

The Graded Geometry of Einstein-Cartan and Palatini-Cartan Gravity

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1. Motivation

Why Consider Graded Gravity?

Graded geometry can be used to

¹Chatzistavrakidis, Karagiannis, Khoo, Roest, Schupp (2017, 2020)

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→ Study Einstein-Cartan and Palatini-Cartan gravity & BV extensions

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2. Graded Geometry

Graded Manifolds

A *graded manifold* is locally modelled on

$$\mathbb{R}^{k_0} \times \mathbb{R}^{k_1}[1] \times \dots \times \mathbb{R}^{k_N}[N],$$

with smooth coordinates (x^{μ_0}) and graded coordinates $(\xi^{\mu_1}, \dots, \xi^{\mu_N})$ (polynomial generators), satisfying

$$[x^\mu, \xi^\nu] = 0, \quad \xi^\mu \xi^\nu = (-1)^{|\xi^\mu||\xi^\nu|} \xi^\nu \xi^\mu.$$

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Example

For a vector bundle $E \rightarrow M$, the shift $E[k]$ may be regarded as a graded manifold with algebra of functions $\mathcal{O}_{E[k]} = \Gamma(M, S(E^\vee[k]))$.

Graded Geometry: A Dictionary

Tensor Calculus	Graded Geometry
Tensor fields	Functions
Metric $g_{\mu\nu}$	Poisson bracket $\{-, -\}$
Infinitesimal rotations $\lambda_\mu{}^\nu$	Hamiltonian vector fields $\{\lambda, -\}$
Metric connection ∇_μ	Poisson derivation D
Tensor contractions	Differential operators

Table: Tensor calculus vs. graded geometry

3. Graded Einstein-Cartan Gravity

The Graded Manifold

We work on $\mathcal{M} = T[1]M \oplus T[1]M$ with holonomic coordinates

$$(x^\mu, \theta^\nu = dx^\nu, \tilde{\theta}^\kappa = \widetilde{dx^\kappa}).$$

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$$g = g_{\mu\nu} \theta^\mu \tilde{\theta}^\nu \in \mathcal{O}_{\mathcal{M}}^{1,1}.$$

To integrate functions on $\mathcal{M}$, we require a *Berezin measure*

$$\mu_g := |\det(g_{\mu\nu})|^{-1/2} [dx \, d\theta \, d\tilde{\theta}].$$

Extending d to $\mathcal{M}$

On $T[1]M$, $d = \theta^\mu \partial_\mu$ defines a cohomological vector field of degree $+1$.

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$$Dx^\mu = \theta^\mu, \quad D\theta^\mu = 0, \quad D\tilde{\theta}^\mu = \varphi_{\mu\nu}{}^\kappa \theta^\mu \tilde{\theta}^\nu \frac{\partial}{\partial \tilde{\theta}^\kappa}.$$

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Equivariance under a change of $\tilde{\theta}$ coordinates implies

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D is generally not homological:

$$D^2 = \frac{1}{2} R_{\mu\nu\kappa}{}^\lambda \theta^\mu \theta^\nu \tilde{\theta}^\kappa \frac{\partial}{\partial \tilde{\theta}^\lambda},$$

where R corresponds to the curvature R^∇ .

Poisson Structures

We are interested in a degree $(0, -2)$ Poisson structure

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Furthermore, we are interested in d -extensions D which are Poisson derivations, that is,

$$D\{a, b\} = \{Da, b\} + (-1)^{|a|}\{a, Db\}.$$

Graded Einstein-Cartan Gravity

The graded Einstein-Cartan action is

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and the field equations are

$$\frac{1}{(n-3)!} g^{n-3} R_D = \mathcal{T}, \quad \frac{1}{(n-3)!} g^{n-3} Dg = \mathcal{S},$$

where $\mathcal{T}$ and $\mathcal{S}$ are stress-energy and spin of S_m .

4. Graded Palatini-Cartan Gravity

We work on $\mathcal{N} = T[1]M \oplus T^*[1]M$ with holonomic coordinates

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Note that we have a canonical function $e = \theta^\mu \chi_\mu \in \mathcal{O}_{\mathcal{N}}^{1,1}$.

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$\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic as graded manifolds:

$$\Phi^*(x^\mu) = x^\mu, \quad \Phi^*(\theta^\mu) = \theta^\mu, \quad \Phi^*(\chi_\mu) = g_{\mu\nu} \tilde{\theta}^\nu.$$

Graded Geometry on $\mathcal{N}$

Fix a background $h_{\mu\nu}$. As before, we have a Berezin measure

$$\nu_h = |\det(h_{\mu\nu})|^{1/2} [dx d\theta d\chi],$$

a Poisson bracket

$$\{\chi_\mu, \chi_\nu\} = h_{\mu\nu},$$

and a Poisson d -extension D .

Graded Palatini-Cartan Gravity

Fix metric background h (and hence also $\{-, -\}$ and ν_h) and define the graded Palatini-Cartan action

$$S(q, D) = \frac{1}{(n-2)!} \int_{\mathcal{N}} q^{n-2} R_D \nu_h,$$

where $q = q_\mu{}^\nu \theta^\mu \chi_\nu$ is the Vielbein and satisfies $\{q, e\} = 0$.

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This theory is equivalent to graded Einstein-Cartan upon identifying

$$g_{\mu\nu} = q_\mu{}^\kappa h_{\kappa\lambda} q_\nu{}^\lambda.$$

Minimal BV Extension

The theory is invariant under $\text{Diff}(M)$, which is infinitesimally generated by $\mathfrak{X}(M)$ via Lie derivatives. The *anchor map*

$$\rho : \mathcal{O}_{\mathcal{N}}^{0,1} \rightarrow \mathfrak{X}(M)[1], \quad \rho(c^\mu \chi_\mu) = c^\mu [1] \partial_\mu =: Q$$

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Define antifields $q^\dagger, \varphi^\dagger, c^\dagger$, and the minimal BV extended action reads

$$S_{BV} = \int_{\mathcal{N}} \left(\frac{1}{(n-2)!} q^{n-2} R_D \nu_h + \sum_{\gamma \in \{q, \varphi, c\}} \gamma^\dagger Q(\gamma^\dagger) \right).$$

5. Conclusion & Outlook

Summary and Future Work

So far:

- Intrinsic graded geometric structures on curved spaces
- Formalism describes gravity & BV extension concisely
- New interpretation of $(g_{\mu\nu}, \nabla_\mu)$ as Poisson derivation

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Outlook:

- Jet geometry via $T[1](T[1]M)$
- Replace $T[1]M \oplus T^*[1]M$ by exact Courant algebroids

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