



FACULTY OF MATHEMATICS,
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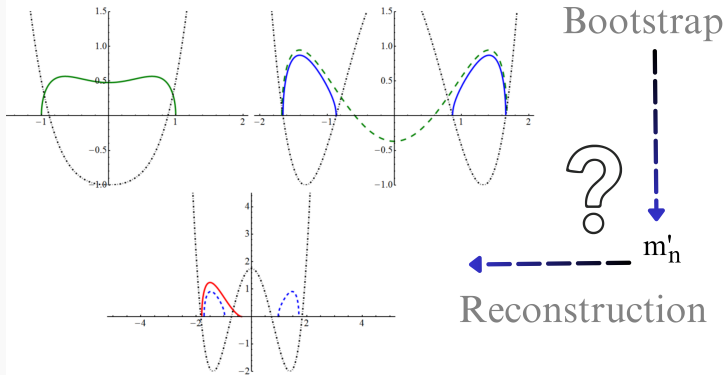
Eigenvalue distribution from bootstrap estimates

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[S. Kováčik, K. Magdolenová (2025)]



Expected value

$$\langle f \rangle = \frac{1}{Z} \int d[\phi] f(\phi) e^{-N^2 S(\phi)}$$

1. Dyson-Schwinger equations

$$\int d[\phi] \frac{\partial}{\partial \phi} \left(\phi^s e^{-N^2 S(\phi)} \right) = 0$$

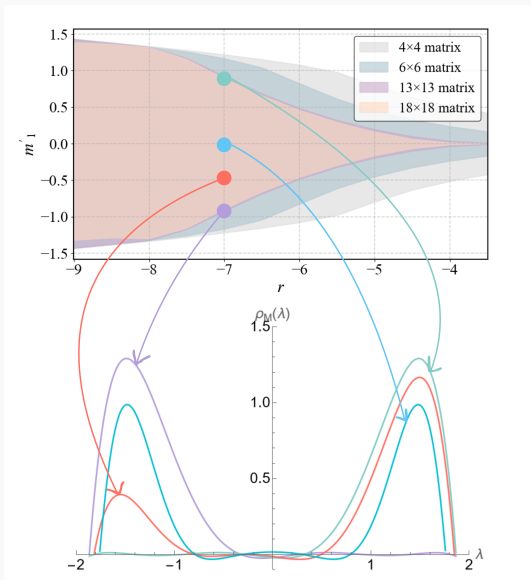
$$\langle \text{Tr}(\phi^m V'(\phi)) \rangle = \sum_{q=0}^{m-1} \langle \text{Tr}(\phi^q) \rangle \langle \text{Tr}(\phi^{m-1-q}) \rangle$$

$$m_n = \frac{1}{N} \langle \text{Tr}(\phi^n) \rangle$$

2. Hamburger moment problem

$$\begin{pmatrix} m_0 & m_1 & m_2 & \dots \\ m_1 & m_2 & m_3 & \dots \\ m_2 & m_3 & m_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0$$

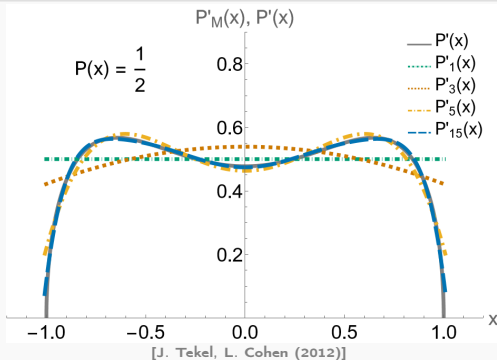
Bootstrap moments





$$\int K_n(x)K_m(x)P(x) dx = N_n\delta_{nm}$$

$$P'_M(x) = P(x) \sum_{n=0}^M \frac{1}{N_n} \langle K_n(x) \rangle' K_n(x)$$

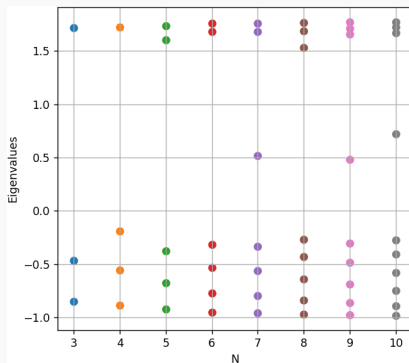




Projected operator

$$M_n = H_0^{-1} H_1$$

$$(H_0)_{ij} = m_{i+j}, \quad (H_1)_{ij} = m_{i+j+1}, \quad V_n = \text{span}\{1, x, \dots, x^{n-1}\}$$

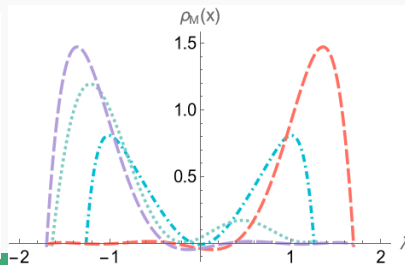
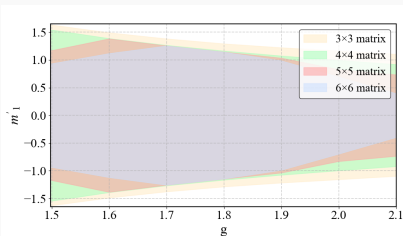


Cubic asymmetric multitrace model

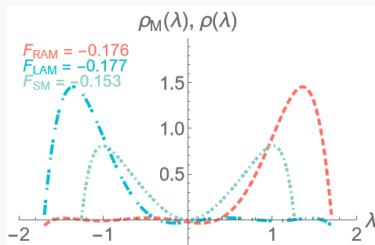


$$S = \frac{1}{N} \left(\frac{1}{2} r \text{Tr}(\phi^2) + g \text{Tr}(\phi^4) \right) + \frac{1}{N^2} t \text{Tr}(\phi) \text{Tr}(\phi^3)$$

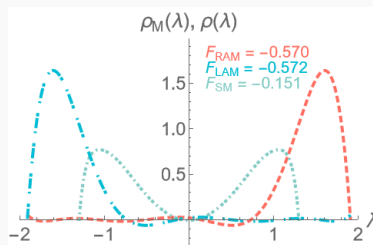
$$\sum_{q=0}^{k-1} \langle \text{Tr}(\phi^q) \rangle \langle \text{Tr}(\phi^{k-1-q}) \rangle - N r \langle \text{Tr}(\phi^{k+1}) \rangle - 4 N g \langle \text{Tr}(\phi^{k+3}) \rangle - t (\langle \text{Tr}(\phi^3) \rangle \langle \text{Tr}(\phi^k) \rangle - 3 \langle \text{Tr}(\phi) \rangle \langle \text{Tr}(\phi^{k+2}) \rangle) = 0$$



$$F = \int_C d\lambda V(\lambda) \rho(\lambda) - \iint_{C \times C \setminus \lambda = \lambda'} d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \log |\lambda - \lambda'|$$



(a) $r=-5.265$ $g=1.732525$

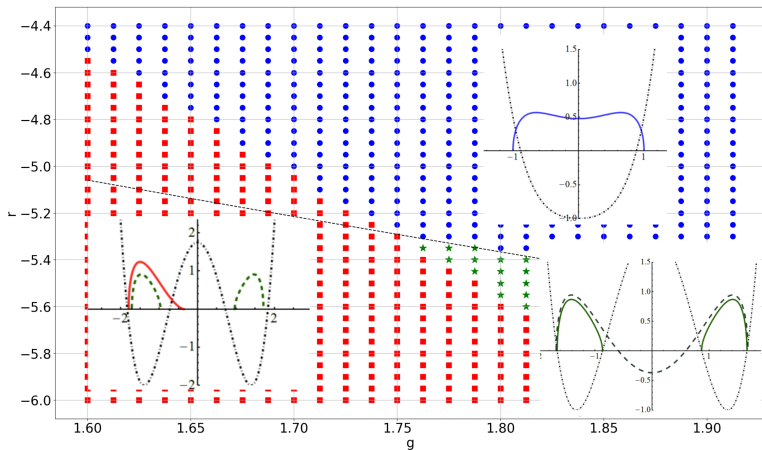


(b) $r=-5$ $g=1.5$

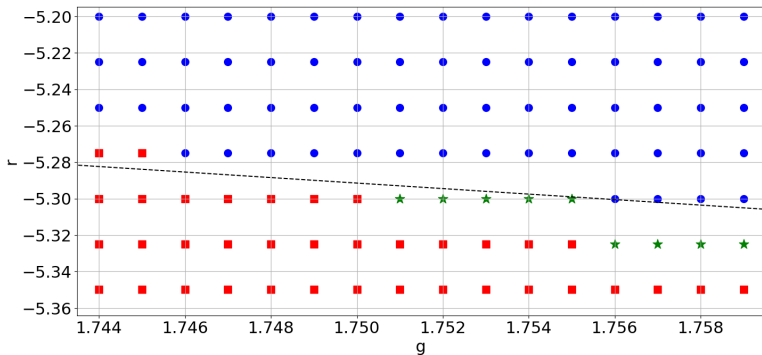
Phase diagram



6



Triple point



BM + RM	$g=1.75$ $r=-5.29$
HMC	$g=1.77$ $r=-5.54$
theory	$g=1.73$ $r=-5.27$



Dirac operators

$$D_{(1,0)} = \{H, \cdot\} \qquad D_{(0,1)} = [H, \cdot]$$

Quartic Dirac ensemble

$$\frac{1}{Z} e^{-S(D)} dD$$

$$S(D) = g \operatorname{Tr}(D^2) + \operatorname{Tr}(D^4)$$

Equivalent matrix model

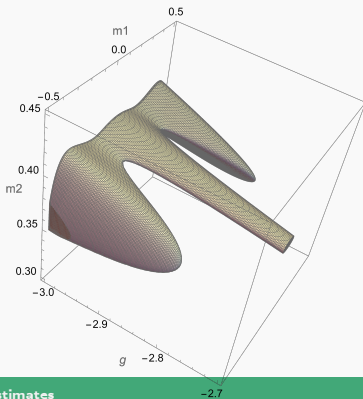
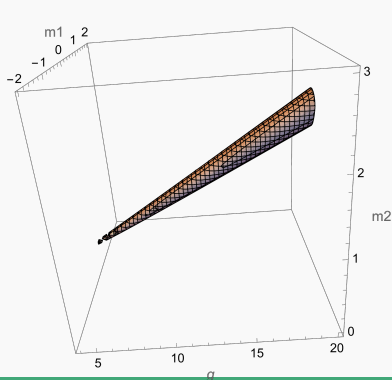
$$\frac{1}{Z} \exp \left[-2g \left(N \operatorname{Tr}(H^2) + \varepsilon (\operatorname{Tr} H)^2 \right) \right. \\ \left. - 2 \left(N \operatorname{Tr}(H^4) + \varepsilon 4 \operatorname{Tr}(H) \operatorname{Tr}(H^3) + 3 (\operatorname{Tr}(H^2))^2 \right) \right] dH$$

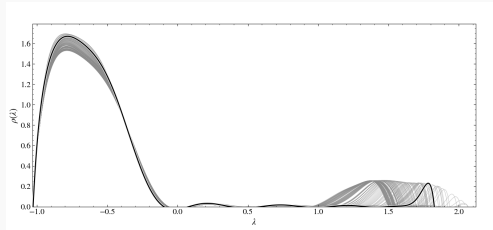


Recurrence relation

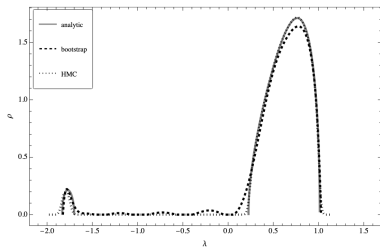
$$\sum_{k=0}^{l-1} m_k m_{l-k-1} = g (4m_{l+1} + 4\varepsilon m_1 m_l)$$

$$+ 8 (m_{l+3} + \varepsilon m_3 m_l + 3\varepsilon m_1 m_{l+2} + 3m_2 m_{l+1})$$

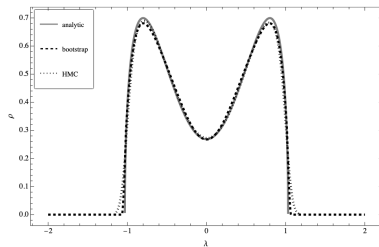




$g = -4$



(a) Comparison for $g = -4$.



(b) Comparison for $g = -3.15$.

Reduced symmetric model

$$S = N \left[-g_r \text{Tr} R \Phi^2 - g_2 \text{Tr} \Phi^2 + g_4 \text{Tr} \Phi^4 \right]$$

Modified loop equation

$$m_{2p+2} = \frac{\sum_{j=0}^{p-1} m_{2j} m_{2p-2-2j} + 2g_2 m_{2p} + 2g_r d_{2p}}{4g_4}$$

External matrix

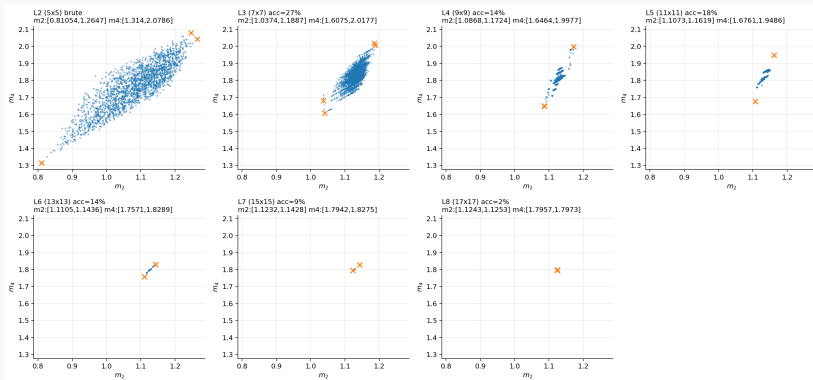
$$R \underset{N \gg 1}{\simeq} -\frac{16}{N} \text{diag}(1, 2, \dots, N)$$

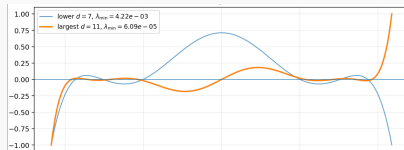
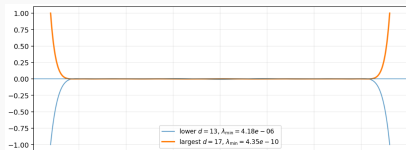
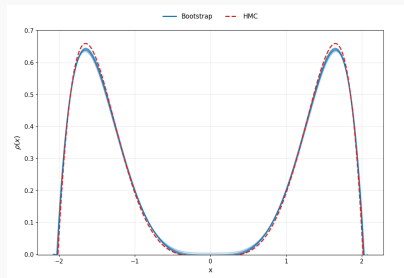
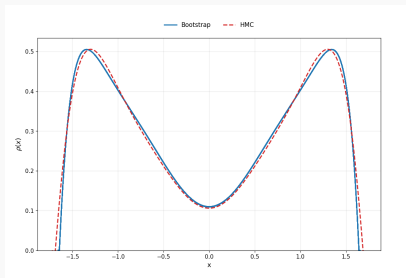
Additional correlator

$$d_{2p} = \frac{1}{N} \text{Tr}(R \Phi^{2p})$$

$$d_{2p} = \lambda_{\min}(R) m_{2p} u_p$$

GW bootstrap: increasing truncation





[H. W. Lin (2020)]

Words

$$A, B, AB, BA, A^2, B^2, B^2A, A^2B, ABA, \dots$$

$$\#\{\text{words of length} \leq \Lambda\} \sim 2^\Lambda$$

Positivity

$$M_{ij} = \left\langle \frac{1}{N} \text{Tr} \left(O_i^\dagger O_j \right) \right\rangle \succeq 0, \quad O_i \in \{I, A, B, AB, \dots\}$$

Loop equations – quadratic, but no longer closed

$$\langle \text{Tr} (\text{Word}_l \times \partial_M V(A, B)) \rangle = \sum_{l_1=1}^l \langle \text{Tr Word}_{l_1-1}^{(M)} \rangle \langle \text{Tr Word}_{l-l_1}^{(M)} \rangle$$



Large- N factorization $\Rightarrow$ quadratic constraints

$$x_i = \langle \text{Tr } O_i \rangle, \quad X_{ij} = x_i x_j$$

Nonlinear (non-convex)

$$X = x x^T \\ x^T A_i x + b_i^T x + a_i = 0$$

Relaxation (convex)

$$X \succeq x x^T \quad \Longleftrightarrow \quad R = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$$

Resulting linear SDP

$$M_0 + \sum_{j=1}^L M_j x_j \succeq 0, \quad \text{Tr } X A_i + b_i^T x + a_i = 0, \quad R \succeq 0$$

convex $\Rightarrow$ solvable (SDPA, Clarabel, SCS)

Recent improvement

Add an additional positivity constraint from the correlation matrix:

$$\boxed{M - X \succeq 0} \quad \Rightarrow \quad \text{stronger relaxation, fewer spurious solutions, tighter bounds}$$

Increasing complexity

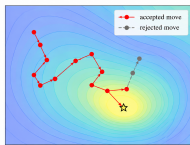
more moments



larger search space

Metropolis-style exploration

Random proposals + probabilistic acceptance



1. Propose a new point

$$x' = x + \delta$$

2. Compute change in loss

$$\Delta\mathcal{L} = \mathcal{L}(x') - \mathcal{L}(x)$$

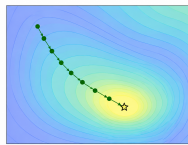
3. Accept with probability

$$P_{\text{acc}} = \min\left(1, e^{-\Delta\mathcal{L}/T}\right)$$

- Random exploration of the search space
- Can escape local minima
- Good for mapping the allowed region

Gradient-based search

Deterministic descent towards lower loss



1. Compute the gradient

$$g = \nabla\mathcal{L}(x)$$

2. Update parameters

$$x_{n+1} = x_n - \eta g$$

3. Repeat until convergence

$$\|g\| < \epsilon$$

- Directed search along the gradient
- Efficient convergence near a minimum
- May get trapped in local minima

→ **Complementary approaches:** Metropolis for global exploration, gradient descent for fast refinement.

Main results

- Eigenvalue reconstruction from bootstrap moments
- Free-energy selection of thermodynamic phases
- Cubic asymmetric multitrace model
- Dirac ensembles: types $(1, 0)$ and $(0, 1)$
- Grosse–Wulkenhaar model
- Support estimation for improved reconstruction
- Relaxation bootstrap for multimatrix models

Current work & outlook

- Two- and three-matrix models
- Increasingly complex multimatrix systems
- Efficient exploration of the bootstrap search space
- Metropolis and gradient-based approaches
- Applications to other areas of physics

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- [3] H. Hessam, M. Khalkhali, N. Pagliaroli, "Bootstrapping Dirac ensembles", J. Phys. A 55:335204, 2022.
- [4] B. Bukor, J. Tekel, "Cubic asymmetric multitrace matrix model", J. Phys. A 58:255203, 2025.
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Thank you for your attention