

Phase structure of a Dirac-Type Random Matrix Model

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Introduction

In this work, we apply saddle point method to investigate properties of simple Dirac ensembles

Random Matrix Models

Random matrix model = set of matrices + probability weight
Expected value of a function is

$$\langle f \rangle = \int dM f(M) e^{-N^2 S(M)}$$

Random Matrix Models

If there is a symmetry under conjugation $M \rightarrow U M U^\dagger$ with a unitary matrix $U \in SU(N)$, we can write M in terms of eigenvalues λ and unitary matrix U .

In that case

$$dM = \left(\prod_{i < j} (\lambda_i - \lambda_j)^2 \right) \left(\prod_{i=1}^N d\lambda_i \right) dU$$

and integral turns into

$$\langle f \rangle = \frac{1}{Z} \int \left(\prod_{i=1}^N \lambda_i \right) f(\lambda_i) e^{-N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i < j} \log |\lambda_i - \lambda_j| \right]} \times \int dU$$

Saddle point equation

We are interested in case of infinitely large matrices $N \rightarrow \infty$

$$\langle f \rangle = \frac{1}{Z} \int \left(\prod_{i=1}^N \lambda_i \right) f(\lambda_i) e^{-N^2 \left[\frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i < j} \log |\lambda_i - \lambda_j| \right]}$$

Configurations of eigenvalues λ_i that minimize action $S_{\text{eff}}(M)$ dominate.

Saddle point condition

$$\frac{\partial S_{\text{eff}}}{\partial \lambda_i} = 0$$

$$V'(\lambda_i) - \frac{2}{N} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} = 0$$

We can switch from λ_i to eigenvalue distribution $\rho(\lambda)$

$$V'(\lambda) - 2 \text{PV} \int_{\mathcal{C}} d\tilde{\lambda} \frac{\rho(\tilde{\lambda})}{\lambda - \tilde{\lambda}} = 0$$

How to find eigenvalue distribution from saddle point equation?

To solve it, we introduce resolvent function

$$\omega(z) = \int d\lambda \frac{\rho(\lambda)}{z - \lambda}$$

It is possible to turn this into Riemman-Hilbert-like problem

Saddle point equation

Function that satisfies the equation is

$$\omega(z) = \frac{1}{2} \left[V'(z) - |M(z)| \sqrt{\sigma(z)} \right]$$

and therefore

$$\rho(\lambda) = \frac{1}{2\pi i} |M(\lambda)| \sqrt{\sigma(\lambda)}$$

$$\sigma(z) = \prod_i (z - z_i) \quad z_i \text{ are endpoints of intervals}$$

Saddle point equation

Resolvent function must satisfy

$$\omega(z) = \int d\lambda \frac{\rho(\lambda)}{z - \lambda} \quad \omega(z) \rightarrow \frac{1}{z} \quad \text{as } z \rightarrow \infty$$

We expand $\omega(z)$ around infinity

$$\omega(z) = A z^2 + B z^1 + C z^0 + D \frac{1}{z} + E \frac{1}{z^2} + \dots$$

Free energy

There can be many solutions to saddle point equation.

Only those with lowest free energy dominate

$$\mathcal{F} = \frac{1}{N} \sum_i V(\lambda_i) - \frac{2}{N^2} \sum_{i < j} \log |\lambda_i - \lambda_j|$$

Effective action with multitrace terms

Any function of moments c_n added to action still satisfies the $M \rightarrow U M U^\dagger$ symmetry

$$S(M) = f(c_1, c_2, \dots, c_n) + \frac{1}{2} r \operatorname{Tr}(M^2)$$

These are called *multitrace matrix models*.

We can write saddle point condition for them

$$\sum_n \frac{\partial f}{\partial c_n} n \lambda_i^{n-1} + r \lambda_i = \frac{2}{N} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j}$$

It is saddle point equation of pure potential model with effective constants!

$$r_{\text{eff}} \lambda_i + \sum_{n \neq 2} g_{n,\text{eff}} \lambda_i^{n-1} = \frac{2}{N} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j}$$

Effective action with multitrace terms

Moments of this distribution become functions of themselves $\rightarrow$ self-consistent equations

$$c_1 = f(c_1, c_2, \dots)$$

$$c_2 = f(c_1, c_2, \dots)$$

$$\vdots$$

Additional set of equations that must be satisfied

Dirac operator

Dirac operator encodes geometry of the underlying space

In a Dirac ensemble, we allow the Dirac operator — and therefore the geometry — to vary according to a probability distribution.

$$p(D) = \frac{1}{Z} dD e^{-S[D]}$$

Simple example

$(1, 0)$ - type Dirac operator

$$D_{(1,0)} = \{H, \cdot\}$$

$$D_{(1,0)}(X) = \{H, X\}$$

$(0, 1)$ - type Dirac operator

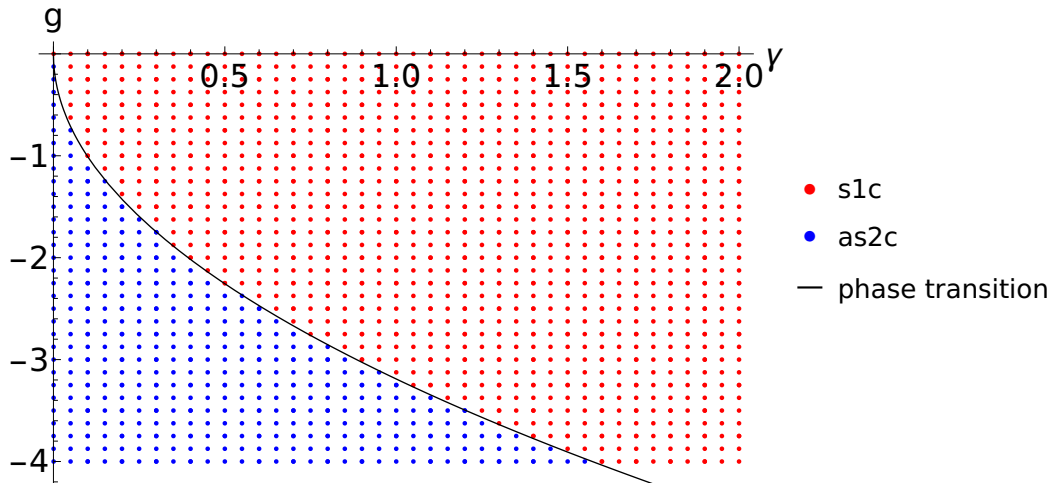
$$D_{(0,1)} = [H, \cdot]$$

$$D_{(0,1)}(X) = [H, X]$$

We considered the action

$$S(D) = g \operatorname{Tr}(D^2) + \operatorname{Tr}(D^4)$$

Phase diagram for (1,0)-type Dirac operator



[B. Bukor et al., arXiv:2609.19898 (2026)]

Mixed Dirac operator

We define mixed Dirac operator as

$$D = \mathbf{a}(\Gamma_1 \otimes [H, \cdot]) + \mathbf{b}(\Gamma_2 \otimes \{iH, \cdot\})$$

with Γ 's being the 2×2 spinor representation

The action can be rewritten in terms of matrix H as

$$S(H) = N \left[4g(a^2 + b^2) \text{Tr}(H^2) - 4g(a^2 - b^2) \text{Tr}(H)^2 + 4(a^2 + b^2)^2 \text{Tr}(H^4) \right. \\ \left. + 4(3a^4 - 2a^2b^2 + 3b^4)(\text{Tr}(H^2))^2 - 16(a^4 - b^4) \text{Tr}(H^3) \text{Tr}(H) \right]$$

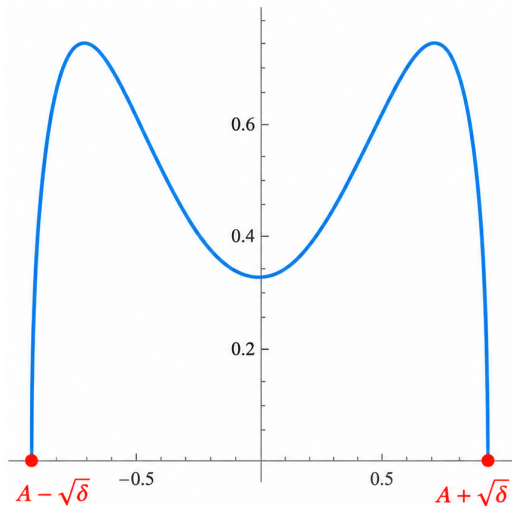
How did we look for $\rho(\lambda)$

Saddle point condition $\Rightarrow \rho(\lambda)$ + equations for ends of interval of support

Solution with lowest free energy $\rho(\lambda) \rightarrow$ Preferred solution

How did we look for $\rho(\lambda)$

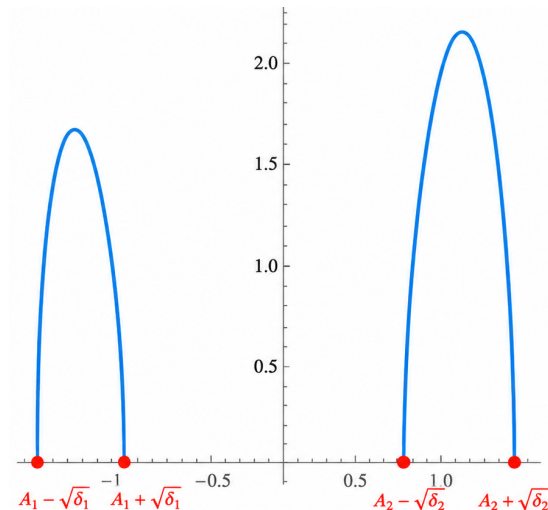
One-cut solution ansatz



$$\sigma(z) = \left(z - \left(A - \sqrt{\delta}\right)\right) \left(z - \left(A + \sqrt{\delta}\right)\right)$$

How did we look for $\rho(\lambda)$

Two-cut solution ansatz

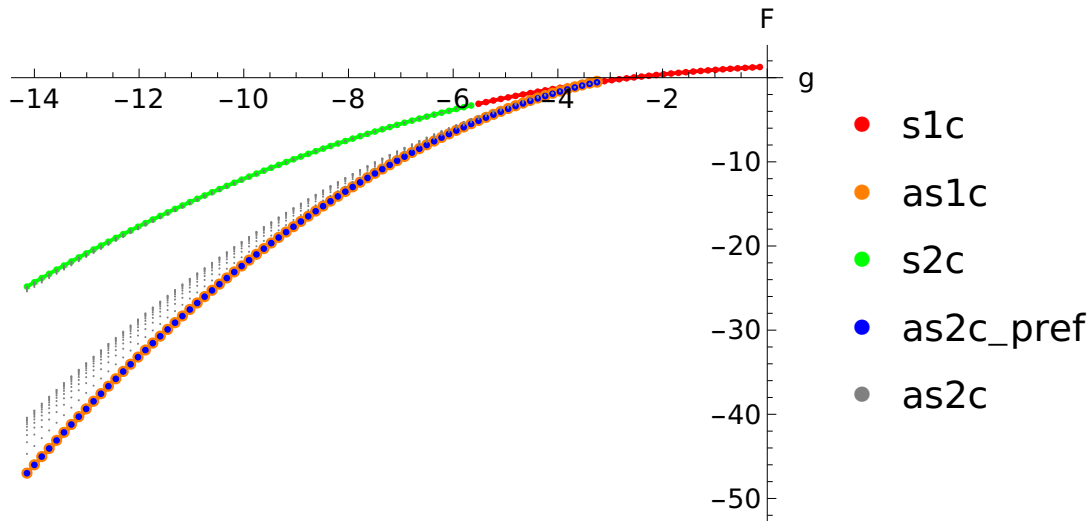


$$\sigma(z) = \left(z - \left(A_1 - \sqrt{\delta_1} \right) \right) \left(z - \left(A_1 + \sqrt{\delta_1} \right) \right) \\ \times \left(z - \left(A_2 - \sqrt{\delta_2} \right) \right) \left(z - \left(A_2 + \sqrt{\delta_2} \right) \right)$$

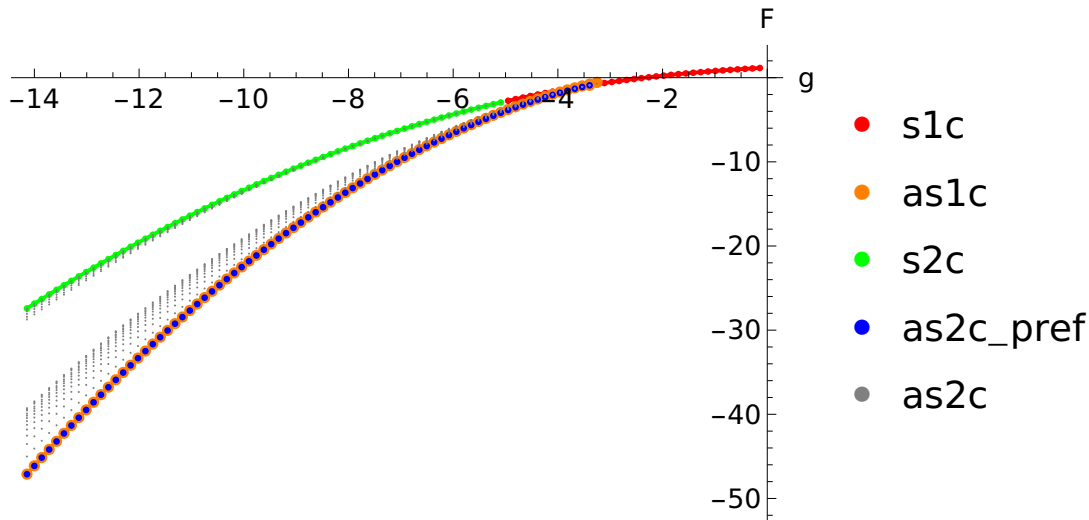
Not enough equations!

$$\delta_1 = k \delta_2, \quad k \in (0, 1)$$

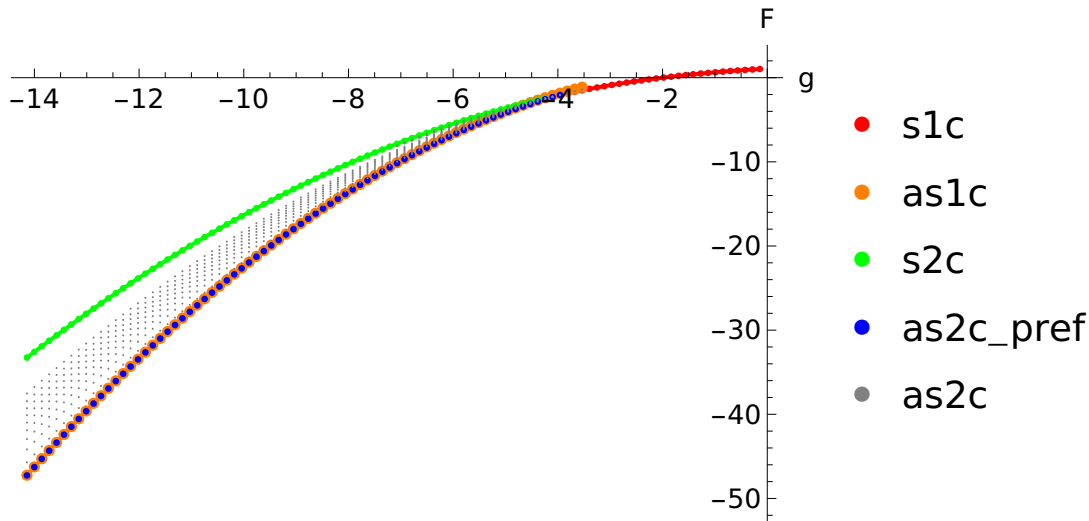
Phase diagram for $D = \mathbf{0.1} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.9} (\Gamma_2 \otimes \{H, \cdot\})$



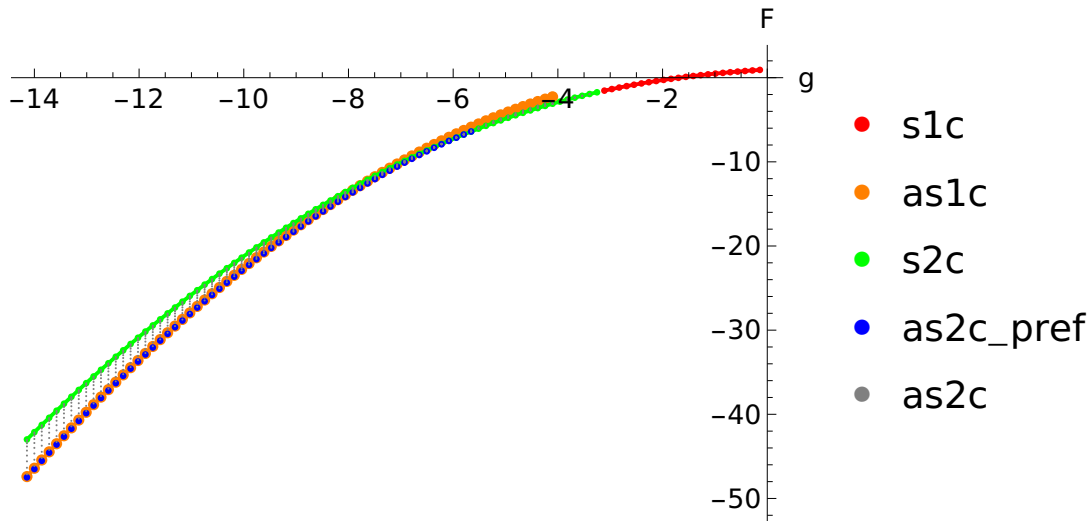
Phase diagram for $D = \mathbf{0.2} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.8} (\Gamma_2 \otimes \{H, \cdot\})$



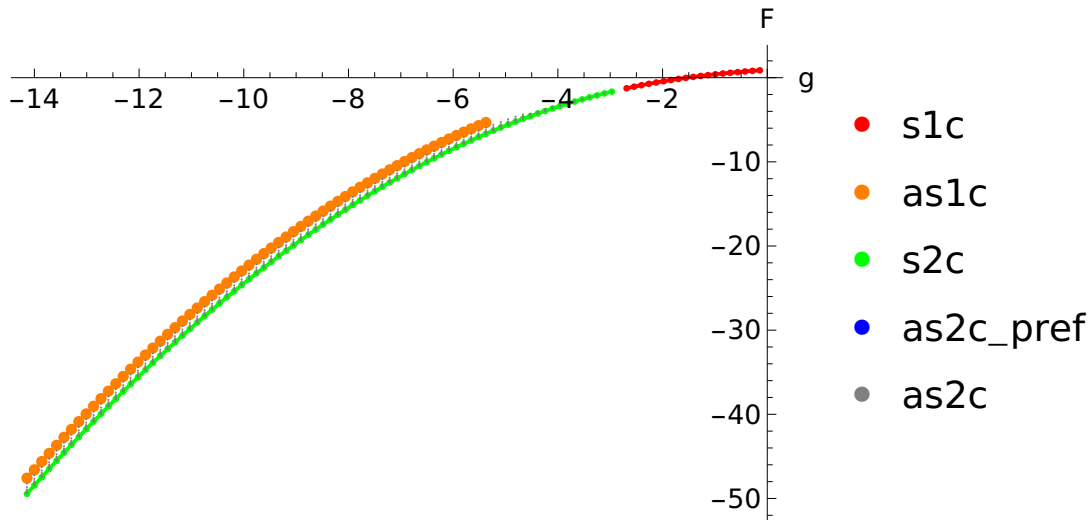
Phase diagram for $D = \mathbf{0.3} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.7} (\Gamma_2 \otimes \{H, \cdot\})$



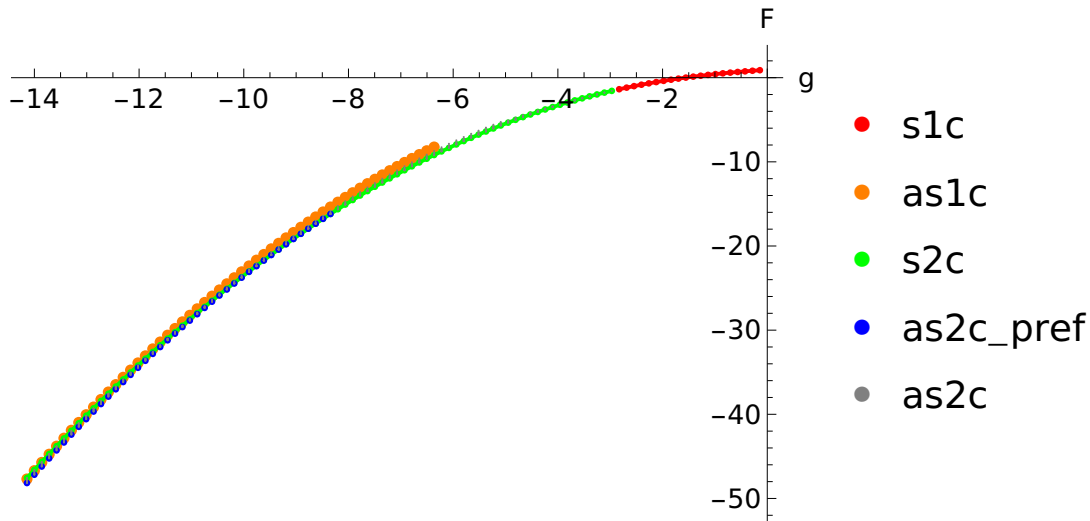
Phase diagram for $D = \mathbf{0.4} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.6} (\Gamma_2 \otimes \{H, \cdot\})$



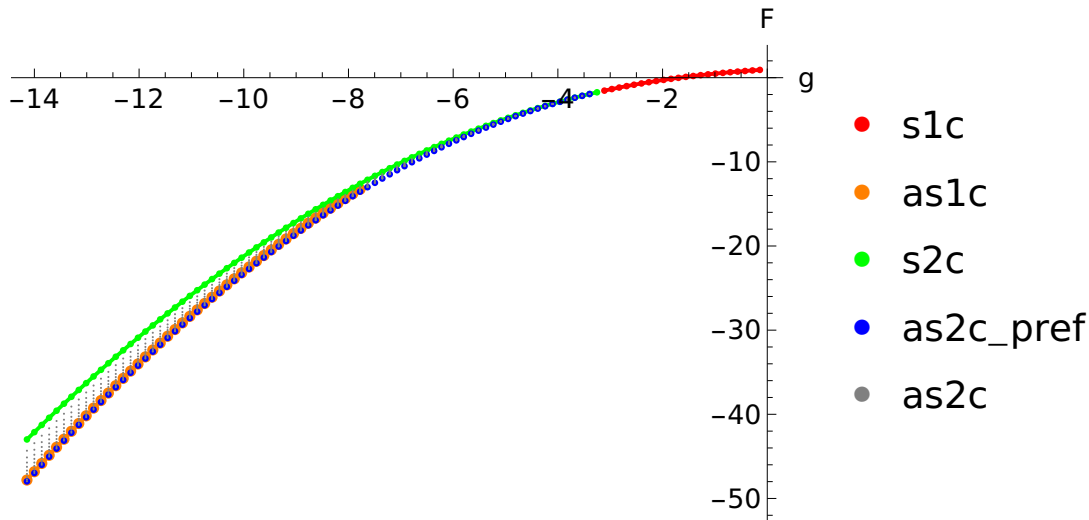
Phase diagram for $D = \mathbf{0.5} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.5} (\Gamma_2 \otimes \{H, \cdot\})$



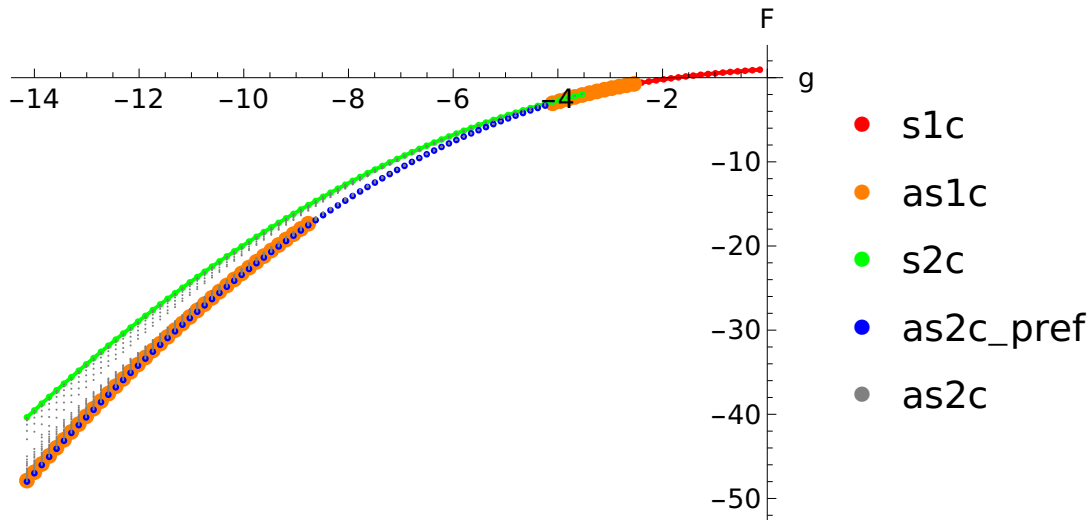
Phase diagram for $D = \mathbf{0.55} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.45} (\Gamma_2 \otimes \{H, \cdot\})$



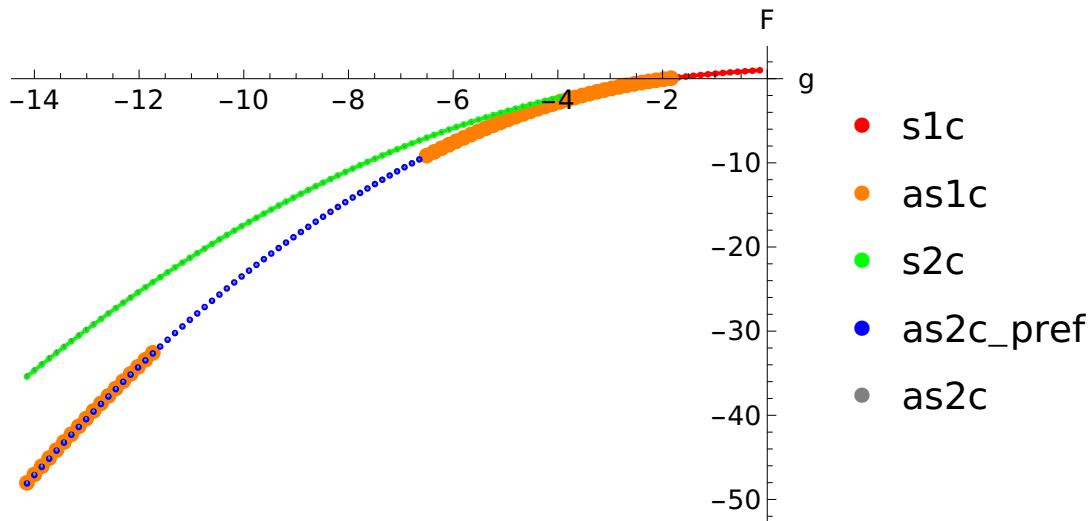
Phase diagram for $D = \mathbf{0.6} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.4} (\Gamma_2 \otimes \{H, \cdot\})$



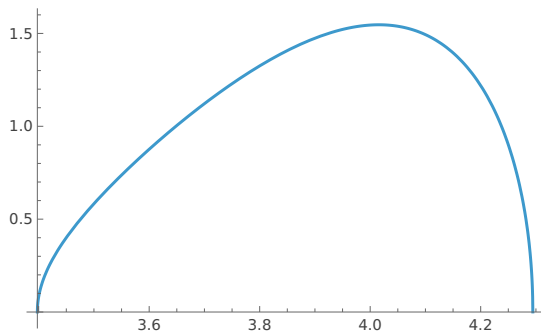
Phase diagram for $D = \mathbf{0.625} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.375} (\Gamma_2 \otimes \{H, \cdot\})$



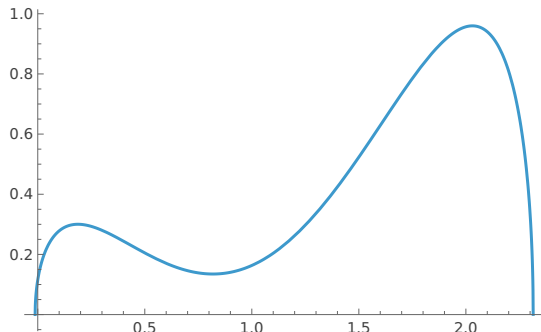
Phase diagram for $D = \mathbf{0.675} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.325} (\Gamma_2 \otimes \{H, \cdot\})$



Phase diagram for $D = \mathbf{0.675} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.325} (\Gamma_2 \otimes \{H, \cdot\})$

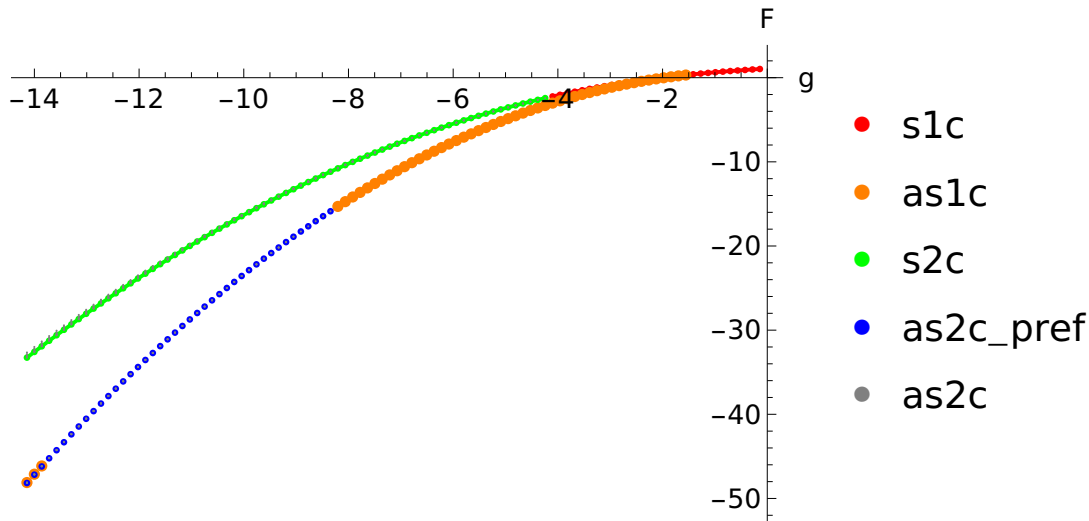


This AS1C phase disappears ($g = -14$)

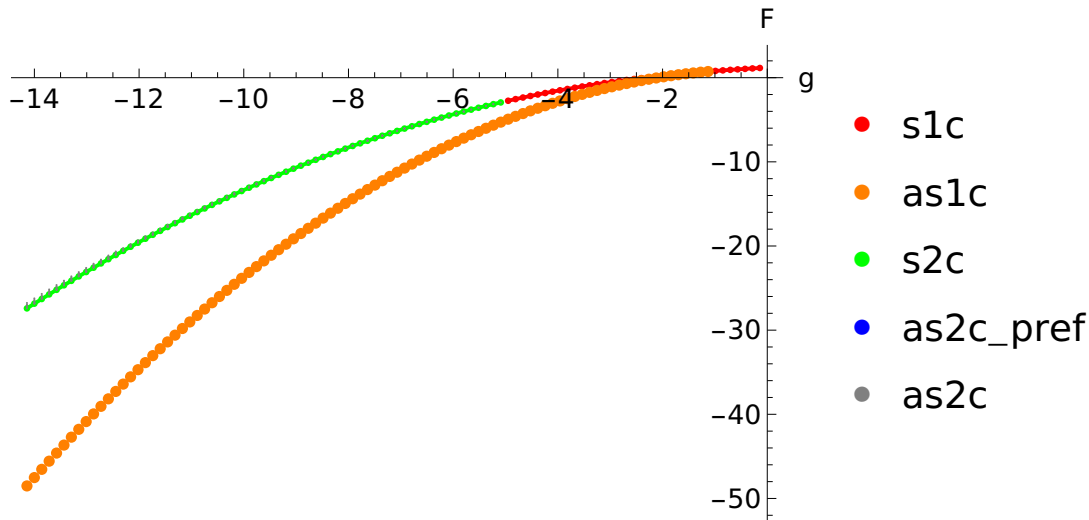


This AS1C phase emerges ($g = -4$)

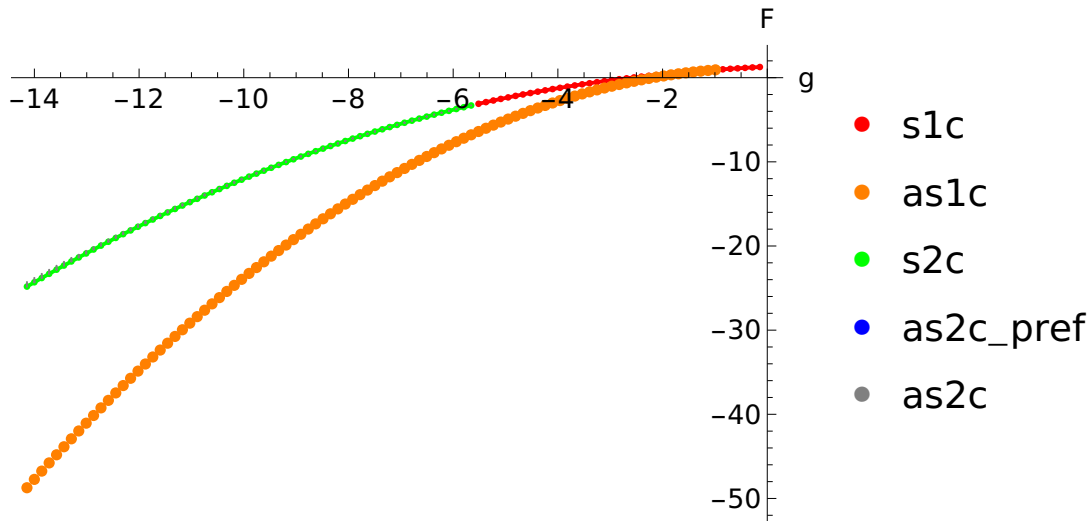
Phase diagram for $D = \mathbf{0.7} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.3} (\Gamma_2 \otimes \{H, \cdot\})$



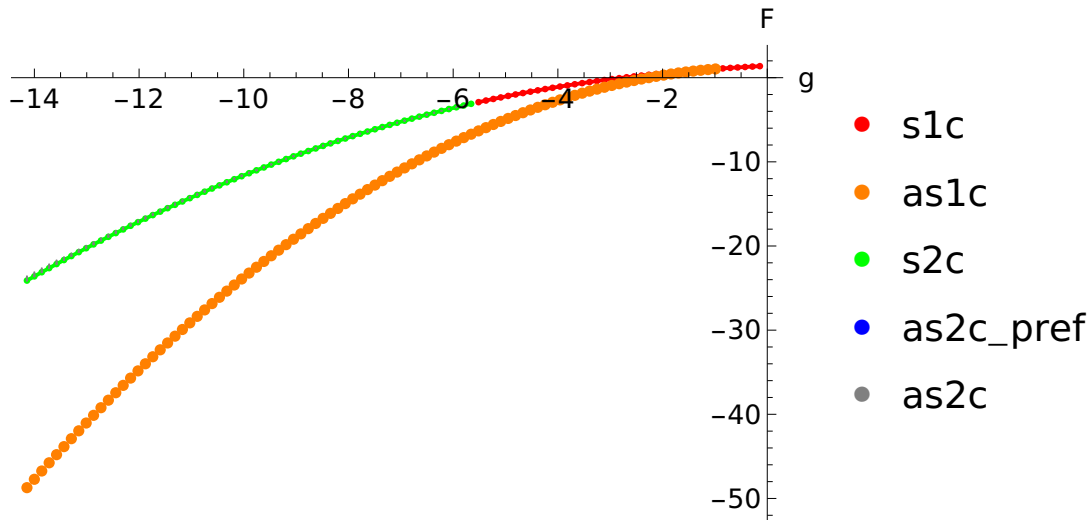
Phase diagram for $D = \mathbf{0.8} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.2} (\Gamma_2 \otimes \{H, \cdot\})$



Phase diagram for $D = \mathbf{0.9} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.1} (\Gamma_2 \otimes \{H, \cdot\})$



Phase diagram for $D = \mathbf{0.999} (\Gamma_1 \otimes [H, \cdot]) + \mathbf{0.001} (\Gamma_2 \otimes \{H, \cdot\})$



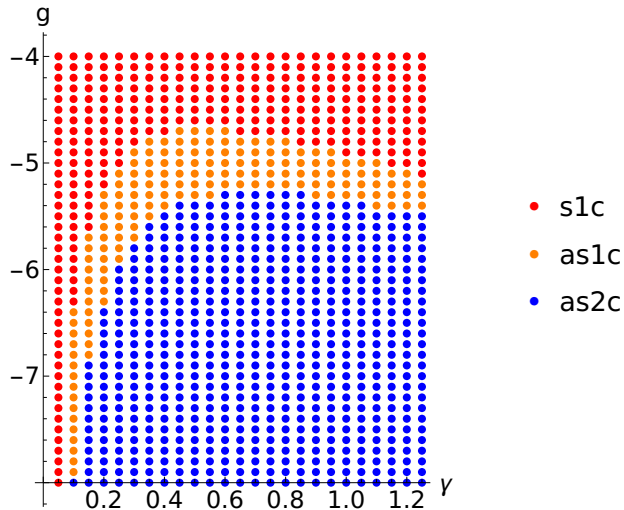
Action with additional multitrace term

$$S(D) = g \operatorname{Tr}(D^2) + \gamma \operatorname{Tr}(D^4) + \operatorname{Tr}(D^2)^2$$

Modified action for $(1, 0)$ Dirac operator is

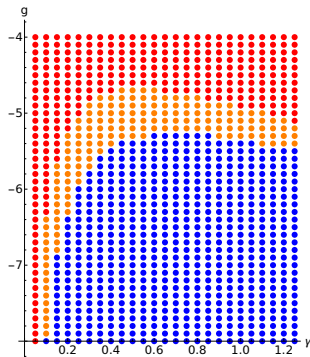
$$\begin{aligned} S(H) = & 2gN \operatorname{Tr}(H^2) + 2g(\operatorname{Tr} H)^2 \\ & + 2\gamma N \operatorname{Tr}(H^4) + 8\gamma \operatorname{Tr}(H) \operatorname{Tr}(H^3) \\ & + 6\gamma(\operatorname{Tr}(H^2))^2 + 4N^2(\operatorname{Tr}(H^2))^2 \\ & + 8N \operatorname{Tr}(H^2)(\operatorname{Tr} H)^2 + 4(\operatorname{Tr} H)^4 \end{aligned}$$

Phase diagram of $(\text{Tr}(D^2))^2$ model

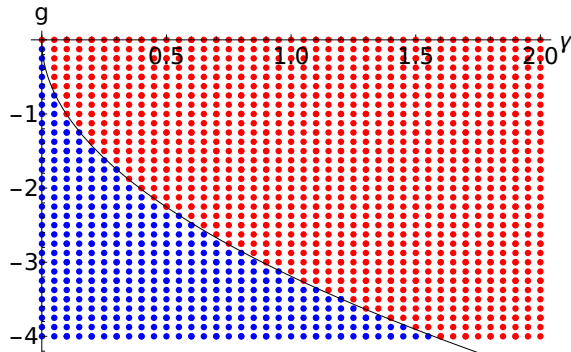


Comparison with previous model

Multitrace model

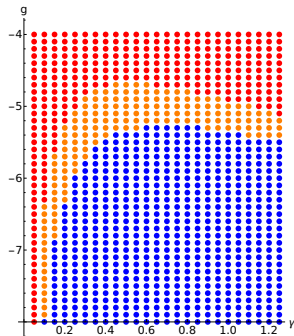


Previous model

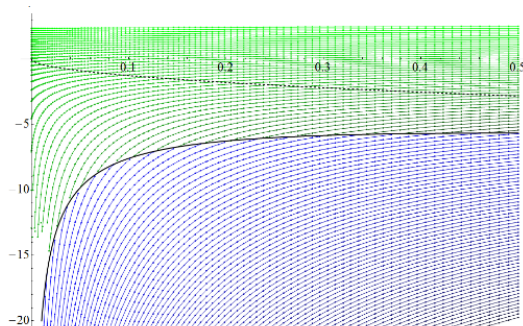


Comparison with c_2^2 model

Multitrace model



c_2^2 model



[J. Tekel (2015)]

Conclusions

- We analysed the phase structure of two Dirac ensembles in the large- N limit using the saddle-point method.
- For a linear combination of the $(1, 0)$ and $(0, 1)$ Dirac operators, we studied the model with

$$S(D) = g \operatorname{Tr}(D^2) + \operatorname{Tr}(D^4).$$

- For the $(1, 0)$ Dirac operator, we obtained the phase diagram of the multitrace model

$$S(D) = \beta \operatorname{Tr}(D^2) + \gamma \operatorname{Tr}(D^4) + (\operatorname{Tr}(D^2))^2.$$

References

- ① B. Bukor, M. Khalkhali, S. Kováčik, A. Kubiš, K. Magdolenová, N. Pagliaroli, J. Tekel, “Asymmetric phase transitions in random noncommutative geometries”, arXiv:2609.19898, 2026.
- ② M. Šubjaková, J. Tekel, “Fuzzy field theories and related matrix models”, *PoS CORFU2019* 189, 2020.
- ③ J. Tekel, “Phase structure of fuzzy field theories and multitrace matrix models”, *Acta Phys. Slovaca* 65:369, 2015.

Thank you for your attention!