



FAKULTA MATEMATIKY,
FYZIKY A INFORMATIKY
Univerzita Komenského
v Bratislave

Numerical Study of Scalar Field Theory on the Fuzzy Onion

[2608.08855]

Matej Hrmo, Samuel Kováčik, Juraġ Tekel

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The fuzzy sphere can be viewed as a quantisation of the commutative sphere:

$$\sum_{i=1}^3 \hat{x}_i^2 = r^2 \quad [\hat{x}_i, \hat{x}_j] = i\theta \varepsilon_{ijk} \hat{x}_k. \quad (1)$$

This can be realized by taking $\hat{x}_i$ to be $N \times N$ Hermitian matrices. Therefore, the fields on the fuzzy sphere can be expressed in terms of matrices.



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This can be realized by taking $\hat{x}_i$ to be $N \times N$ Hermitian matrices. Therefore, the fields on the fuzzy sphere can be expressed in terms of matrices. On the fuzzy sphere we have the polarisation tensors $Y_{lm}^{(N)}$, which form a basis for Hermitian matrices.

$$\Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^l c_{lm}^{(N)} Y_{lm}^{(N)}. \quad (2)$$

These are the eigen-matrices of some (kinetic) operator $\mathcal{K}$ on the fuzzy sphere.



...would be defined by the matrix action

$$S \left[\Phi^{(N)} \right] = \text{tr}_N \left(a \Phi^{(N)} \mathcal{K}^{(N)} \Phi^{(N)} + b \Phi^{(N)2} + c \Phi^{(N)4} \right), \quad (3)$$

with the trace playing the role of the integral. We know three phase configurations on the fuzzy sphere. Let us note that the potential is a quartic function with two minima.

Disordered phase — the field oscillates around zero.

Uniform phase — the field inhabits one of the potential minima.

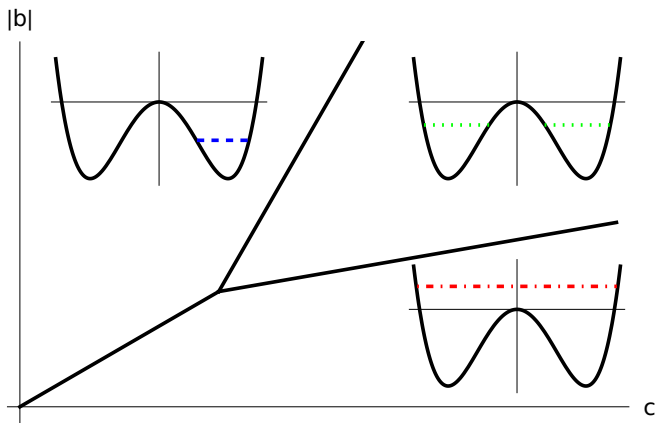
Non-uniform phase — the field inhabits both minima of the potential.

Scalar field theory on the fuzzy sphere



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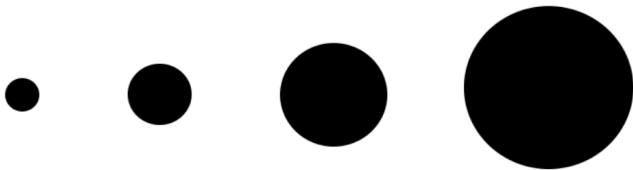




We study the fuzzy onion model [KOVÁČIK, TEKEL '23.04 '23.09], defined as a set of **concentric fuzzy spheres with increasing radii**. The radius of the innermost being $\lambda = \frac{\theta}{2}$ and the radius of the N -th sphere from the inside is $N\lambda$. But with each larger sphere, we also increase the size of the matrix that encodes the field on it.



$$(\cdot), \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

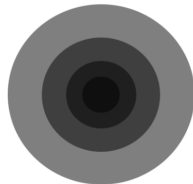


Fuzzy onion: the field



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$$\psi = \begin{pmatrix} (.) \\ \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \\ \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \\ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \end{pmatrix}$$





... is defined very similarly to the fuzzy sphere one.

$$S[\Psi] = 4\pi\lambda^2 \text{Tr} R \left(a \Psi (\mathcal{K}_R + \mathcal{K}_L) \Psi + b \Psi^2 + c \Psi^4 \right), \quad (4)$$

$R = \text{diag}(\lambda \mathbb{I}_{1 \times 1}, 2\lambda \mathbb{I}_{2 \times 2}, \dots, M\lambda \mathbb{I}_{M \times M})$ being a matrix of radial distance, and the angular part of the kinetic operator $\mathcal{K}_L$ being

$$\mathcal{K}_L \Psi = R^{-2} \begin{pmatrix} \mathcal{K}^{(1)} \Phi^{(1)} & & & \\ & \mathcal{K}^{(2)} \Phi^{(2)} & & \\ & & \ddots & \\ & & & \mathcal{K}^{(M)} \Phi^{(M)} \end{pmatrix} \quad (5)$$

and the radial part $\mathcal{K}_R$ being a little more tricky.



We wish to study the field theory with the action given by (4), for example in terms of observables:

$$\langle \mathcal{O}(\Psi) \rangle = \frac{1}{Z} \int d\Psi e^{-S(\Psi)} \mathcal{O}(\Psi). \quad (6)$$

For a numerical approach, we would need to have a large number of matrices that simulate such distribution well. To obtain them, we use the Hamiltonian Monte Carlo method.

Our attempt was to search the parameter space of the fuzzy onion model for phase transitions and phases themselves. This brings several questions. What do we even expect the phases to be? What is a suitable way to describe them?



We know three phase configurations on the fuzzy sphere. Let us note that the potential is a quartic function with two minima.

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Non-uniform phase — the field inhabits both minima of the potential.

This means that each layer of the fuzzy onion could independently assume one of these phases



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This means that each layer of the fuzzy onion could independently assume one of these phases, but it does not! One phase dominates the whole onion, layer by layer.

Phases on the onion — uniform

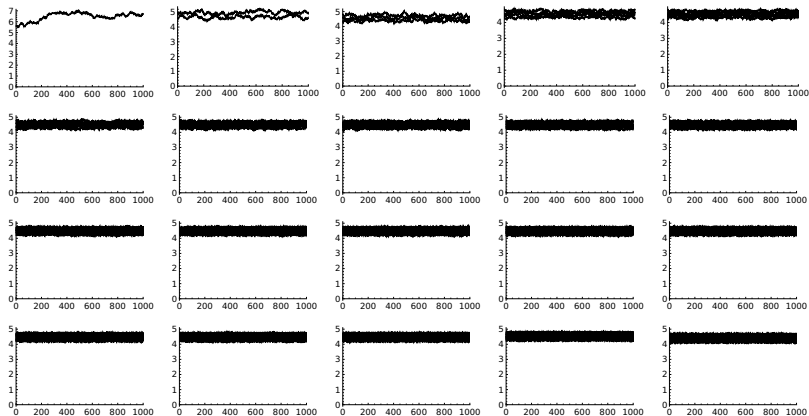


Figure: Eigenvalue trajectories on the fuzzy onion space with twenty layers in the uniform phase. The layer number grows from left to right and from top to bottom.

Phases on the onion — non-uniform

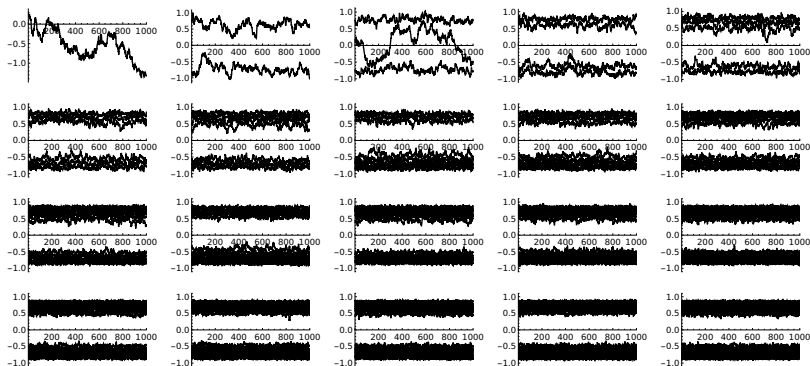


Figure: Eigenvalue trajectories on the fuzzy onion space with twenty layers in the non-uniform phase. The layer number grows from left to right and from top to bottom.

Phases on the onion — disordered

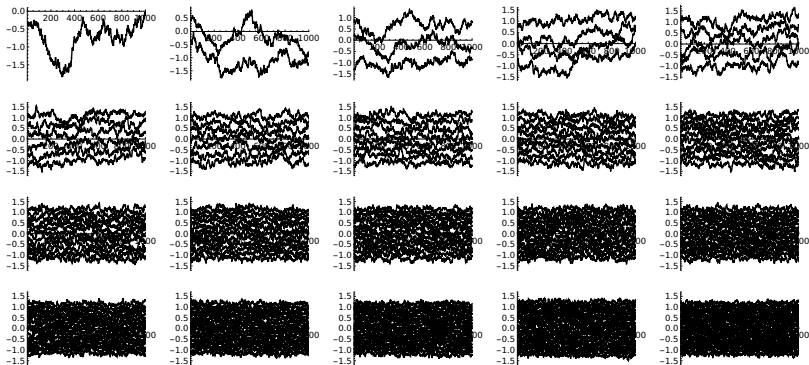


Figure: Eigenvalue trajectories on the fuzzy onion space with twenty layers in the disordered phase. The layer number grows from left to right and from top to bottom.

Preliminary phase diagram

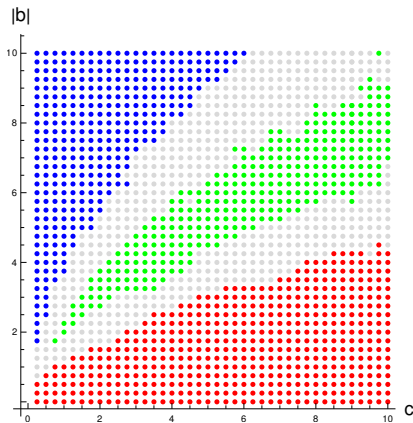


Figure: A preliminary phase diagram for the fuzzy onion with ten layers obtained via numerical HMC simulations. We note three distinct phases (blue — uniform, green — non-uniform, red — disordered) and transition areas between these phases, where we see features of two phases simultaneously.

Phase transition examples



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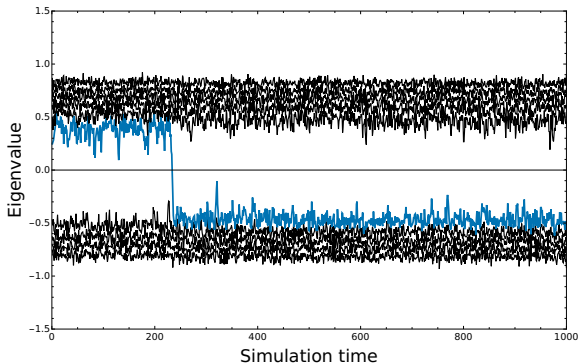


Figure: Eigenvalue trajectories depicting the dynamical transition between the uniform phase and the non-uniform phase. The highlighted eigenvalue spontaneously shifts to the other minimum during the simulation.

Phase transition examples

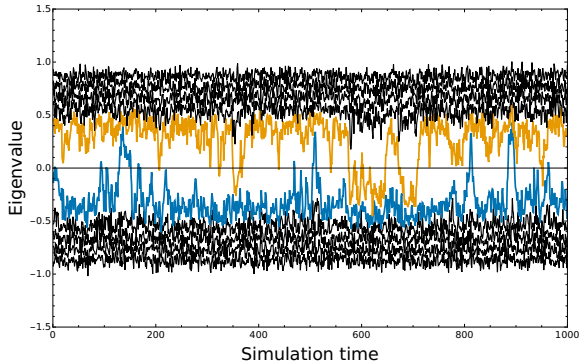


Figure: Eigenvalue trajectories depicting the dynamical transition between the disordered phase and the non-uniform phase. The highlighted eigenvalues shift between the minima back and forth.

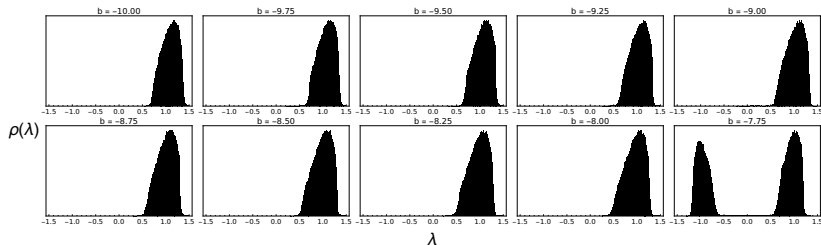
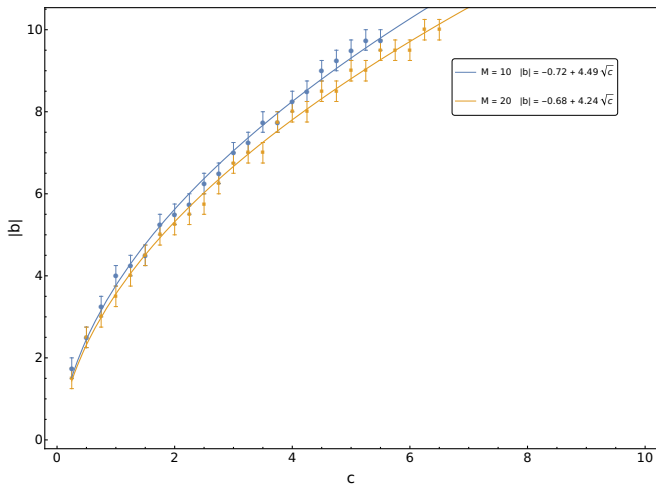


Figure: Eigenvalue histograms close to the expected uniform/non-uniform boundary as we decrease parameter $|b|$ for a fixed c . For some value, the phase assumes a 2-cut (non-uniform) solution.



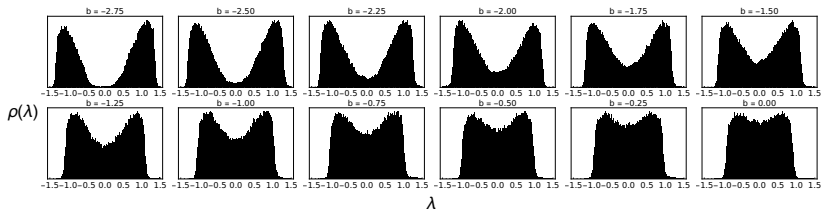


Figure: Eigenvalue histograms close to the expected disordered/non-uniform boundary as we decrease parameter $|b|$ for a fixed c . The central support linearly drops.

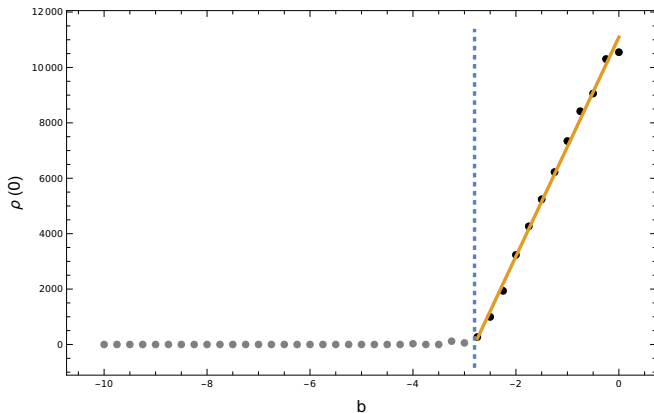


Figure: The central support drops linearly, yielding a critical value of the $|b|$ parameter.

Critical disordered boundary

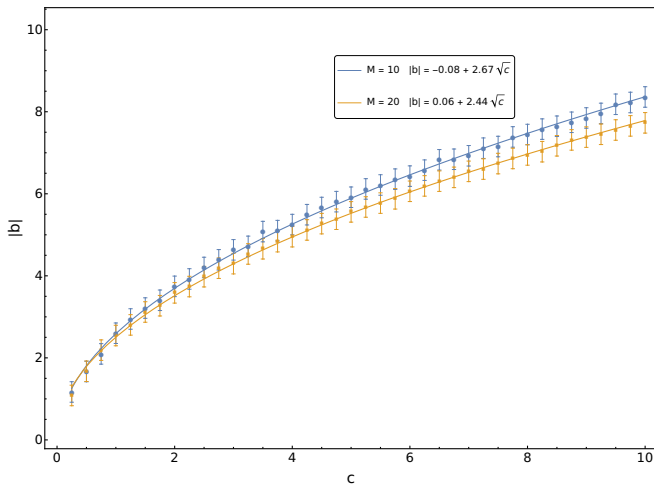
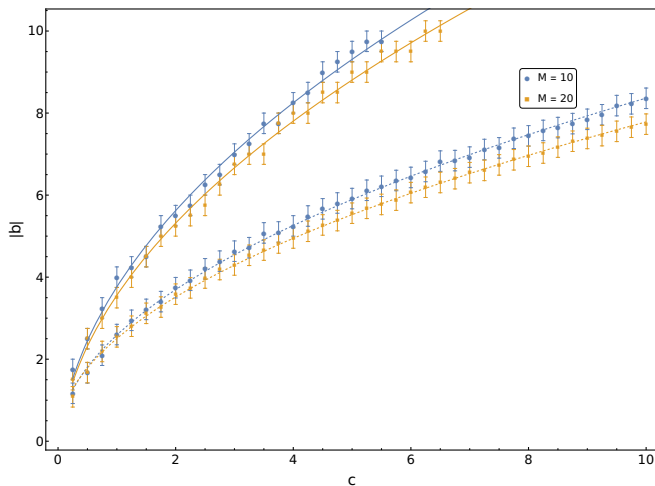


Figure: Critical boundary estimate for the disordered phase.

Phase diagram boundaries



To conclude, the fuzzy onion model:

- is a three-dimensional noncommutative framework that is well-accessible to numerical treatment,
- the scalar field theory exhibits the well-explored fuzzy sphere phases,
- ... and shows interesting dynamical properties related to phase changes,
- seems to have a similar (but not identical) phase structure as the fuzzy sphere.
- We have been able to obtain some boundaries on its complicated phase structure.
- Seems to be a stable configuration in the $D = 3$ YMCS matrix model.

Part of the research was obtained using the computational resources procured in the national project National competence centre for high performance computing within the Operational programme Integrated infrastructure (project code: 311070AKF2).



Thank you for your attention!



Fuzzy sphere and matrix models (some references, the list would be long):

HOPPE '82 [PhD. Thesis]

MADORE '96 [DOI: 10.1088/0264-9381/9/1/008]

BALACHANDRAN, KURKCUOGLU, VAIDYA '05 [hep-th/0511114]

FLORES, MARTIN, O'CONNOR '09 [arxiv:0903.1986]

STEINACKER '11 [arxiv:1109.5521]

TEKEL '15 [arxiv:1512.00689]

and many other

Fuzzy Onion model and related constructions:

HAMMOU, LAGRAA, SHEIKH-JABARI '02 [arxiv:hep-th/0110291]

KUPRIYANOV, VITALE '15 [arxiv:1502.06544]

GÁLIKOVÁ, KOVÁČIK, PREŠNAJDER '15 [arxiv:1510.04496]

KOVÁČIK, TEKEL '23 [arxiv:2309.00576]

KOVÁČIK, TEKEL, MH '26 [arxiv:2608.08855]

Backup slides



The commutator matches the structure of $su(2)$ algebra, one can use the spin- j representation to define coordinate “functions” – in fact matrices.

$$[L_i, L_j] = i\varepsilon_{ijk}L_k \quad \sum_i^3 L_i^2 = j(j+1), \quad (7)$$

With

$$\hat{x}_i = \frac{2r}{\sqrt{N^2-1}}L_i. \quad (8)$$

$N = 2j + 1$ is the dimension of the representation and $\theta = \frac{2r}{\sqrt{N^2-1}}$.

On a two-sphere we have the spherical harmonics $Y_{lm}(\theta, \varphi)$

$$r^2 \Delta Y_{lm} = l(l+1)Y_{lm}. \quad (9)$$

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l c_{lm} Y_{lm}(\theta, \varphi). \quad (10)$$

On the fuzzy sphere we have the polarisation tensors $Y_{lm}^{(N)}$

$$\Phi = \sum_{l=0}^{N-1} \sum_{m=-l}^l c_{lm}^{(N)} Y_{lm}^{(N)}, \quad (11)$$

$$\left[L_i^{(N)}, \left[L_i^{(N)}, Y_{lm}^{(N)} \right] \right] = l(l+1)Y_{lm}^{(N)} \quad \left[L_i^{(N)}, Y_{lm}^{(N)} \right] = mY_{lm}^{(N)}. \quad (12)$$

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We define the kinetic operator:

$$\Delta \leftrightarrow \mathcal{K}^{(N)} = \left[L_i^{(N)}, \left[L_i^{(N)}, \cdot \right] \right] \quad (13)$$

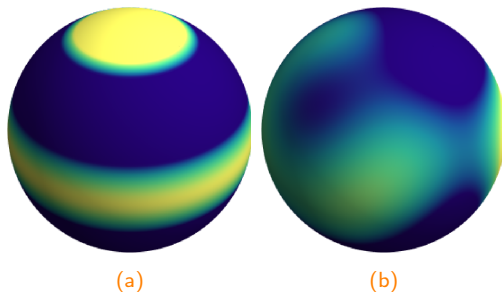


Figure: Visualisation of the map between fields and matrices. Fig. 12a depicts a field on the sphere that maps to a matrix $\text{diag}(1, -2, 3, -4, 5)$ while Fig. 12b depicts a field obtained by mapping a random 5×5 hermitian matrix.



The radius increases by λ between two layers, the radius of the N -th layer is λN . Integral becomes akin to

$$\sum_{N=1}^M \frac{4\pi(\lambda N)^2}{N} \lambda \text{tr}_N \Phi^{(N)} = \text{Tr}(4\pi\lambda^2 R \Psi)$$

with $R = \text{diag}(\lambda \mathbb{I}_{1 \times 1}, 2\lambda \mathbb{I}_{2 \times 2}, \dots, M\lambda \mathbb{I}_{M \times M})$ being a matrix of radial distance.



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with $R = \text{diag}(\lambda\mathbb{I}_{1\times 1}, 2\lambda\mathbb{I}_{2\times 2}, \dots, M\lambda\mathbb{I}_{M\times M})$ being a matrix of radial distance. Kinetic operator is trickier — we need radial communication.

Let us have

$$\Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^l c_{lm}^{(N)} Y_{lm}^{(N)}$$

$$\Phi^{(N+1)} = \sum_{l=0}^N \sum_{m=-l}^l c_{lm}^{(N+1)} Y_{lm}^{(N+1)}$$

We define two maps. The first is $\mathcal{D} : (N+1) \rightarrow (N)$ defined as

$$\mathcal{D} : \Phi^{(N+1)} \rightarrow \Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^l c_{lm}^{(N+1)} Y_{lm}^{(N)}$$

And the second is $\mathcal{U} : (N) \rightarrow (N+1)$ defined as

$$\mathcal{U} : \Phi^{(N)} \rightarrow \Phi^{(N+1)} = \sum_{l=0}^{N-1} \sum_{m=-l}^l c_{lm}^{(N)} Y_{lm}^{(N+1)}$$

This allows us to define the radial derivative (first and second) as

$$\partial_r^{(N)} \Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - \mathcal{U}\Phi^{(N-1)}}{2\lambda} \quad (14)$$

$$\partial_r^{2(N)} \Phi^{(N)} = \frac{\mathcal{D}\Phi^{(N+1)} - 2\Phi^{(N)} + \mathcal{U}\Phi^{(N-1)}}{\lambda^2} \quad (15)$$

and the radial part of the (spherical) Laplacian as

$$\mathcal{K}_R \Psi = \partial_r^2 \Psi + 2R^{-1} \partial_r \Psi \quad (16)$$

with

$$\partial_r \Psi = \begin{pmatrix} \partial_r^{(1)} \Phi^{(1)} & & & \\ & \partial_r^{(2)} \Phi^{(2)} & & \\ & & \ddots & \\ & & & \partial_r^{(M)} \Phi^{(M)} \end{pmatrix}$$