

Gauge theories for G-structures

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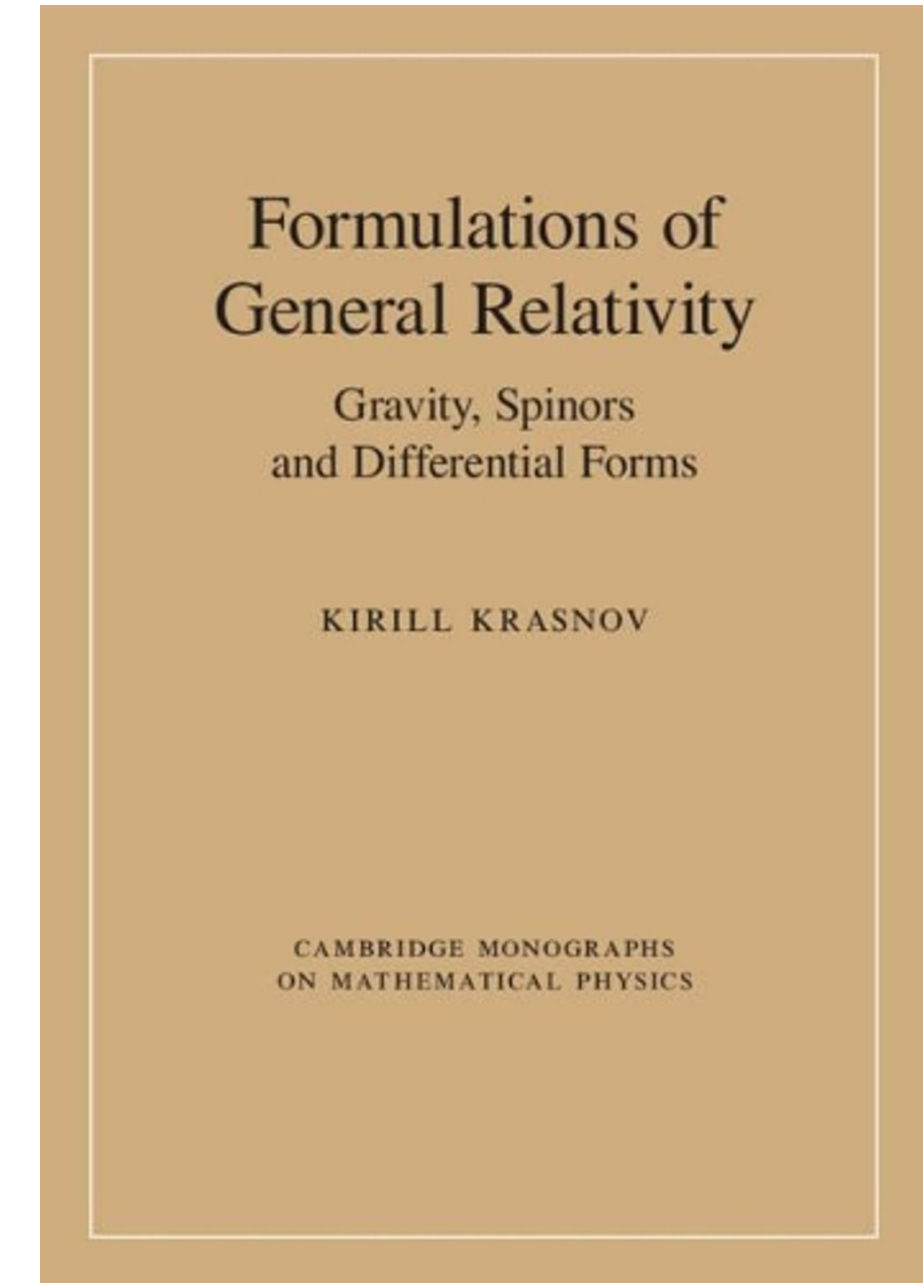
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Appetiser

I wrote a book about “Formulations of General Relativity”

Here is a formulation of (Euclidean) 4D GR that is **not** in the book

Let $\theta^{1,2,3,4}$ be an orthonormal frame 1-forms, and build $\Psi = \begin{pmatrix} \theta^1 + i\theta^2 \\ \theta^3 + i\theta^4 \end{pmatrix}$



Let A be an $SU(2)$ connection - 1-form with values in traceless anti-Hermitian matrices

$$S[A, \Psi] = \int_M (D_A \Psi)^\dagger \wedge (D_A \Psi)$$

Here $D_A \Psi = d\Psi + A \wedge \Psi$

Proposition: The metrics $g = \text{Re}(\Psi^\dagger \otimes \Psi)$ defined by Ψ that are critical points of this action are Ricci-flat.

In the reverse direction, start with Ricci-flat metric, take its orthonormal frame, form Ψ .

It will satisfy EL equations obtained by extremising this action

G-structures

G-structures provide possibly the most natural generalisation of Riemannian geometry - invented by Cartan

The general question that this talk addresses: What replaces Einstein-Hilbert action and Einstein equations?

Definition: A G-structure on M is a reduction of the principal $GL(n, \mathbb{R})$ bundle of frames to a G-subbundle

Concretely: A G-structure can be realised as a tensor $t \in \Gamma((TM)^{\otimes p} \otimes (T^*M)^{\otimes q})$, or a collection of tensors, such that $\text{Stab}_{GL(n, \mathbb{R})}(t) = G$

Examples:

- Metric is an $O(n)$ -structure, with $g \in \Gamma(S^2 T^*M)$
- Almost complex structure on a $2n$ -dimensional manifold is $J \in \Gamma(TM \otimes T^*M) \cong \text{End}(TM) : J^2 = -\mathbb{I}$

It is a $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ structure

- Almost-Hermitian structure is both structures combined (compatibly $g(J \cdot, J \cdot) = g(\cdot, \cdot)$)

It is a $U(n) \subset GL(2n, \mathbb{R})$ structure

Metric G-structures

Definition: A G-structure on M is called metric if $G \subset O(n) \subset GL(n, \mathbb{R})$

Almost-Hermitian structure is metric $U(n) \subset O(2n)$, but almost-complex structure is not

The space of G-structures at a point is the group coset $GL(n, \mathbb{R})/G$, which for metric G-structures becomes

$$GL(n, \mathbb{R})/G = GL(n, \mathbb{R})/O(n) \times O(n)/G$$

Metric G-structures = Metric + Additional geometric data

The basic question of interest to us here is: What is the most natural dynamical principle for G-structures?

I will not answer this question in general, but in some cases the answer is interesting

Why care? 4D Riemannian geometry as explained in the first slide. But also geometry in higher dimensions

Cartan's viewpoint - soldering forms

There are several canonical objects associated with a G -structure. Usually described from the principal G -bundle

$$Fr \rightarrow M \quad \text{point of view}$$

But I will use the physics local description in terms of differential forms on M

Let $E \rightarrow M$ be a vector bundle with fibres V isomorphic to $\mathbb{R}^n$

Soldering form $\theta \in \Gamma(E \otimes \Lambda^1 M)$ - an $\mathbb{R}^n$ -valued 1-form θ_μ^I on M .

Solders $T_x M$ with V : for $v \in T_x M$ we have $\theta(v) \in V \cong \mathbb{R}^n$

Then the G -invariant tensors on M can be described as pull-backs of constant G -invariant tensors on V

$$\text{E.g. metric } g_{\mu\nu} = \delta_{IJ} \theta_\mu^I \theta_\nu^J$$

For a more general (metric) G -structure, only G acts on the soldering forms, rather than $O(n)$

Example: Almost-Hermitian structure

Example relevant for what comes below: Almost-Hermitian $U(n)$ structure in $2n$ real dimensions

The soldering form in this case can be described as an n -component column of complex $(1,0)$ -forms

$U(n)$ acts on this as in a vector in its fundamental representation

$$\Psi = \begin{pmatrix} \theta^1 + iJ(\theta^1) \\ \theta^2 + iJ(\theta^2) \\ \vdots \\ \theta^n + iJ(\theta^n) \end{pmatrix}$$

Both the metric and the Kahler 2-form $U(n)$ -invariant tensors can be recovered as

$$\Psi^\dagger \otimes \Psi = g + i\omega$$

G-connections and torsion

The second geometric object for any G-structure is a G-connection ω .

Starts its life as an arbitrary connection in the principal G-bundle $Fr \rightarrow M$

But concretely, can view it as a $\mathfrak{g}$ -valued 1-form $\omega \in \Omega^1(M, \mathfrak{g})$ on M

There are many G-connections. What distinguishes them is their **torsion** $d\theta + \omega \wedge \theta = \frac{1}{2}T(\theta \wedge \theta)$

$$T \in V \otimes \Lambda^2 V^*$$

Changing the connection changes the torsion, but there is in general a part of T that cannot be killed by changing ω

Intrinsic torsion

Fundamental theorem of Riemannian geometry: For $O(n)$ -structure, there exists a unique torsion-free connection - Levi-Civita

Alternatively, the intrinsic torsion is zero in this case, can kill torsion by choosing the connection well

Torsion of a metric G-structure

For a metric G-structure the most natural concept of torsion comes from the split

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$$

Then $d\theta + w_{\mathfrak{g}}^{LC} = -w_{\mathfrak{g}^\perp}^{LC} \wedge \theta$

$$w^{LC} = w_{\mathfrak{g}}^{LC} + w_{\mathfrak{g}^\perp}^{LC}$$

And $w_{\mathfrak{g}^\perp}^{LC}$ becomes the torsion of the $\mathfrak{g}$ part of the Levi-Civita connection,

and gives the canonical representative for the intrinsic torsion of a metric G-structure

Remark: Intrinsic torsion, or more concretely the torsion of a metric G-structure, is a **first-order invariant**

I.e., it is a tensor that is constructed from the first derivatives of the tensors defining G-structure

For $O(n)$ -structure, no intrinsic torsion, no first-order invariant!

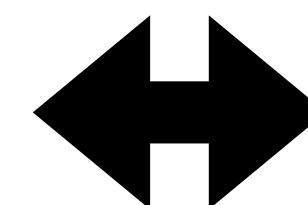
But for a general G-structure, such an invariant is there

Remark: The vanishing of the intrinsic torsion of a G-structure is the most natural first-order condition that can be imposed on it

Special holonomy, very rich subfield of differential geometry

Example:

Torsion of Almost-Hermitian structure vanishes



Kähler geometry

Towards a second-order condition

Because torsion is a first-order invariant, can decompose $w_{\mathfrak{g}^\perp}^{LC}$ into irreducibles of $\mathfrak{g}$ $w_{\mathfrak{g}^\perp}^{LC} = \bigoplus_{i \in \text{Irreps}(\mathfrak{g})} (w_{\mathfrak{g}^\perp}^{LC})_i$

And form the torsion squared action

$$S[\text{G-structure}] = \int_M \sum_i c_i |(w_{\mathfrak{g}^\perp}^{LC})_i|^2$$

In all known examples, it is known that the Einstein-Hilbert functional, which only depends on the metric part of the G-structure, can be written in this (torsion)² way, with very specific coefficients

But this answer is so general that it is not useful

It is not clear if there are any preferred choices of the coefficients, apart from those giving the EH functional

Remark: While (torsion)² actions exist for G-structures, there is no such action for O(n)-structure!

Instead, EH functional is integral of a second-order Ricci scalar, rather than of square of first-order torsion

Gauge theories for G-structures

Idea: construct the most general functional $S[w, \theta]$ built from the soldering form θ ,
its exterior covariant derivative $D_w \theta$ and $F(w)$

Recall that the usual Einstein-Cartan formalism for gravity is exactly of this type

$$S_{\text{EH}}[w, \theta] = \int_M \epsilon_{IJKL} \theta^I \wedge \theta^J \wedge F(w)^{KL}$$

Example: Almost-Hermitian U(2) structures in dimension 4 $\mathfrak{u}(2) = \mathfrak{s}\mathfrak{u}(2) \oplus \mathfrak{u}(1)$
[2603.21930](#) [math.DG] $w = A + a$ $F(A) = dA + \frac{1}{2}A \wedge A$

Modulo integration by parts, the only terms you can write are

$$S[A, a, \Psi] = \int \lambda \Psi^\dagger \wedge F(A) \wedge \Psi + \mu da \wedge (\Psi^\dagger \wedge \Psi) + \nu (\Psi^\dagger \wedge \Psi)^2$$

Where λ, μ, ν are real parameters

Proposition: Critical points are constant scalar curvature almost-Kahler 4-manifolds

SU(2) structures in 4D

I can now explain the functional from the first slide

Let Ψ be a 2-component column of complex-valued 1-forms, and let A be an SU(2) connection 1-form

The only functional one can write

$$S[A, \Psi] = \int_M \Psi^\dagger \wedge F(A) \wedge \Psi \approx \int_M (D_A \Psi)^\dagger \wedge (D_A \Psi)$$

Here $\approx$ means modulo integration by parts (and also modulo sign), so this is the functional from the first slide

Introduce $\Sigma = (\Psi \wedge \Psi^\dagger)_0$ Can then rewrite $S[A, \Psi] = \int_M \text{Tr}(\Sigma \wedge F(A))$ **Chiral Einstein-Cartan formalism**

The EL equations for the connection are $D_A \Sigma = 0$. Can be shown to imply $A = w_+^{LC}$

Where $w^{LC} = w_+^{LC} + w_-^{LC}$ is the decomposition of the Levi-Civita connection corresponding to $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$

Substituting this solution back one can show that the integrand is a multiple of the scalar curvature

So, this is Einstein-Hilbert action in 4D in disguise!

The reason why it works is that the action has hidden enlarged full SO(4) gauge invariance. The other SU(2) acts trivially on Σ

Spin(7)-structures in 8D

Spin(7)-structure in 8D defines a Spin(7)-invariant 4-form $\Phi \in \Omega^4 M$

Here there are many Spin(7)-invariant terms that one can write in the action.

But the number is small if one requires that the EL equations for the connection w are algebraic

$$S[w, \theta] = \int_M \Phi \wedge [\mu F(w)^{IJ} \wedge \theta^I \wedge \theta^J + \nu D_w \theta^I \wedge D_w \theta^I]$$

Result: There is a relative coefficient here for which this becomes a multiple of the Einstein-Hilbert action

In particular, independent of the Spin(7)-structure, just functional of the metric in disguise

But it is not clear if there is any other relative value of the coefficients that is special (in an interesting way)

SU(3)-structures in 6D

Attempting to generalise the story that works so well in 4D

Let Ψ be a 3-component column of complex-valued 1-forms. Let A be an SU(3) connection

Many terms can be written, but the number is small if require that the EL equations for A are algebraic

Define the Kahler 2-form as
$$\omega := \frac{1}{i} \Psi^\dagger \wedge \Psi$$

Then the most interesting action (and also analog of the 8D action) is

$$S[A, \Psi] = \int_M \omega \wedge [\mu \Psi^\dagger \wedge F(A) \wedge \Psi + \nu (D_A \Psi)^\dagger \wedge (D_A \Psi)]$$

Preliminary result: It appears that this time it is not possible to choose the relative value of the parameters to obtain EH

The resulting action with w integrated out is of $(\text{torsion})^2$ type, but the arising coefficients are not ones that can give EH

Summary

- Can write down gauge-theoretic actions $S[w, \theta]$, functionals of the soldering form describing a G-structure and a G-connection w
This generalises the familiar Einstein-Cartan description of GR to arbitrary G-structures
In higher dimensions many terms are possible, the number is smaller if requires the EL equations for w to be algebraic
- The simplest application of this to SU(2)-structures in 4D produces a very useful alternative way of thinking about 4D Riemannian signature Einstein equations
$$S[A, \Psi] = \int_M (D_A \Psi)^\dagger \wedge (D_A \Psi)$$

There is a lot of beautiful geometry here, related to self-duality in 4D
- The status of this construction for G-structures in higher dimensions is not yet clear
In some cases can reproduce the Einstein-Hilbert functional in this way.
But it appears that does not work in most of the cases
The geometric interpretation of the arising dynamical principle for G-structures is not yet clear

Outlook

Question that motivated this line of work:

There is a clean answer to the question of what the most natural first-order condition on a G-structure is:

Vanishing of the intrinsic torsion

What is “the most natural” set of second-order PDE’s that a G-structure should satisfy?

For the $O(n)$ -structure, the answer is Einstein equations

What replaces this, and does the question even make sense?

Σας ευχαριστώ