

The Microcanonical Ensemble and Negative Heat Capacity

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Based on DOC, S. Ramgoolam: arXiv:2606.09521, arXiv:2312.12397,
arXiv:2312.12398, arXiv:2405.13150, arXiv:2506.18813

Dedicated to the memory of Rafael Sorkin



5 October 1945 – 12 September 2026

A friend, teacher and mentor.

The BFSS Matrix model

BFSS Action

$$S_{BFSS} = \frac{1}{g^2} \int dt \operatorname{Tr} \left\{ \frac{1}{2} (\mathcal{D}_t X^A)^2 + \frac{1}{4} [X^A, X^B]^2 + \frac{1}{2} \Psi^T i D_t \Psi + \frac{1}{2} \Psi^T \Gamma_A [X^A, \Psi] \right\}.$$

The Hamiltonian

$$H_{BFSS} = \operatorname{Tr} \left\{ \frac{\Pi_A \Pi^A}{2} - \frac{1}{4} [X^A, X^B]^2 + \Theta^T \Gamma_A [X^A, \theta] \right\}$$

X^A are $N \times N$ matrices with $A = 1, \dots, 9$ and Θ are Majorana fermions.

M. Baake, P. Reinicke and V. Rittenberg *J. Math. Phys.* **26** (1985) 1070.

R. Flume, *Ann. Phys.* **164** (1985) 189.

B. deWit J. Hoppe and H. Nicolai, *Nucl. Phys.* B305 (1988) 545.

M. Claudson and M. B. Halpern, *Nucl. Phys.* **B250** (1985) 689.

T. Banks, W. Fischler, S.H. Shenker and L. Susskind *Phys. Rev* **D55** (1997) 5112.

The gravity dual and its geometry

Gauge/gravity duality predicts strong coupling is described by the II_A supergravity and in a thermal background has black hole metric:

Dual Geometry

$$ds^2 = -ar^{7/2}F(r)dt^2 + \frac{b}{r^{7/2}}\left(\frac{dr^2}{F(r)} + r^2 d\Omega_8^2\right)$$
$$e^\phi = \frac{c}{r^{7/28}}, \quad C \sim r^7, \quad F(r) = 1 - \frac{r_H^7}{r^7} \quad (1)$$

$z \sim r^{-\frac{5}{2}}$ and $\tau \sim t$ the metric without the black hole becomes

$$\frac{ds^2}{\alpha'} = \left(\frac{5}{2}z\right)^{3/5} \left[\left(\frac{2}{5}\right)^2 \left(\frac{-d\tau^2 + dz^2}{z^2}\right) + d\Omega_8^2 \right]$$

conformal to $AdS^2 \times S^8$.

N. Itzhaki, J. Maldacena, J. Sonnenschein and S. Yankielowicz *Phys.Rev.***D58**(1998) 046004

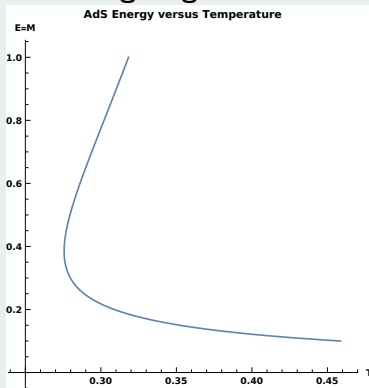
M. Hanada, Y. Hyakutake, G. Ishiki and J. Nishimura, *Science* **344**, (2014) 882 (2014) [arXiv 1311.5607].

A. Biggs and A. Herderschee, arXiv:2503.14685

V. Filev and D.O'C.[arXiv:1506.01366, 1512.02536,1605.01611,1710.02565]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2 \quad f(r) = 1 + \frac{r^2}{L^2} - \left(\frac{r_h}{r}\right)^{D-3}$$

Hawking-Page transition



● Small BH: $C_V < 0$

Large BH: $C_V > 0$

How do the Matrices Encode the Geometry?

- The observables of the matrix model are $SU(N)$ invariants, e.g. $O_{a_1 \dots a_k} = \text{tr}(X_{a_1} \dots X_{a_k})$
- These form the commutative algebra with relations, $C_{9,N}$.
- $\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta} H_{BFSS} + \text{constraint}$
- Q_α a Dirac like operator on the invariant algebra.

The Space of Invariants

Matrix Quantum Mechanics

Consider d hermitian $N \times N$ matrices X_a , $a = 1, \dots, d$, with gauge symmetry

$$X_a \longrightarrow g X_a g^{-1}, \quad g \in U(N).$$

Physical states are those invariant under this gauge action.

The canonical partition function

$$Z = \text{Tr}_{\text{Inv}} \left(e^{-\beta H} \right) = \sum_k Z(N, d, k) x^k; \quad x = e^{-\beta},$$

where $Z(N, d, k)$ is the **microcanonical degeneracy**—the number of gauge-invariant states at energy level k .

- If $H = \hat{N}$ the number operator it does the counting.
- Counting $Z(N, d, k)$ is a problem in **invariant theory**.

Simplified Setting

We work with

- The commutative algebra $C_{d,N}$
- The operator $H = \hat{N}$
- The microcanonical ensemble energy $E = k$.

Identifying and Counting Invariants

Single Hermitian matrix: $U(N)$ adjoint action

One matrix X has N^2 degrees of freedom, but only N invariants: the eigenvalues, or equivalently the traces $\text{Tr}(X^k)$ for $k = 1, \dots, N$.

$$Z[N, 1; x] = \prod_{n=1}^N \frac{1}{1 - x^n} = \sum_{k=0}^{\infty} p_N(k) x^k$$

with $p_N(k)$ the number of partitions of k into no more than N parts.

$$Z(N, 1, k) = p_N(k)$$

F. Dolan, “Counting BPS Operators in $\mathcal{N} = 4$ SYM”
arXiv:0704.1038.

Multi-matrix Invariants

Invariants at degree k

For d matrices $X_1, \dots, X_d$ invariants of degree k are spanned by

$$\text{Tr}(X_{a_1} X_{a_2} \cdots X_{a_k}), \quad a_i \in \{1, \dots, d\}.$$

$Z(N, d, k)$ is the number of linearly independent degree k multi traces.

In $\mathcal{N} = 4$ SYM at weak coupling, $Z(N, 2, k)$ counts **half-BPS operators**. Dual to **graviton states** in $\text{AdS}_5 \times S^5$.

Witten, arXiv:hep-th/9802150;

Aharony, Gubser, Maldacena, Ooguri, Oz arXiv:hep-th/9905111;

Corley, Jevicki, Ramgoolam, arXiv:hep-th/0111222.

The d -Matrix Harmonic Oscillator

Matrix Variables

$$X_{a,j}^i \quad \text{with } a = 1, \dots, d, \quad i, j = 1, \dots, N$$

$N \times N$ hermitian matrices: $X_{a,j}^i = (X_{a,i}^j)^*$

Lagrangian

$$\mathcal{L} = \sum_{a=1}^d \sum_{i,j=1}^N \frac{1}{2} (|\partial_t X_{a,j}^i|^2 - |X_{a,j}^i|^2)$$

Symmetries

- $U(N)$: adjoint action $X_a \rightarrow UX_a U^\dagger$
- $SO(d)$ or $O(d)$: vector action on a -index

Counting Invariant States

States at Energy Level k

Apply k creation operators to vacuum:

$$(A^\dagger)_{a_1 j_1}^{i_1} \cdots (A^\dagger)_{a_k j_k}^{i_k} |0\rangle$$

Gauge Invariant Combinations

- Contract $U(N)$ indices with Kronecker deltas: δ_j^i
- Contract $SO(d)$ indices with deltas and epsilons
- Different contraction patterns = different invariants

Example: $U(N) \times SO(d)$, $k = 2$ invariants

$$A_a^\dagger \rightarrow X_a$$

$$(\text{Tr} X_a)^2, \text{Tr}(X_a^2)$$

Path Integral Derivation of Molien-Weyl

Gauge: Make the group invariance local

$$\mathcal{L} = \text{Tr} \frac{1}{2} (\mathcal{D}_t X)^2 - X^2 \}$$

$$\mathcal{D}_t = \partial_t + \omega - i[A, \cdot]$$

A is a Lagrange multiplier

Implements the Gauss law constraint.

The Hamiltonian is the number operator + constraint.

Euclidean Action

$$S[X] = \int_0^\beta d\tau \operatorname{Tr} \left(\frac{1}{2} (\mathcal{D}_\tau X)^2 + \frac{1}{2} X^2 \right)$$

$$\mathcal{D}_\tau = \partial_\tau + \omega - i[A, \cdot]$$

D.O'C & S. Ramgoolam arXiv:2312.12397

Lattice Discretization

$$\mathcal{D}_\tau X \rightarrow \frac{R_{n,n+1} g_{n,n+1} X_{n+1} g_{n+1,n} - X_n}{a}$$

$g_{n,n+1}$: $U(N)$ link variables,

$R_{n,n+1}$: $SO(d)$ link variables

Partition Function

$$Z_\Lambda = \int \mu_d(R) \mu_N(g) \frac{z_d^{N^2 \Lambda}}{\det[\mathbb{1} - z_d^\Lambda R \otimes g \otimes g^{-1}]}$$

Derivation of Molien-Weyl: Continuum Limit

Taking $\Lambda \rightarrow \infty$

With $z_{\pm} = 1 + \frac{\beta^2}{2\Lambda^2} \pm \sqrt{\dots}$ and $z_+ z_- = 1$:

$$Z_{\infty}(N, d) = \int \mu_d(R) \mu_N(g) \frac{e^{-dN^2\beta/2}}{\det[\mathbb{1} - e^{-\beta} R \otimes g \otimes g^{-1}]}$$

Remove Zero-Point Energy

$$\mathcal{Z}(N, d; x) = \int \mu_d(R) \mu_N(g) \frac{1}{\det[\mathbb{1} - xR \otimes g \otimes g^{-1}]}$$

$$x = e^{-\beta}.$$

Molien-Weyl Formula

Counts polynomial invariants of degree k

Equivalent to Hilbert-Poincaré series

Exact for all N, d, k .

The Molien-Weyl Formula

Generating Function

For $U(N) \times SO(d)$ invariants:

$$\mathcal{Z}(N, d; x) = \int \mu_{SO(d)}(R) \mu_{U(N)}(g) \frac{1}{\det[\mathbb{1} - x R \otimes g \otimes g^{-1}]}$$

Where

- $\mu_{SO(d)}(R)$: Haar measure on orthogonal group
- $\mu_{U(N)}(g)$: Haar measure on unitary group
- $x = e^{-\beta}$
- **det** is over $d \times N^2$ dimensional representation

Expansion

$$\mathcal{Z}(N, d; x) = \sum_{k=0}^{\infty} \mathcal{Z}(N, d, k) x^k$$

Exact Results for Small N

$N = 1$: d -oscillators

$$\mathcal{Z}(1, d; x) = \frac{1}{1 - x^2} \quad \text{for all } d$$

$N = 2$:

$$\mathcal{Z}(2, 2; x) = \frac{1 + x^4}{(1 - x^2)^2(1 - x^4)^2}$$

$$\mathcal{Z}_O(2, s; x) = \frac{1}{(1 - x^2)^2(1 - x^4)^2}$$

Low Energy Stable Counting

Regime: $N \geq k \gg d \sim \mathcal{O}(1)$

Counting becomes independent of N : **stable range**

Large k Asymptotics

$$\mathcal{Z}(N, d, k) \sim \frac{c(d)}{\phi(\frac{1}{d})} \frac{d^k}{k^{d(d-1)/4}}$$

- $\phi(q) = (q; q)_\infty$: Euler function
- $c(d)$: constant from $SO(d)$ group integrals

High Energy Counting

Large N Free Energy

From the path integral form, using $U(N)$ eigenvalue density:

$$-\frac{1}{N^2} \ln \mathcal{Z}(N, d; x) = \beta F = \begin{cases} 0, & xd < 1 \\ \frac{1}{2} - \frac{1}{2s^2} - \frac{1}{2} \ln s^2, & x > \frac{1}{d} \end{cases}$$

where $s^2 = 1 - \sqrt{1 - \frac{1}{xd}}$

Energy

$$E = \frac{k}{N^2} = -x \frac{d(\beta F)}{dx}$$

The Hagedorn transition at $k_{\text{crit}} = N^2/4$

Large- N saddle-point analysis above the Hagedorn transition

For $x > x_H = d^{-1}$, the eigenvalue distribution of $g \in U(N)$ satisfies

$$E = \frac{k}{N^2} = \frac{1}{4} \left(\frac{2}{s^2} - 1 \right) \xrightarrow{s \rightarrow 0} \frac{1}{4} + \sqrt{\frac{d}{2}} \sqrt{x - \frac{1}{d}} + \dots$$

The **transition energy** $E_H = k_{\text{crit}}/N^2 = 1/4$, i.e. $k_{\text{crit}} = N^2/4$.

$F = E - TS$ becomes $-\beta F = S(E) - \beta(E)E$

Entropy across the transition

$$S\left(\frac{k}{N^2}\right) \simeq \begin{cases} \frac{k}{N^2} \ln d - \nu(d) \ln \frac{4k}{N^2} + \dots & k \leq \frac{N^2}{4}, \\ \frac{k}{N^2} \ln d - \nu(d) \ln \frac{4k}{N^2} - \frac{4}{3} \left(\frac{k}{N^2} - \frac{1}{4} \right)^3 + \dots & k \geq \frac{N^2}{4}. \end{cases}$$



Asymptotic Counting

$$\mathcal{Z}(N, d, k)$$

$$\mathcal{Z}(N, d, k) = \begin{cases} \frac{d^k}{\phi(\frac{1}{d})} \frac{c(d)}{k^{d(d-1)/4}} & \text{for } k \leq \frac{N^2}{4}, \\ \frac{d^k}{\phi(\frac{1}{d})} \frac{c(d)}{k^{d(d-1)/4}} e^{-N^2 \frac{4}{3} (\frac{k}{N^2} - \frac{1}{4})^3 + \dots} & \text{for } k \geq \frac{N^2}{4}. \end{cases}$$

Microcanonical Ensemble

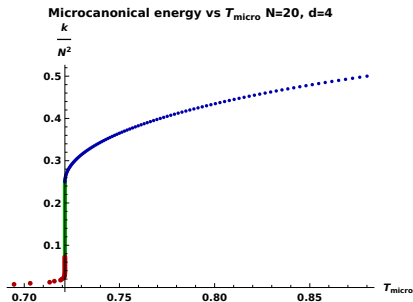
Microcanonical Temperature

$$T_{\text{micro}}^{-1} = \frac{dS}{dE}, \quad S(N, d, k) = \ln \mathcal{Z}(N, d, k)$$

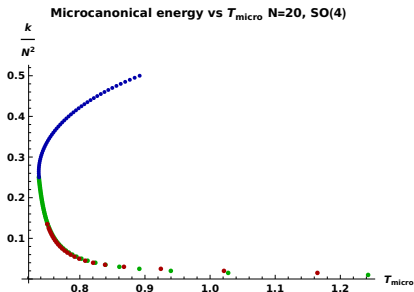
Numerically:

$$T_{\text{micro}}(N, d, k) = \frac{2}{\ln \left(\frac{\mathcal{Z}(N, d, k+1)}{\mathcal{Z}(N, d, k-1)} \right)}$$

A Caloric Fold



Ungauged $d = 4$ model



$SO(4)$ -gauged model

Red: exact finite- N microcanonical data.

Green: asymptotic large N but $k \leq N^2/4$ low energy counting

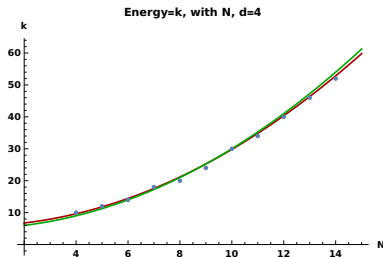
Blue: analytic approximation from the eigenvalue density.

Curves match **surprisingly well** already at $N = 20$, merging at $k = N^2/4$.

Key Observation

- **Low energy:** T decreases as E increases $\Rightarrow$ **negative heat capacity**
- **High energy:** T increases with $E \Rightarrow$ positive heat capacity
- Turning point at $k_{\text{crit}} \sim N^2/4$

The Turning Point k_{crit} : Exact counting



Fits

$$SO(4) : k_{\text{crit}} = 4.5 + \frac{N^2}{4}$$

$$SO(6) : k_{\text{crit}} = 7.2 + \frac{(N+1)^2}{4}$$

Black Holes in AdS vs Matrix Models

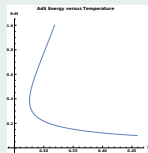
Schwarzschild-AdS

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2$$

$$f(r) = 1 + \frac{r^2}{L^2} - \left(\frac{r_h}{r}\right)^{D-3}$$

- Small BH: $C_V < 0$
- Large BH: $C_V > 0$

Hawking-Page transition



$SO(d)$ Matrix Model

- $T_H = 1/\ln d$
- Low energy: $C_V < 0$
- High energy: $C_V > 0$
- $k_{\text{crit}} = N^2/4$
- Caloric fold at k_{crit}

Conjecture

$O(d)$ invariant sectors capture **microscopic degrees of freedom** of dual AdS black holes.

Summary and Outlook

Main Results

- Derived exact finite N counting formula for counting matrix invariants of $U(N) \times SO(d)$.
- Character-pairing formula: $\mathcal{Z}(N, d, k) = \sum_p \Psi\Phi/\text{Sym}(p)$
- **Negative heat capacity** in microcanonical ensemble
- Caloric fold at $k_{\text{crit}} \sim N^2/4$

Extensions & Broader Context

- Fermionic matrices (vector vs spinor representations)
- Strong coupling SYM: minimum black hole temperature
- Testbed for AdS/CFT thermodynamics
- New connections between representation theory and black hole physics
- Potential for numerical lattice simulations

Ευχαριστώ