

A Symmetric Deformation of the JT Lie–Poisson Sigma Model

Self-induced metric sector, affine Casimir flow and boundary dynamics

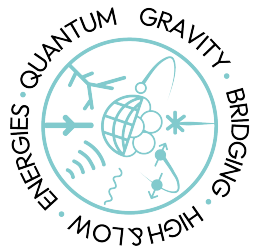
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Corfu2026 - WNGG&PA, September 18, 2026

BridgeQG COST Action: context for this talk



BridgeQG



COST Action CA23130

BridgeQG COST Action CA23130: the big picture

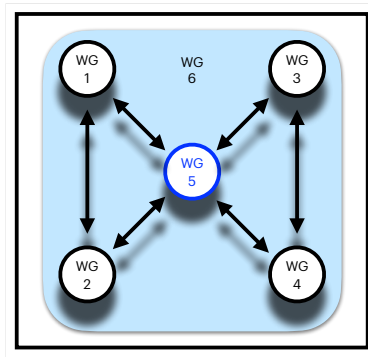
Main aim

BridgeQG investigates the interface between **high-energy quantum gravity** and **quantum aspects of gravity at low energies**, combining theoretical and experimental viewpoints toward a phenomenologically viable description of quantum gravity.

Questions highlighted by BridgeQG

- Is gravity quantised, and what constitutes a quantum signature of gravity? [e.g. entanglement]
- What are the relevant symmetries at the Planck scale?
- Is there a clean separation of scales, or can ultraviolet effects survive at low energies?
- How should observers and reference frames be defined in quantum gravity?
- How does gravity (both classical and quantum) affect the dynamics of quantum systems? [Decoherence, modified Schrodinger equation,]

BridgeQG is organized around the *interplay*



Working-group architecture

WG1 High-energy quantum gravity theory

WG2 High-energy quantum gravity experiment

WG3 Low-energy gravitational effects in quantum systems

WG4 Low-energy high-precision experiment

WG5 Connection between low-energy and high-energy quantum gravity

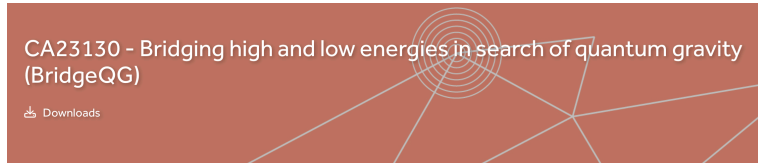
WG6 Dissemination and Diversity

Interpretation of the structure

The high- and low-energy theory/experiment sectors are not treated as isolated programs. **WG5 sits at the interface**, emphasizing the transfer of concepts, constraints and candidate signatures between regimes.

How to join BridgeQG COST Action

- Website: <https://www.cost.eu/actions/CA23130>



[Home](#) > [Browse Actions](#) > Bridging high and low energies in search of quantum gravity (BridgeQG)

Description

Management Committee

Main Contacts and Leadership

Working Groups and Membership



BridgeQG

 **cost**
EUROPEAN COOPERATION
IN SCIENCE & TECHNOLOGY

Where this talk sits inside BridgeQG

Connection to the Action

The emphasis is on an **effective model for quantum-gravity corrections**: a controlled deformation tests what bulk/target-space information can survive in low-energy gravitational and boundary data.

Our concrete laboratory

$$\text{JT} \leftrightarrow SL(2, \mathbb{R}) \text{ PSM/BF}$$

We introduce a symmetric, metric-induced deformation and ask separately what happens to

- local AdS_2 geometry and gauge structure,
- conserved Casimir/global data,
- boundary Schwarzian dynamics,
- topological and entropy sectors.

Motivation and undeformed baseline

PSM baseline and the linear/BF sector

A Poisson sigma model (PSM) is a **2d TQFT**: fields

$$X : \Sigma \rightarrow M, \quad A \in \Omega^1(\Sigma, T^*M)$$

are maps from a worldsheet Σ into a target Poisson manifold (M, Π) with dynamics governed by Poisson geometry,

$$S_{\text{PSM}} = \kappa \int_{\Sigma} \left(A_I \wedge dX^I + \frac{1}{2} \Pi^{IJ}(X) A_I \wedge A_J \right).$$

The equations are

$$dX^I + \Pi^{IJ} A_J = 0, \quad dA_I + \frac{1}{2} \partial_I \Pi^{JK} A_J \wedge A_K = 0.$$

If Π is **linear**, $\Pi^{IJ} = f^{IJ}{}_K X^K$, then $M = \mathfrak{g}^*$ with $f^{IJ}{}_K$ the structure constants of a Lie algebra $\mathfrak{g}$, and

$$D_A X = 0, \quad F(A) = 0,$$

i.e. the linear PSM is a 2d non-abelian BF theory up to total derivative

JT as an $\mathfrak{sl}(2, \mathbb{R})^*$ Lie–Poisson model

The field A becomes a $\mathfrak{g}$ -valued connection 1-form: for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, choosing the basis $\{J, P_a\}$ with

$$[J, P_a] = \epsilon_a{}^b P_b, \quad [P_a, P_b] = \ell^{-2} \epsilon_{ab} J,$$

and

$$A = \omega J + e^a P_a, \quad X = \Phi J^* + X_a P^{a*}.$$

Then PSM describes the **Jackiv-Teitelboim (JT)** 2d topological **gravity** model

Jackiw1985, Teitelboim1983

$$S_{\text{PSM}} \rightarrow S_{JT} = \kappa \int_{\Sigma} \left[X_a T^a + \Phi d\omega + \frac{\Phi}{2\ell^2} \epsilon_{ab} e^a \wedge e^b \right], \quad \kappa = (8\pi G_2)^{-1}.$$

The field equations imply **local AdS_2** geometry

$$F(A) = 0 \quad \Rightarrow \quad T^a = 0, \quad R[g] = -\frac{2}{\ell^2},$$

For $\mathfrak{sl}(2, \mathbb{R})^*$, the Lie–Poisson structure possesses the quadratic **Casimir**

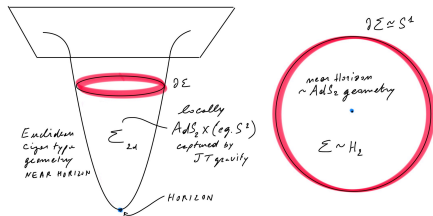
$$C = \frac{1}{2} h_{ab} X^a X^b - \frac{\Phi^2}{2\ell^2}.$$

Since $\Pi^{IJ} \partial_J C = 0$, the PSM equations imply $dC = 0$.

Conservative character key for holographic map JT/SYK (equilibrium)

In the gravitational interpretation this **conserved Casimir** is related to the **conserved black-hole mass parameter** of the JT solution: Ikeda 1994, Schaller & Strobl 1994, Grumiller 2002

- classical solutions are organized by $SL(2, \mathbb{R})$ coadjoint orbits $\subset \mathfrak{sl}(2, \mathbb{R})^*$ labelled by the Casimir



- holographic viewpoint:** correspondingly boundary Schwarzian theory is defined on (exceptional) Virasoro coadjoint orbits $\mathcal{O}_b = \frac{\text{Diff}(S^1)}{\text{PSL}(2, \mathbb{R})} \subset \mathfrak{Vir}^*$

Main point: deform to describe non/near-equilibrium

Question

Can one describe flows among different BH solution regimes via a deformation of the PSM geometry? Breaking conservative description in a controlled way...

Metriplectic inspiration

In classical mechanics a *symmetric* phase space bracket can drive motion relative to the undeformed symplectic foliation while preserving the conservative dynamics induced by the Poisson structure.

[Morrison1984](#), [Morrison1986](#), [Grmela-Ottinger1997](#)

Idea

Deform PSM with a antisymmetric + symmetric target structure reminiscent of metriplectic/GENERIC constructions preserving enough gauge structure to remain solvable and retain gravitational interpretation

Symmetric deformation

$$\Pi^{ij} \rightarrow \Pi^{ij} + \tau G^{ij} \text{ on } M$$

General metric deformation: keep G and S distinct

Start from

$$S_\tau = S_{\text{PSM}} + \frac{\kappa\tau}{2} \int_\Sigma G^{IJ}(X) A_I \wedge *_g A_J \quad \text{with} \quad g_{\mu\nu} = S^{KL}(X) A_{K\mu} A_{L\nu}.$$

Define the mismatch tensor

$$\Theta^{\mu\nu} = H^{\mu\nu} - \frac{1}{2} g^{\mu\nu} Q, \quad \text{with} \quad H_{\mu\nu} = G^{IJ} A_{I\mu} A_{J\nu}, \quad Q = g^{\mu\nu} H_{\mu\nu}.$$

Because $*_g$ is field dependent, varying the deformation gives both direct G terms and metric-induced S terms. In compact form,

$$\begin{aligned} dX^I + \Pi^{IJ} A_J + \tau *_g \mathcal{J}^I &= 0, \quad \text{with} \quad \mathcal{J}_\mu^I = G^{IJ} A_{J\mu} - S^{IJ} \Theta_\mu{}^\nu A_{J\nu}, \\ dA_I + \frac{1}{2} \partial_I \Pi^{JK} A_J \wedge A_K + \tau K_I &= 0, \end{aligned}$$

where K_I contains $\partial_I G$ and $\partial_I S$. Thus:

- even constant tensors generically modify the first equation $\Rightarrow (dC \neq 0)$ Casimir flows
- whereas $K_I = 0$ for target-independent $G, S \Rightarrow (F = 0)$ locally flat AdS_2 geometry preserved

Strobl metric tensor and the JT choice

Let

$$S = S^{IJ} \partial_I \odot \partial_J, \quad g_{\mu\nu} = S^{IJ}(X) A_{I\mu} A_{J\nu}.$$

For a two-dimensional gravitational sector Strobl's conditions require Strobl 2004

$$\text{rank } S^\sharp = 2, \quad {}^E\mathcal{L}_\psi S = 0 \quad \forall \psi \in \Gamma(\ker S^\sharp),$$

with $\det g \neq 0$ on the spacetime field configuration. For JT choose

$$S(J, \cdot) = 0, \quad S(P_a, P_b) = h_{ab},$$

so that

$$\ker S^\sharp \simeq \text{span}\{J\}, \quad g_{\mu\nu} = h_{ab} e_\mu^a e_\nu^b.$$

The compatibility condition is simply Lorentz invariance of h_{ab} :

$$({}^E\mathcal{L}_J S)(P_a, P_b) = -\epsilon_a^c h_{cb} - \epsilon_b^c h_{ac} = 0.$$

Important

The kernel means that the Lorentz connection is invisible to the reconstructed metric; it is essentially local Lorentz invariance; it does *not* reduce the bulk gauge algebra.

Self-induced symmetric deformation

$$\Pi^{ij} \rightarrow \Pi^{ij} + S^{ij} \text{ on } M$$

Self-induced case $G = S$: the deformation becomes volume

Principle of economy: take $G = S$ (aligned case),

$$H_{\mu\nu} = S^{IJ} A_{I\mu} A_{J\nu} = g_{\mu\nu}.$$

In two dimensions,

$$Q = g^{\mu\nu} g_{\mu\nu} = 2, \quad \Theta^{\mu\nu} = 0,$$

so, **off shell**,

$$\boxed{\frac{1}{2} S^{IJ} A_I \wedge *_g A_J = \text{vol}_g} \quad \Rightarrow \quad S_{\text{def}} = \kappa \tau \int_{\Sigma} \text{vol}_g.$$

For the constant JT tensor,

$$\text{vol}_g = \frac{1}{2} \epsilon_{ab} e^a \wedge e^b,$$

and the full first-order action is

$$\boxed{S_{\tau} = \kappa \int_{\Sigma} \left[X_a T^a + \Phi d\omega + \frac{1}{2} \left(\frac{\Phi}{\ell^2} + \tau \right) \epsilon_{ab} e^a \wedge e^b \right].}$$

Crucial $S = \text{const}$: for $S = S(X)$, explicit $\partial_I S$ terms remain in the connection equation.

Field equations: local AdS_2 geometry survives

The deformed equations are

$$\begin{aligned} T^a &= 0, & d\omega + \frac{1}{2\ell^2} \epsilon_{ab} e^a \wedge e^b &= 0, \\ d\Phi + X^a \epsilon_{ab} e^b &= 0, & DX^a + \left(\frac{\Phi}{\ell^2} + \tau \right) \epsilon^a{}_b e^b &= 0. \end{aligned}$$

Hence, in the invertible-zweibein sector,

$$R[g] = -2/\ell^2$$

remains exact.

Hierarchy of restrictions

$$\begin{aligned} G &\neq S : \text{generic mismatch,} \\ G &= S : \text{pure volume,} \\ G &= S = \text{const.} : \text{undeformed connection equation.} \end{aligned}$$

The deformation is therefore **locally mild** but can still act nontrivially on the target-space/global data.

Casimir flow and affine completion

The undeformed JT Casimir starts to flow

Start from

$$C = \frac{1}{2} h_{ab} X^a X^b - \frac{\Phi^2}{2\ell^2}, \quad dC = X_a DX^a - \frac{\Phi}{\ell^2} d\Phi.$$

Using

$$DX^a = -\left(\frac{\Phi}{\ell^2} + \tau\right) \epsilon^a{}_b e^b, \quad d\Phi = -X^a \epsilon_{ab} e^b,$$

we obtain

$$dC = \tau d\Phi.$$

Thus C is **no longer** constant along the deformed solution. Moreover,

$$S^\sharp(dC) = X^a \partial_{X^a}, \quad S(dC, dC) = h_{ab} X^a X^b,$$

so the symmetric sector generically probes **directions transverse** to the undeformed Poisson foliation which are generically non vanishing. Yet, not dissipative.

Exact affine completion: translated dilaton and conserved charge

Set

$$\alpha = \tau \ell^2, \quad \boxed{\tilde{\Phi} = \Phi + \alpha}.$$

Then the target equations become exactly the undeformed JT equations,

$$d\tilde{\Phi} + X^a \epsilon_{ab} e^b = 0, \quad DX^a + \frac{\tilde{\Phi}}{\ell^2} \epsilon^a{}_b e^b = 0.$$

Therefore

$$\boxed{\tilde{C} = \frac{1}{2} h_{ab} X^a X^b - \frac{\tilde{\Phi}^2}{2\ell^2} = C - \tau \Phi - \frac{1}{2} \tau^2 \ell^2, \quad d\tilde{C} = 0.}$$

Equivalently, the deformed Poisson tensor is affine,

$$\Pi_{\tau}^{\Phi a} = \epsilon^a{}_b X^b, \quad \Pi_{\tau}^{ab} = \left(\frac{\Phi}{\ell^2} + \tau \right) \epsilon^{ab},$$

and with the target coordinates shift $Y^{\Phi} = \Phi + \tau \ell^2$, $Y^a = X^a$, one gets

$$\boxed{\Pi_{\tau}(X) = \Pi_0(Y).} \quad \text{with} \quad \Pi_{\tau}^{IJ} = f^{IJ}{}_K X^K + c^{IJ}, \quad c^{IJ} = \alpha f^{IJ}{}_K q^K,$$

so the constant term is a translation coboundary. Poissonity follows from the Lie-algebra Jacobi identity.

Gauge structure: affine target, unchanged $\mathfrak{sl}(2)$ algebra

For a PSM,

$$\delta_\varepsilon X^I = \Pi^{IJ} \varepsilon_J, \quad \delta_\varepsilon A_I = -d\varepsilon_I - \partial_I \Pi^{JK} A_J \varepsilon_K.$$

For the affine tensor, in the linear case

$$\partial_K \Pi_\tau^{IJ} = f^{IJ}{}_K,$$

so the constant affine term drops out of the connection transformation:

$$\boxed{\delta A = -d\varepsilon - [A, \varepsilon]_{\mathfrak{sl}(2)}.$$

In translated variables,

$$F(A) = 0, \quad D_A Y = 0,$$

and Y transforms coadjointly. Since

$$\partial_I \partial_J \Pi_\tau^{KL} = 0,$$

the gauge algebra closes off shell as ordinary $\mathfrak{sl}(2, \mathbb{R})$ BF theory.

Ikeda 1994, Strobl 2004

Precise statement

Gauge invariance follows from the exact affine-PSM rewriting of the *complete* constant aligned theory, not from separate invariance of the isolated metric term.

Boundary completion and Schwarzian

Second-order representative: affine shift becomes Euler completion

The Lorentzian bulk action is

$$S_{\tau, \text{bulk}}^L = \frac{1}{16\pi G_2} \int_{\Sigma} \sqrt{-g} \left[\Phi \left(R + \frac{2}{\ell^2} \right) + 2\tau \right] d^2x.$$

With $\Phi = \tilde{\Phi} - \alpha$ and $\alpha = \tau \ell^2$,

$$\Phi \left(R + \frac{2}{\ell^2} \right) + 2\tau = \tilde{\Phi} \left(R + \frac{2}{\ell^2} \right) - \alpha R.$$

In presence of a boundary, the necessary **GHY term** shifts simultaneously, giving

$$S_{\tau, \text{ren}}^L[\Phi, g] = S_{JT, \text{ren}}^L[\tilde{\Phi}, g] - \frac{\alpha}{16\pi G_2} \mathcal{E}[\Sigma],$$

$$\mathcal{E}[\Sigma] = \int_{\Sigma} \sqrt{|g|} R + 2 \int_{\partial\Sigma} \sqrt{|\gamma|} K = 4\pi\chi(\Sigma). \quad \text{Euler-Gauss-Bonnet completion}$$

Thus the target translation is locally trivial in the affine variables but exposes a topological curvature/boundary term in the gravitational representative.

Grumiller 2002, Mertens & Turiaci 2023

Entropy bookkeeping

Write $\Phi_{\text{full}} = \Phi_0 + \Phi$ and $S_0 = \Phi_0/(4G_2)$.

Wald 1993, Iyer & Wald 1994

The affine Euler term is equivalent to

$$S_0 \longrightarrow S_0^{\text{eff}} = S_0 - \frac{\alpha}{4G_2} = S_0 - \frac{\tau \ell^2}{4G_2}.$$

Meanwhile $\tilde{\Phi}_h = \Phi_h + \alpha$. Hence

$$S_{\text{Wald}} = S_0 + \frac{\Phi_h}{4G_2} = S_0^{\text{eff}} + \frac{\tilde{\Phi}_h}{4G_2}.$$

The volume deformation has no explicit Riemann dependence and therefore no independent Wald term.

Stationary entropy unchanged

The affine shift reorganizes, but does not change, total stationary entropy.

PSM boundary functional and shifted Casimir

For the affine PSM, on shell

$$\delta S_{\text{bulk}}|_{\text{EOM}} = -\kappa \int_{\partial\Sigma} du \langle A_u, \delta Y \rangle.$$

Impose

$$A_u = \lambda(u) Y, \quad \delta\lambda = 0.$$

Since $\delta\tilde{C} = \langle Y, \delta Y \rangle$, add

$$S_{\partial} = \kappa \int_{\partial\Sigma} du \lambda \tilde{C}.$$

Then $\delta(I_{\text{bulk}} + I_{\partial}) = 0$ for the mixed data. On shell, $d\tilde{C} = 0$ and

$$I_{\partial}^{\text{os}} = \kappa \tilde{C} \int du \lambda.$$

This identifies $\tilde{C}$ with the conserved Hamiltonian/global sector, distinct from the Euler/topological sector.

Vassilevich 2013, Valcarcel & Vassilevich 2022

Flatness gives the standard boundary particle

From $A_u = \lambda Y$,

$$Y = \lambda^{-1} A_u, \quad S_{\partial} = \frac{\kappa}{2} \int du \lambda^{-1} \langle A_u, A_u \rangle.$$

The affine translation has disappeared from the *local* boundary functional; it survives in

$$X = \lambda^{-1} A_u - \alpha q$$

and in the global charge map. Since $F(A) = 0$, locally $A = g^{-1} dg$, so

$$S_{\text{part}}[g] = \frac{\kappa}{2} \int du \lambda^{-1}(u) \left\langle g^{-1} \partial_u g, g^{-1} \partial_u g \right\rangle.$$

This is the standard particle on $SL(2, \mathbb{R})$ that underlies the JT boundary reduction.

Grumiller & McNees 2017, Maldacena, Stanford & Yang 2016

Drinfel'd–Sokolov (DS) reduction and the Schwarzian

Use $\mathfrak{sl}(2, \mathbb{R})$ conformal basis $\{L_0, L_{\pm 1}\}$

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0,$$

with $\langle L_1, L_{-1} \rangle = 1$, $\langle L_0, L_0 \rangle = -1/2$. The DS constraint (asymptotically AdS_2 fall-off) fixes

$$a_u = L_1 + \mathcal{L}(u)L_{-1}, \quad \langle a_u, a_u \rangle = 2\mathcal{L}(u).$$

For constant $\lambda = \lambda_b$,

$$S_{\text{red}} = \frac{\kappa}{\lambda_b} \int du \mathcal{L}(u), \quad \mathcal{L}_f = (f')^2 \mathcal{L}_0 - \frac{1}{2} \{f, u\}.$$

For the thermal representative, triviality of boundary holonomy fixes $\mathcal{L}_0 = -\pi^2/\beta^2$,

Maldacena, Stanford & Yang 2016, Stanford & Witten 2017

$$S_{\text{Sch}}[f] = -\frac{\kappa}{2\lambda_b} \int du \left\{ \tan \frac{\pi f(u)}{\beta}, u \right\}.$$

Schwarzian unchanged

Thus the local **Schwarzian functional is unchanged** on the selfinduced affine deformation.

Bulk Casimir $\leftrightarrow$ Virasoro orbit representative

The mixed boundary condition gives

$$\tilde{C} = \frac{1}{2} \langle Y, Y \rangle = \frac{1}{2\lambda_b^2} \langle a_u, a_u \rangle = \frac{1}{\lambda_b^2} \mathcal{L}(u).$$

For a constant orbit representative,

$$\tilde{C} = \frac{1}{\lambda_b^2} \mathcal{L}_0.$$

Hence the **new** conserved finite-dimensional bulk charge fixes the global Virasoro coadjoint sector.

Near the asymptotic boundary, one writes the dilaton perturbation as

$$\Phi = \frac{\Phi_r(u)}{\epsilon} + \Phi_{(0)}(u) + O(\epsilon), \quad \tilde{\Phi} = \Phi + \alpha,$$

so the finite affine shift leaves the **leading source Φ_r unchanged**. Therefore, at fixed asymptotic source,

$$C_{\text{Sch}}^{(\tau)} = C_{\text{Sch}}^{JT}.$$

The deformation changes the global/topological sector, not the local Schwarzian coupling.

Topology, JT/SYK and checks

Topology sum: a locally trivial shift can be quantum-mechanically visible

In Euclidean signature,

$$S_{\tau, \text{ren}}^E = S_{JT, \text{ren}}^E[\tilde{\Phi}, g] + \frac{\tau \ell^2}{4G_2} \chi(\Sigma), \quad S_0^{\text{eff}} = S_0 - \frac{\tau \ell^2}{4G_2}.$$

At fixed topology this is an additive constant. In a topology sum,

Saad, Shenker & Stanford 2019

$$Z_{\tau}(\beta_1, \dots, \beta_b) = \sum_{g \geq 0} e^{\textcolor{red}{S}_0^{\text{eff}} \chi_{g,b}} Z_{g,b}^{\text{dyn}}, \quad \chi_{g,b} = 2 - 2g - b.$$

Adding one handle changes the weight by

$$\frac{\exp[S_0^{\text{eff}} \chi_{g+1,b}]}{\exp[S_0^{\text{eff}} \chi_{g,b}]} = \exp[-2S_0^{\text{eff}}] = \exp(-2S_0) \textcolor{red}{\exp\left(\frac{\tau \ell^2}{2G_2}\right)}.$$

Shift

So τ acts as a shift of the Euler/genus fugacity while leaving the local Schwarzian path integral unchanged.

JT/SYK interpretation: what is established and what is not

For SYK,

Maldacena & Stanford 2016

$$S_{\text{SYK}}(0) = N s_0(q) + O(N^0), \quad C_{\text{Sch}}^{\text{SYK}} = \frac{N \alpha_S(q)}{J} + O(N^0).$$

Our gravitational affine deformation gives instead

$$\delta_\tau S_0^{\text{eff}} \neq 0, \quad \delta_\tau C_{\text{Sch}} = 0.$$

Therefore τ should *not* generically be identified with changing N , q , or J .

Controlled interpretation

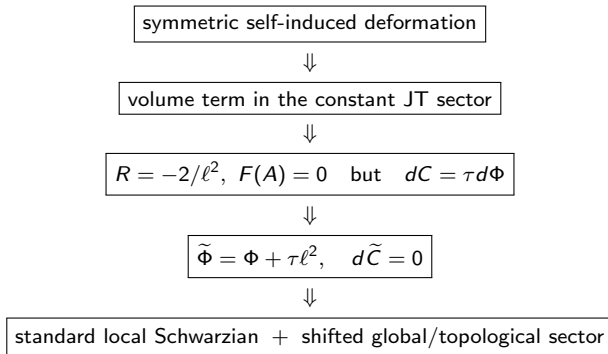
The robust statement is a change of the topological/genus weighting. A microscopic SYK deformation realizing this selective shift is an open problem.

If S_0 is freely adjustable, τ can be absorbed into it; it becomes physically meaningful when S_0 is fixed independently by a microscopic or higher-dimensional completion.

Saad, Shenker & Stanford 2019

Take-home message

The affine model provides the analytically controlled reference point:



Conceptual payoff

The minimal self-induced deformation produces a clean **local/global separation**: local AdS_2 and Schwarzian dynamics survive, whereas the conserved charge and Euler/topology weighting are reorganized.

Beyond the affine reference case

The constant aligned model is deliberately special. For target-dependent G and S ,

$$\partial_I G \neq 0 \quad \text{or} \quad \partial_I S \neq 0,$$

the connection equation receives genuine new terms and the Casimir flow can take the form

$$dC = \tau F_I(X) dX^I.$$

If $F_I dX^I$ is not globally exact, no translated conserved Casimir analogous to $\tilde{C}$ exists.

Holographic question

Can genuine motion across bulk Poisson leaves be mapped to transitions among boundary Virasoro coadjoint sectors?

Thank You

Component form and the mismatch tensor

Set

$$H_{\mu\nu} = G^{IJ} A_{I\mu} A_{J\nu}, \quad Q = g^{\mu\nu} H_{\mu\nu},$$
$$\Theta^{\mu\nu} = H^{\mu\nu} - \frac{1}{2} g^{\mu\nu} Q.$$

Then

$$\delta\left(\sqrt{|g|} g^{\mu\nu} H_{\mu\nu}\right) = \sqrt{|g|} [g^{\mu\nu} \delta H_{\mu\nu} - \Theta^{\mu\nu} \delta g_{\mu\nu}].$$

Θ measures the non-conformal mismatch between the spacetime quadratic forms induced by G and S .

Generic variation: A -equation contribution

Using

$$\delta H_{\mu\nu} = \partial_K G^{IJ} A_{I\mu} A_{J\nu} \delta X^K + 2G^{IJ} A_{J(\nu} \delta A_{I\mu)},$$

$$\delta g_{\mu\nu} = \partial_K S^{IJ} A_{I\mu} A_{J\nu} \delta X^K + 2S^{IJ} A_{J(\nu} \delta A_{I\mu)},$$

one finds the one-form

$$\mathcal{J}_\mu^I = G^{IJ} A_{J\mu} - S^{IJ} \Theta_\mu{}^\nu A_{J\nu}.$$

The first PSM equation becomes

$$\boxed{dX^I + \Pi^{IJ} A_J + \tau *_g \mathcal{J}^I = 0.}$$

Even constant G, S generically modify this equation.

Generic variation: connection equation contribution

The X -variation defines

$$K_I = \frac{1}{2} \left(\partial_I G^{JK} g^{\mu\nu} A_{J\mu} A_{K\nu} - \Theta^{\mu\nu} \partial_I S^{JK} A_{J\mu} A_{K\nu} \right) \text{vol}_g.$$

Thus

$$dA_I + \frac{1}{2} \partial_I \Pi^{JK} A_J \wedge A_K + \tau K_I = 0.$$

If both tensors are target independent,

$$\partial G = \partial S = 0 \quad \Rightarrow \quad K_I = 0,$$

so the connection equation survives. But this alone does *not* imply affine equivalence.

Euclidean cigar: temperature unchanged, conserved charge shifted

Take

$$ds^2 = f(r)d\tau_E^2 + \frac{dr^2}{f(r)}, \quad f(r) = \frac{r^2 - r_h^2}{\ell^2}.$$

Smoothness gives the standard JT relation

$$\beta = \frac{4\pi}{f'(r_h)}, \quad \boxed{T_H = \frac{r_h}{2\pi\ell^2}},$$

because the metric equation is unchanged. With

$$\tilde{\Phi}(r) = \varphi_r \frac{r}{\ell}, \quad \Phi(r) = \varphi_r \frac{r}{\ell} - \tau \ell^2,$$

one has at the horizon

$$\tilde{C} = -\frac{\tilde{\Phi}_h^2}{2\ell^2}, \quad -\tilde{C} = 2\pi^2 \varphi_r^2 T_H^2$$

in the conventions used here, while

$$C(r) = \tilde{C} + \tau \Phi(r) + \frac{1}{2} \tau^2 \ell^2, \quad \frac{dC}{dr} = \tau \frac{d\Phi}{dr}.$$

Thus $\tilde{C}$, not $C(r)$, is the conserved black-hole charge.

Grumiller 2002

Off-shell closure

Generic nonlinear PSM gauge transformations close on A_I only up to terms involving $\partial_I \partial_J \Pi^{KL}$ and the field equations. Here

$$\partial_I \partial_J \Pi_\tau^{KL} = 0,$$

so

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{[\varepsilon_1, \varepsilon_2]_{\mathfrak{sl}(2)}}$$

off shell. Equivalently, in translated variables,

$$F(A) = 0, \quad D_A Y = 0,$$

which are ordinary $SL(2, \mathbb{R})$ BF equations.

Precise statement

Gauge invariance is established by the exact affine-PSM rewriting of the *complete* constant aligned theory, not by claiming that the isolated metric term is separately invariant under the inhomogeneous connection transformation.

Entropy bookkeeping

Write $\Phi_{\text{full}} = \Phi_0 + \Phi$ and $S_0 = \Phi_0/(4G_2)$. The affine Euler term is equivalent to

$$S_0 \longrightarrow S_0^{\text{eff}} = S_0 - \frac{\alpha}{4G_2} = S_0 - \frac{\tau\ell^2}{4G_2}.$$

Meanwhile $\tilde{\Phi}_h = \Phi_h + \alpha$. Hence

$$S_{\text{Wald}} = S_0 + \frac{\Phi_h}{4G_2} = S_0^{\text{eff}} + \frac{\tilde{\Phi}_h}{4G_2}.$$

The volume deformation has no explicit Riemann dependence and therefore no independent Wald term. The affine shift reorganizes, but does not change, total stationary entropy.

PSM boundary functional and shifted Casimir

For the affine PSM, on shell

$$\delta I_{\text{bulk}}|_{\text{EOM}} = -\kappa \int_{\partial\Sigma} du \langle A_u, \delta Y \rangle.$$

Impose

$$A_u = \lambda(u) Y, \quad \delta\lambda = 0.$$

Since $\delta\tilde{C} = \langle Y, \delta Y \rangle$, add

$$I_{\partial} = \kappa \int_{\partial\Sigma} du \lambda \tilde{C}.$$

Then $\delta(I_{\text{bulk}} + I_{\partial}) = 0$ for the mixed data. On shell, $d\tilde{C} = 0$ and

$$I_{\partial}^{\text{os}} = \kappa \tilde{C} \int du \lambda.$$

This identifies $\tilde{C}$ with the conserved Hamiltonian/global sector, distinct from the Euler/topological sector.

Vassilevich 2013, Valcarcel & Vassilevich 2022