}{2} (1 + 12\alpha) \mathcal{C}_\xi^{\mu\nu} \mathcal{F}_{\mu\nu} \mathcal{E}_G^\xi.$$

- My Euler-Lagrange equations have exactly the above structure at $\alpha = -1/4$. Then the above gauge identity becomes the Noether identity, rewritten in a gauge-covariant form.

Summary

- LPE describes the semiclassical limit of noncommutative $U(1)$ gauge theories with Lie-algebra-type noncommutativities.
- I addressed the problem of finding a local gauge-invariant classical action for a generic LPE.
- The main technical tool of my construction is the integrating factor $M_A(x)$ which converts gauge-covariant expressions into gauge-invariant objects (up to a total derivative).
- The deformed Maxwell action and the corresponding field equations have been constructed for a generic Lie-algebra-type noncommutativity.
- In the κ -Minkowski case, my findings agree with the previous first-order results by M. Dimitrijevic on the classical action and with our previous results on the deformed Maxwell equations.