

PHASE TRANSITIONS IN DIRAC ENSEMBLES AND CORRESPONDING GEOMETRIES

Juraj Tekel

Department of Theoretical Physics
Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava



FACULTY OF MATHEMATICS,
PHYSICS AND INFORMATICS
Comenius University
Bratislava



work with: A. Kubiš, P. Rusnák,
B. Bukor, M. Khalkhali, S. Kováčik, K. Magdolenová, N. Pagliaroli

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Quick motivation - physics



- We need a quantum theory of gravity, unified theory would be great.
- We have reasons to believe that a future theory of quantum gravity will have a different notion of spacetime, with no distinction between points under certain length scales. [[Hossenfelder 1203.6191](#)]
- Further problems with quantization: which DOFs to quantize, superpositions, etc.



Quick motivation - math

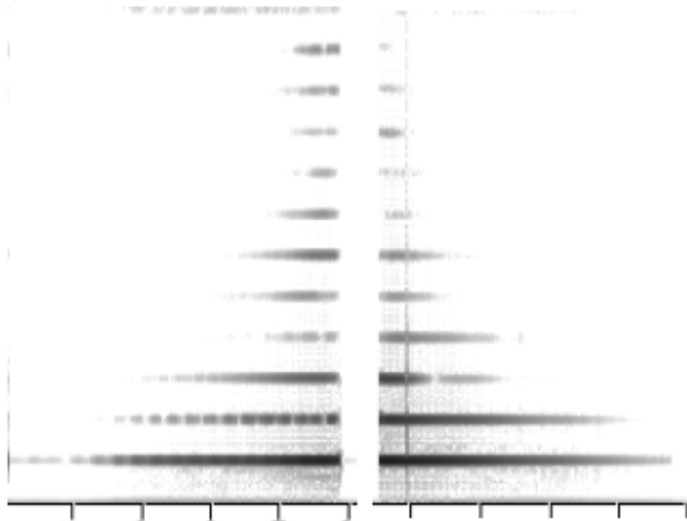


CAN YOU HEAR THE SHAPE OF A DRUM? [KAC '66]

$$\Delta u + \lambda u = 0, \quad u|_{\partial\Omega} = 0$$



CAN YOU HEAR THE LENGTH OF A STRING?



Take home message



TAKE HOME MESSAGE

- Dirac ensembles are examples of random fuzzy geometries with a well defined path integral.
- These are described by random matrix ensembles.
Analyzing them is technically challenging, but doable.
- Interesting things happen even in the simplest toy models.
Mostly related to asymmetric regime of the model.



Spectral triples and Dirac ensembles



[Connes '94, van Suijlekom '14]

- A way to generalize the geometry of smooth manifolds to non-commutative case.
-

$$(\mathcal{A}, \mathcal{H}, D)$$

- algebra $\mathcal{A}$ - functions on a manifold $\mathcal{M}$
- Hilbert space $\mathcal{H}$ - spinors on a manifold $\mathcal{M}$
- self-adjoint operator D - Dirac operator $\not{D}$
- Examples: $\mathcal{A} = C^\infty(\mathcal{M})$, $\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, $\mathcal{H} = \mathbb{C}^2$.



- Points are encoded as pure states of the algebra ϕ_x, ϕ_y . Metric distance between them may be recovered spectrally

$$d(\phi_x, \phi_y) = \sup_{a \in A} \{ |\phi_x(a) - \phi_y(a)| : \|[D, a]\| \leq 1 \}.$$

- Other properties encoded in the heat trace

$$Z(t) = \text{Tr}(e^{-tD^2}) \stackrel{t \rightarrow 0}{\sim} (4\pi t)^{-d/2} (A_0 + A_2 t + \dots) + (4\pi t)^{-(d-1)/2} (b_0 + \dots)$$

where d is dimension and A, B 's depend on the geometry of (the boundary) of $\mathcal{M}$.

- d can be recovered as

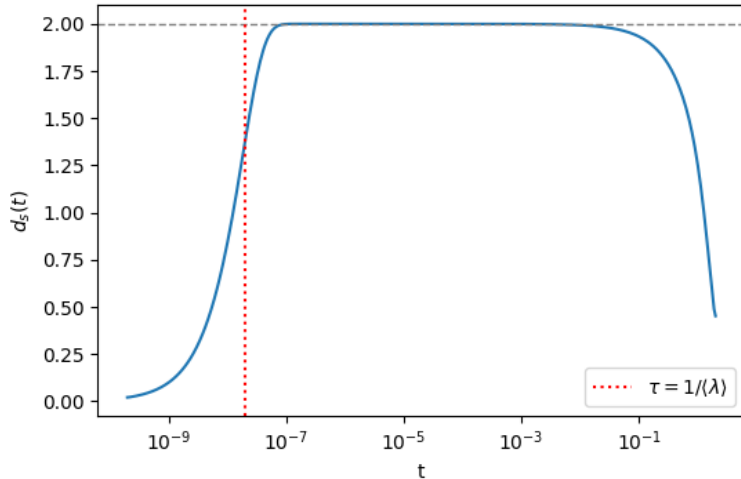
$$d_s(t) = -2 \frac{d \log Z(t)}{d \log t}.$$

other parameters in a similar fashion.



SPECTRAL GEOMETRY

- $d_s(t)$ for the fuzzy sphere



[Barrett '15]

- Finite fuzzy spectral triples of signature (p, q)

$$(M_N(\mathbb{C}), V \otimes M_N(\mathbb{C}), D)$$

and Dirac operator given by

$$D = \sum_{i=1}^p \tau^i \otimes \{H_i, \cdot\} + \sum_{j=1}^q \alpha^j \otimes [L_j, \cdot] \quad , \quad D = \sum_{i=1}^p \tau^i \otimes \{H_i, \cdot\} + \sum_{j=1}^q \alpha^j \otimes [L_j, \cdot]$$

given by a set of $p + q$ matrices.

- The key idea: consider ensemble of geometries with probability distribution

$$P(D) \sim dD e^{-S(D)} .$$

[Barrett, Glaser '16; Glaser '17; Barrett, Druce, Glaser '19; Glaser, Stern '19 '21; Khalkhali, Pagliaroli '21 '22 '25; Pérez-Sánchez '19-'22] [D'Arcangelo '22; D'Arcangelo, Gnutzmann '26]



One matrix $(0, 1)$ and $(1, 0)$ ensembles



$(0, 1)$ AND $(1, 0)$ MULTITRACE MODELS

- The simplest cases include just one matrix. The Dirac operator is given by

$$D_{(1,0)} = \{H, \cdot\} \quad \text{and} \quad D_{(0,1)} = [H, \cdot],$$

the probability distribution $S(D) = t_2 \operatorname{Tr} D^2 + t_4 \operatorname{Tr} D^4$ becomes

$$S(H) = 2t_2(N \operatorname{tr} H^2 \pm (\operatorname{tr} H)^2) + 2t_4(N \operatorname{tr} H^4 \pm 4 \operatorname{tr} H \operatorname{tr} H^3 + 3(\operatorname{tr} H^2)^2) .$$

Simpler $\operatorname{tr} H \operatorname{tr} H^3$ model analyzed before [\[Bukor, JT '25\]](#).

- We have the standard measure on the space of hermitian matrices

$$dD = dH = \prod_{i=1}^N dH_{ii} \prod_{1 \leq i < j \leq N} d(\operatorname{Re}(H_{ij})) d(\operatorname{Im}(H_{ij})).$$



THREE SLIDES ABOUT RANDOM MATRICES

[M.L. Mehta 2004; B. Eynard, T. Kimura, S. Ribault 2015; G. Livan, M. Novaes, P. Vivo 2017]

- Matrix model = ensemble of random matrices, e.g. ensemble of $N \times N$ hermitian matrices with

$$P(M) \sim e^{-N\text{Tr}(V(M))} = e^{-S(M)} , \text{ usually } V(x) = \frac{1}{2}r x^2 + g x^4$$

- Expectation values

$$\langle f \rangle = \frac{1}{Z} \int dM P(M) f(M)$$

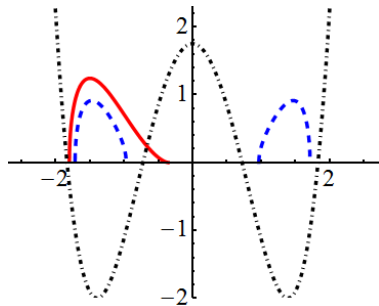
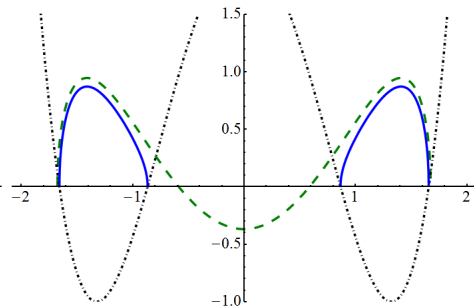
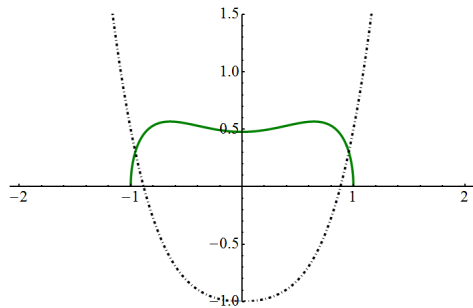
- Saddle point approximation leads to

$$r\lambda_i + 4g\lambda_i^3 = \frac{2}{N} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} .$$

N interacting particles in external potential.

- We introduce eigenvalue distribution $\rho(x)$.





MULTITRACE MATRIX MODELS

[Das, Dhar, Sengupta, Wadia '90; Cicuta, Montaldi '90]

- Varying multitrace action

$$S(M) = f(c_1, c_2, \dots) + \frac{1}{2} r \text{Tr} (M^2) + g \text{Tr} (M^4) \quad , \quad c_n = \frac{1}{N} \sum_i \lambda_i^n$$

leads to saddle point equation

$$\sum_n \frac{\partial f}{\partial c_n} n \lambda_i^{n-1} + r \lambda_i + 4g \lambda_i^3 = \frac{2}{N} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \quad .$$

- Multitrace terms introduce a new kind of interaction among the eigenvalues.



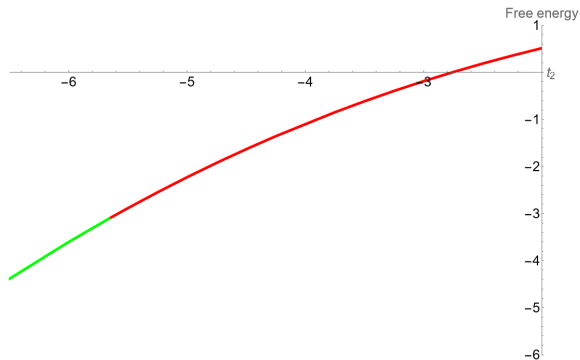
$(0, 1)$ DIRAC ENSEMBLE

[Khalkhali, Pagliaroli '21; D'Arcangelo '22]

- The model

$$S(M) = N(2t_2 c_2 + 2c_4) - 2t_2 c_1^2 - 8c_1 c_3 + 6c_2^2$$

has only symmetric solutions.



[D'Arcangelo, Gnutzmann '26; Bukor, Khalkhali, Kováčik, Kubiš, Magdolenová, Pagliaroli, JT '26]

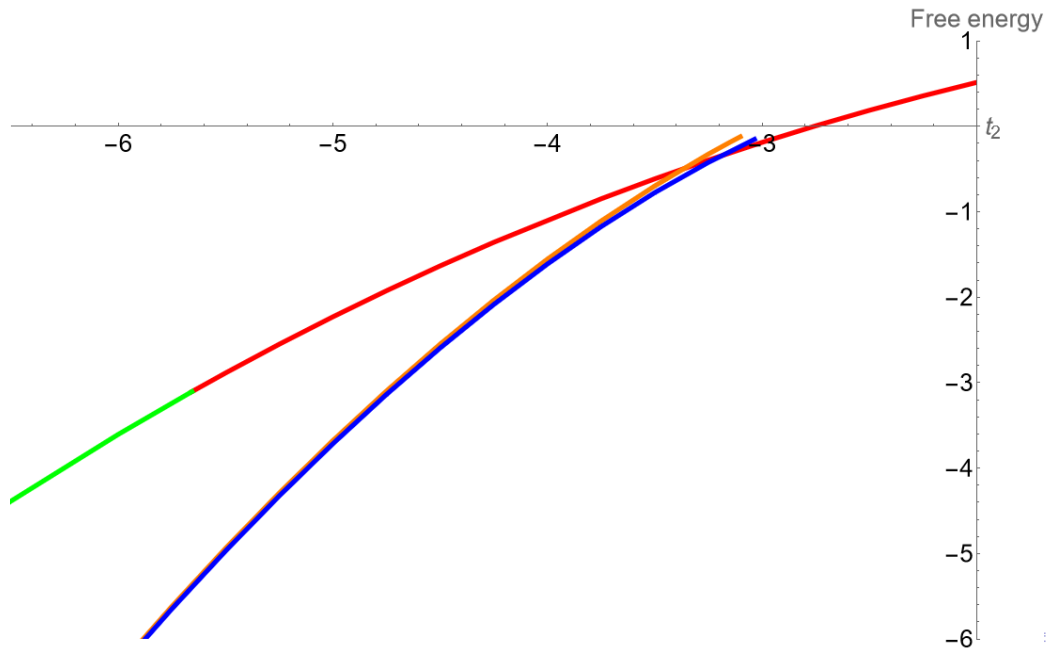
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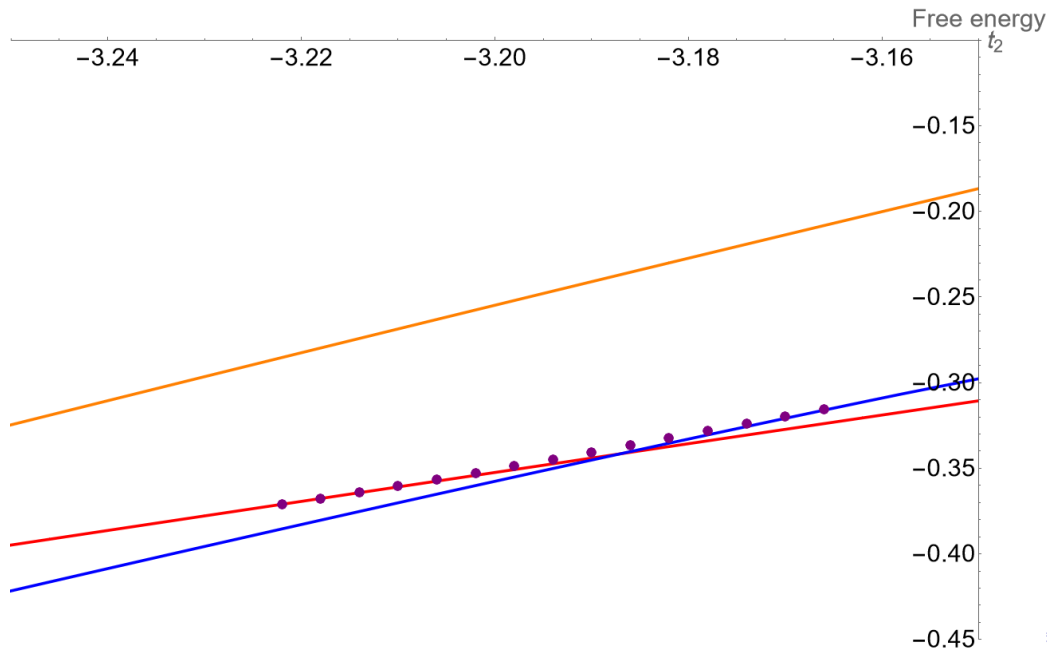
$$S(M) = N(2t_2 c_2 + 2c_4) + 2t_2 c_1^2 + 8c_1 c_3 + 6c_2^2$$

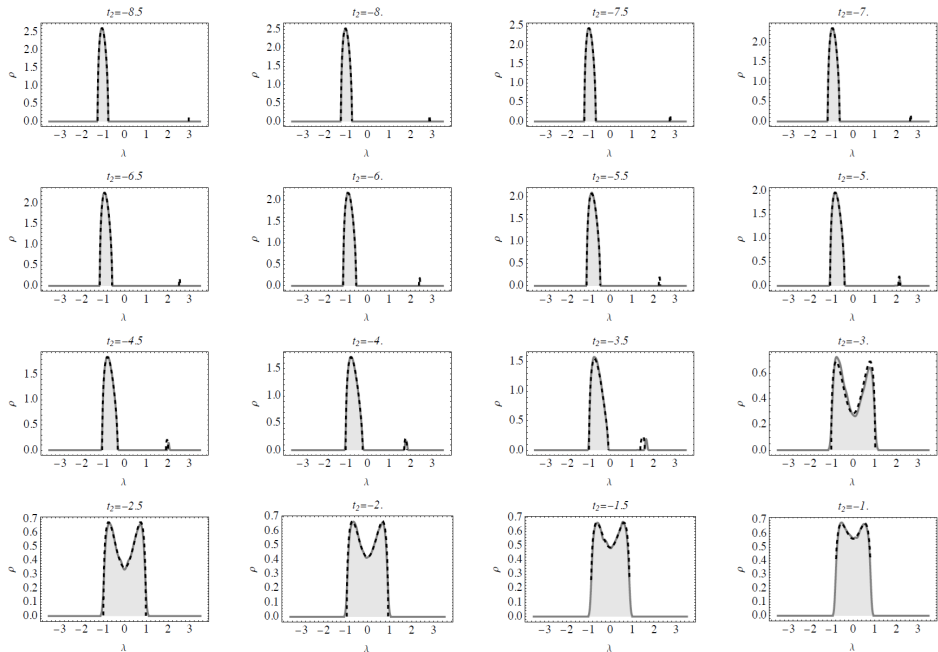
is much richer.

- A variety of asymmetric solutions.
- For $t_2 < -3.18703$ a preferred asymmetric two cut solution.

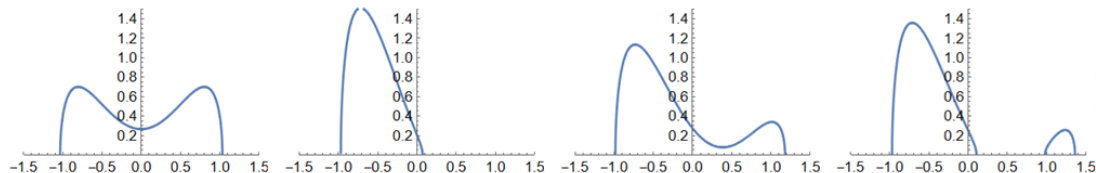




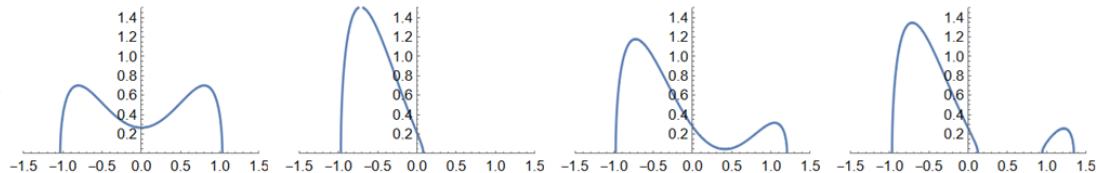




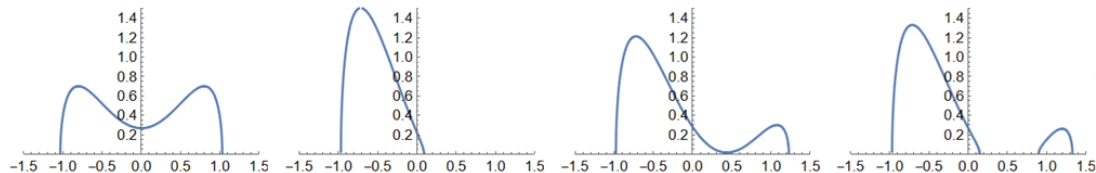
-3.19



-3.18



-3.17



Geometry of the solutions of $(0, 1)$ and $(1, 0)$ ensembles



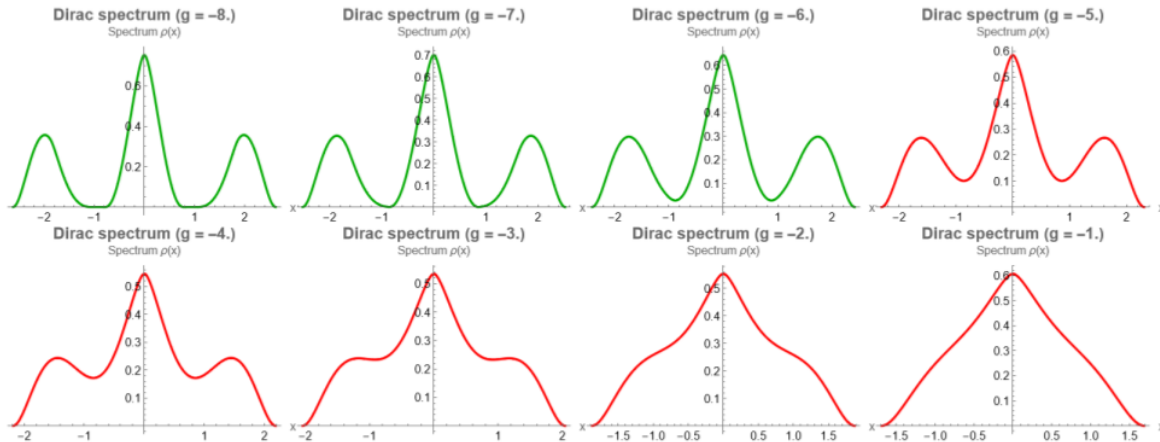
[P. Rusnák, JT work in progress]

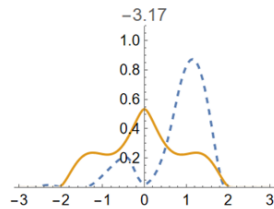
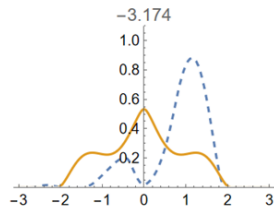
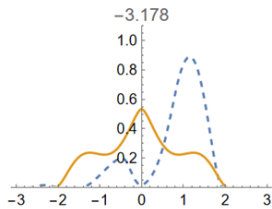
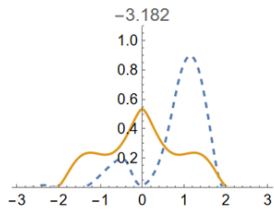
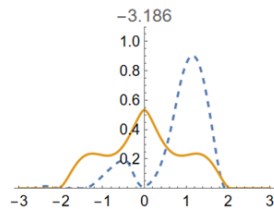
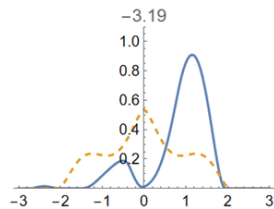
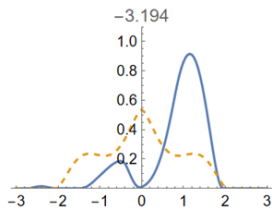
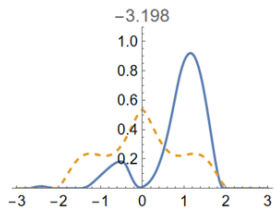
- Spectrum of the Dirac operator in terms of the eigenvalues of H

$$\omega(\lambda) = \int d\mu \rho(\mu) \rho(\lambda \mp \mu) .$$

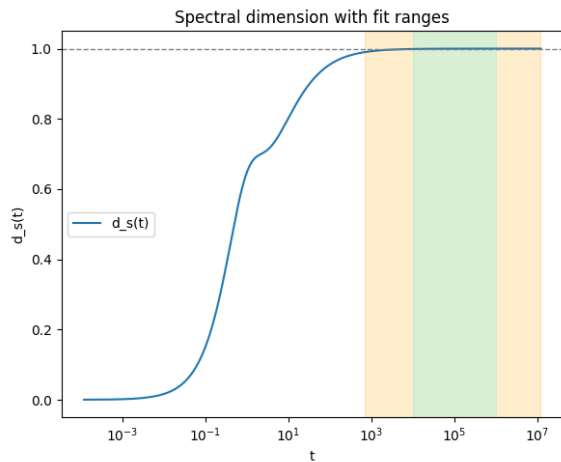
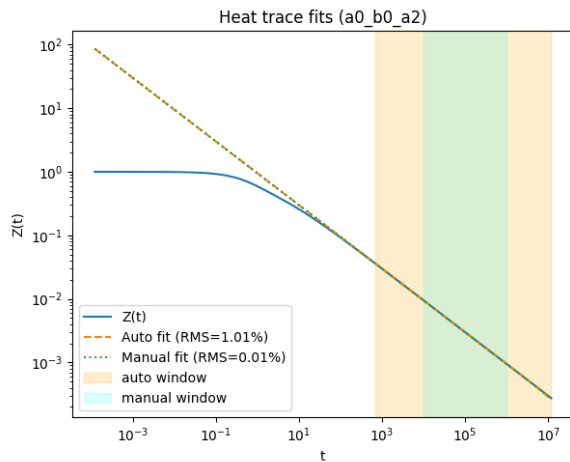
- From this we can evaluate the heat trace and from that the other things.



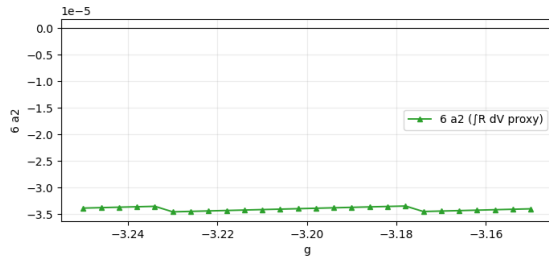
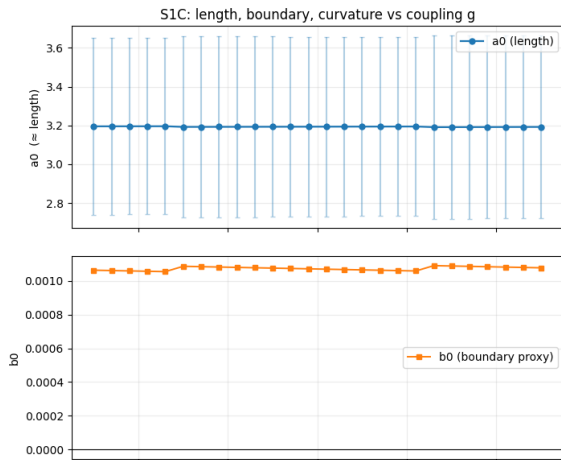




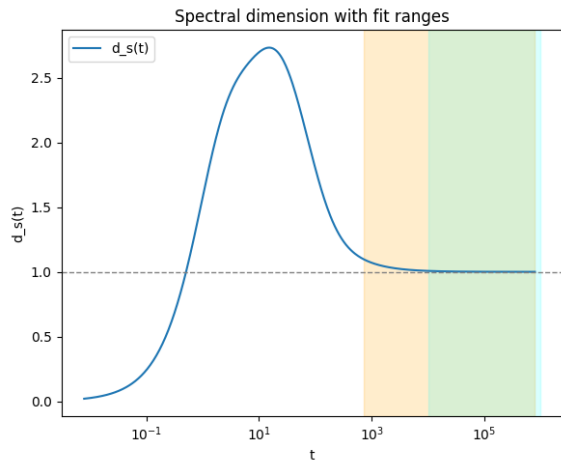
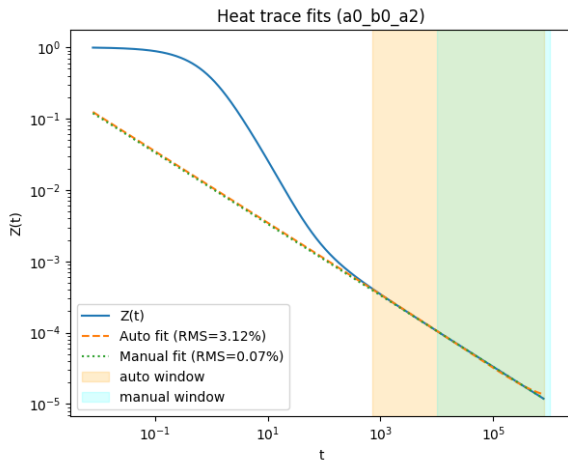
SYMMETRIC ONE CUT GEOMETRY



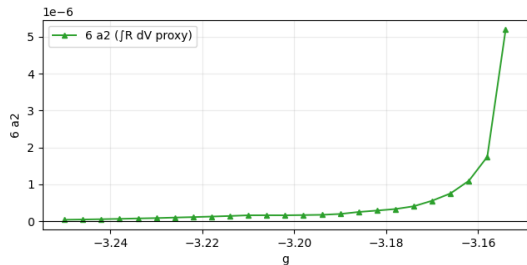
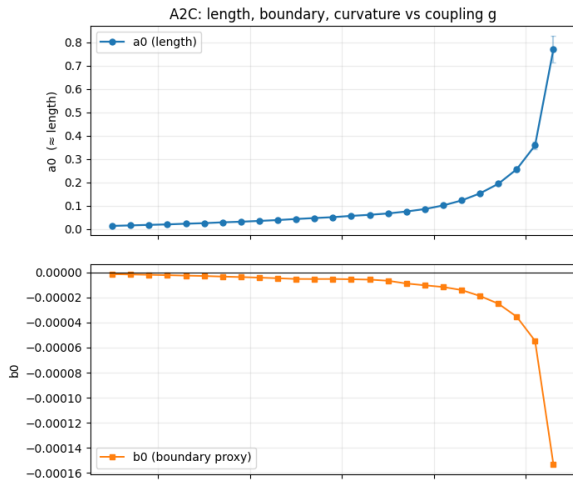
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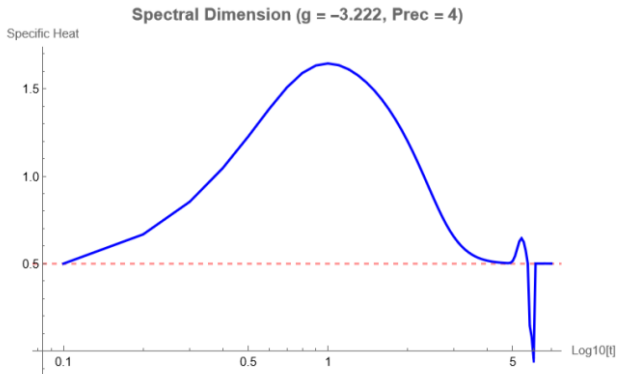
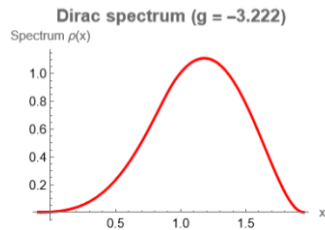
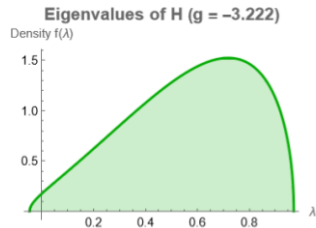
ASYMMETRIC TWO CUT GEOMETRY



ASYMMETRIC TWO CUT GEOMETRY

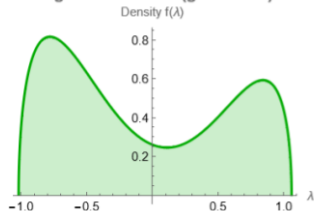


ASYMMETRIC ONE CUT GEOMETRIES

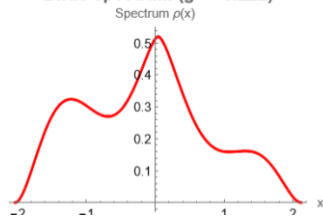


ASYMMETRIC ONE CUT GEOMETRIES

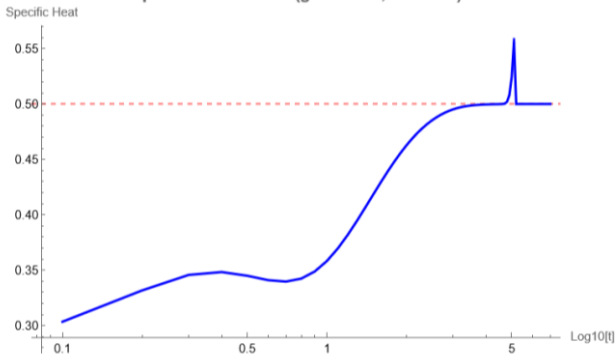
Eigenvalues of H ($g = -3.222$)



Dirac spectrum ($g = -3.222$)

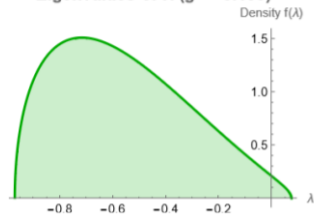


Spectral Dimension ($g = -3.222$, $\text{Prec} = 4$)

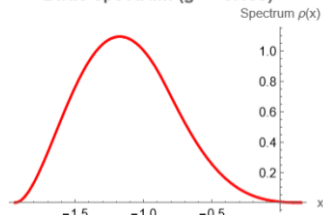


ASYMMETRIC ONE CUT GEOMETRIES

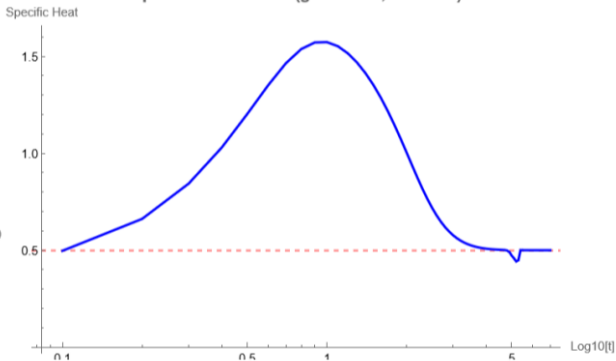
Eigenvalues of H ($g = -3.186$)



Dirac spectrum ($g = -3.186$)

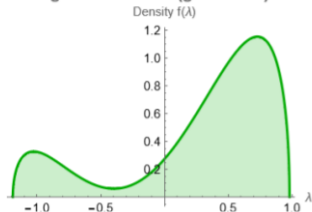


Spectral Dimension ($g = -3.186$, $\text{Prec} = 4$)

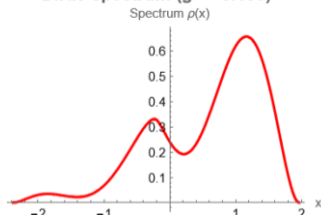


ASYMMETRIC ONE CUT GEOMETRIES

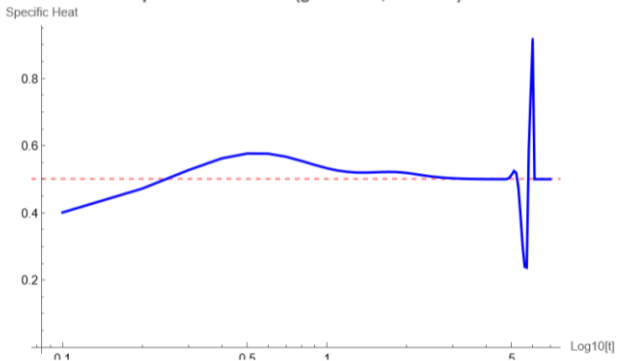
Eigenvalues of H ($g = -3.186$)



Dirac spectrum ($g = -3.186$)



Spectral Dimension ($g = -3.186$, $\text{Prec} = 4$)



ZETA DISTANCE ANALYSIS

- Measure of the geometric "distance" between two spaces by comparing the ratios of their spectral zeta functions. [\[Barrett, Druce, and Glaser '19\]](#)
- Spectral zeta function

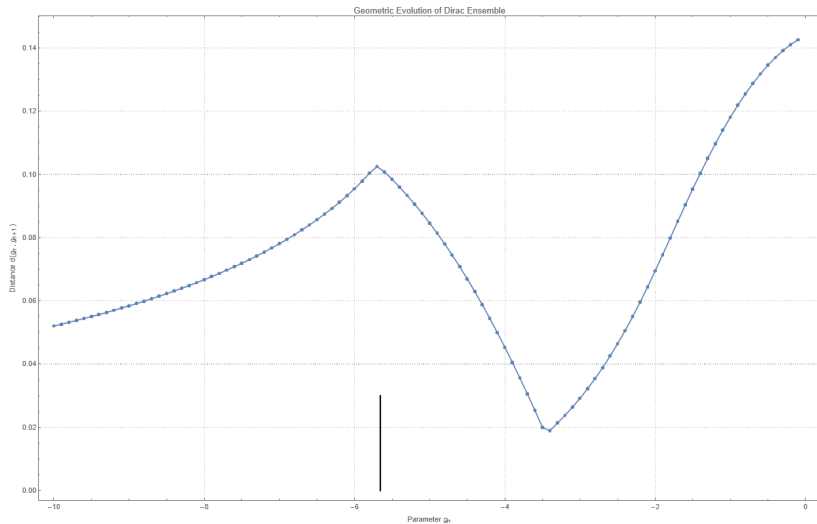
$$\zeta(s) = \sum_{\lambda \neq 0} (\lambda^2)^{-s} = \sum_{\lambda \neq 0} |\lambda|^{-2s}$$

- Distance of two geometries

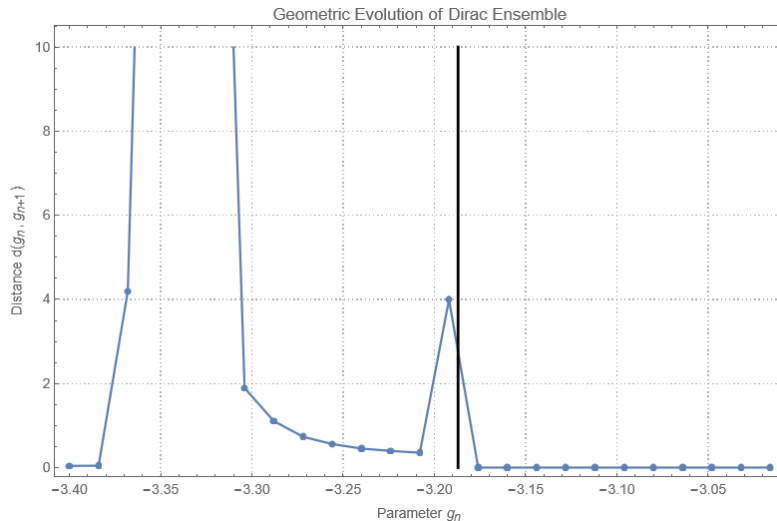
$$\sigma(D_1, D_2) = \sup_{\gamma \leq s \leq \gamma+1} \left| \log \left(\frac{\zeta_1(s)}{\zeta_2(s)} \right) \right|$$



ZETA DISTANCE IN THE $(0, 1)$ MODEL



ZETA DISTANCE IN THE $(1,0)$ MODEL



Future plans - beyond the simple pure geometry



MORE COMPLICATED ONE MATRIX ENSEMBLE

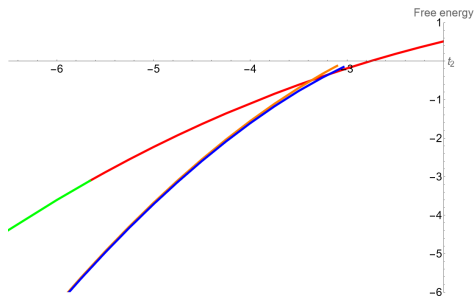
[A. Kubiš, JT work in progress]

- A combination of commutator and anticommutator with just one hermitian matrix H

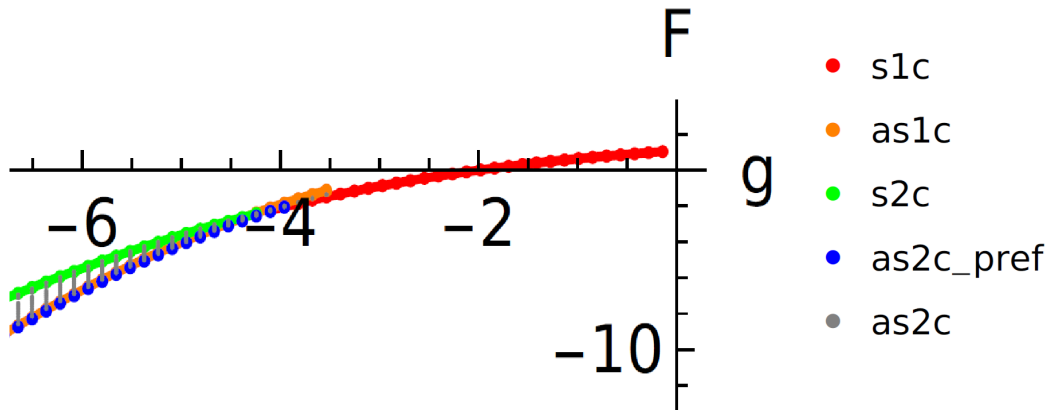
$$D = a\Gamma_1 \otimes [iH, \cdot] + b\Gamma_2 \{H, \cdot\}$$

with Γ 's being the 2×2 spinor representation.

- What kinds of geometries and phase transitions can we get with different a and b ?



$$D = 0.3(\Gamma_1 \otimes [H, \cdot]) + 0.7(\Gamma_2 \otimes \{H, \cdot\}), \quad S = \frac{1}{2} [g \operatorname{Tr}(D^2) + \operatorname{Tr}(D^4)]$$



MORE STRUCTURE ON THE SPACE

- Within this procedure one can naturally include scalar fields with

$$S(\phi) = \text{tr} \left(\frac{1}{2} \phi [D, [D, \phi]] + V(\phi) \right) ,$$

fermions with [\[Barrett '24; Khalkhali, Pagliaroli, Verhoeven '25\]](#)

$$S(\psi) = \langle \psi, D\psi \rangle$$

and gauge fields as fluctuations $D \rightarrow D + A + JAJ^{-1}$. [\[Barrett, Burridge '26\]](#)

- The path integral is then naturally

$$\int dD d\phi d\psi dA e^{-S(D, \phi, \psi)}$$



BEYOND SIMPLE POTENTIAL FOR D

[A. Kubiš, JT work in progress]

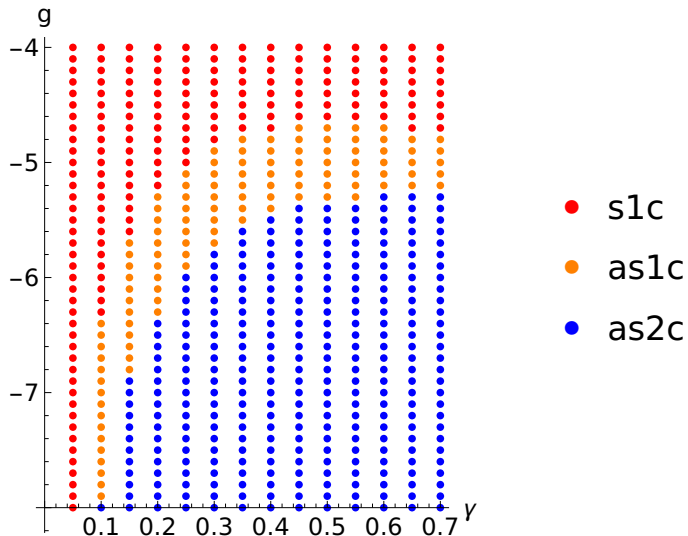
- We can add a multitrace term at the level of $S(D)$

$$S(D) = t_2 \operatorname{Tr} D^2 + t_4 \operatorname{Tr} D^4 + (\operatorname{Tr} D^2)^2 .$$

- This mimics a remnant of interaction of D with some other structure which was integrated out.



$$D = \{H, \cdot\}, \quad S = g \operatorname{Tr}(D^2) + \gamma \operatorname{Tr}(D^4) + \operatorname{Tr}(D^2)^2$$



- Can we get a better idea about the geometry by using some visualization procedure?
[Glaser, Stern '21]
- What about multi-matrix ensembles? Fuzzy sphere in the $(0, 3)$ ensemble.
- Can we get towards more "realistic" theories?



Take home message



TAKE HOME MESSAGE AND 2DO LIST

- Dirac ensembles are examples of random fuzzy geometries with a well defined path integral.
- These are described by random matrix ensembles.
Analyzing them is technically challenging, but doable.
- Interesting things happen even in the simplest toy models.
Mostly related to asymmetric regime of the model.



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- Introduction of matter fields and study of geometry and matter interaction.
- Visualization of the space and further properties of the fluctuating geometry.
- Dirac ensembles with more matrices.



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Thank you for your attention!

