

Kodaira–Spencer theory in physics

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Deformation of complex structures

A change in complex structure $J \rightarrow J'$ infinitesimally induces the isomorphisms

$$TM_{(0,1)} \xrightarrow{\sim} TM'_{(0,1)}, \quad TM'_{(0,1)} \xrightarrow{\sim} TM_{(1,0)}$$

so is captured by the *Beltrami differential*

$$\mu \in \Omega^{(0,1)}(M, TM_{(1,0)})$$

which needs to be integrable:

$$\bar{\partial}\mu + \frac{1}{2}[\mu, \mu]_{SN} = 0.$$

So $(\Omega^{(0,*)}(M, \Lambda^* TM_{(1,0)}), \bar{\partial}, [\cdot, \cdot]_{SN})$ is the dgla for the deformation problem of complex structures. [\[Kodaira–Spencer\]](#).

Deformation of Calabi–Yau

If you have a holomorphic volume form $\Upsilon \in \Omega_{hol}^{n,0}(M)$, there is an isomorphism

$$\Upsilon : \Omega^{0,p}(M, \Lambda^k TM_{(1,0)}) \xrightarrow{\sim} \Omega^{n-k,p}(M)$$

so the differential ∂ gets enhanced to a Υ -dependent divergence operator or *BV Laplacian* [Bogomolov–Tian–Todorov]:

$$[\mu, \mu]_{SN} = \operatorname{div} \Upsilon (\mu \wedge \mu) - 2 \operatorname{div} \Upsilon \mu \wedge \mu .$$

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Note: by Grothendieck's definition, a divergence is a second order differential operator on *multivector fields*, hence a Laplacian.

Upcoming work B.-Fiorenza on BV Laplacians :-)

Deformation of Calabi–Yau

We may ask for complex structure deformations that also preserve the holomorphic volume form $\Upsilon \in \Omega_{hol}^{n,0}(M)$

$\rightsquigarrow$ then if $g \in \Omega^{0,0}(M) = C^\infty(M)$ we shall also impose the condition:

$$\bar{\partial}g + \operatorname{div}_\Upsilon \mu + [g, \mu]_{SN} = 0.$$

In topological string theory

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$$\Omega^{0,*}(M, \Lambda^* TM_{(1,0)})[[u]] \equiv PV^{*,*}(M)[[u]], \quad \bar{\partial} + u \operatorname{div}$$

Note:

- div is Connes' operator for cyclic Hochschild cochains.

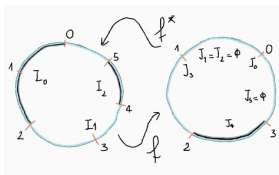


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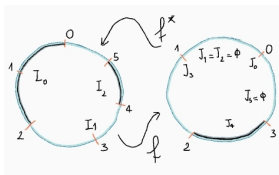


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Note:

- ▶ div is Connes' operator for cyclic Hochschild cochains.
- ▶ $U(1)$ appears because it is homotopy equivalent to $\operatorname{Diff}_0(D)$.
- ▶ If M is compact CY, the dgla with 2-product given by $\wedge$ is formal (= no multiproducts in $\bar{\partial}$ -cohomology)

[Deligne–Griffith–Morgan–Sullivan] [Barannikov–Kontsevich]

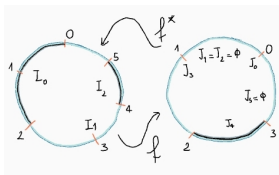


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Plan of the talk

Relation to supergeometry

A BV field theory for the deformation

A primer on BV

Relation to supergeometry

Alternative worldline model - Our work

The dgla can be reclaimed from another underlying model than the model of [Witten].

Locally in coordinates

$$\begin{array}{ccc} M & TM^*_{(0,1)} & TM^*_{(1,0)} \\ z, \bar{z} & z, \bar{z}, \bar{\theta} \cong dz & z, \bar{z}, \bar{\psi} \cong d\bar{z} \end{array}$$

so that $PV^{*,*}(M) \cong C^\infty(M) \otimes S^*(V^\vee) \cong C^\infty(\mathcal{M})$ and a form is a function of the supermanifold, e.g.

$$\alpha(z)_{\bar{k}\bar{m}}{}^i \frac{\partial}{\partial z^i} \wedge d\bar{z}^{\bar{k}} \wedge d\bar{z}^{\bar{m}} = \alpha(z)_{\bar{k}\bar{m}}{}^i \theta_i \bar{\psi}^{\bar{k}} \bar{\psi}^{\bar{m}}$$

Observables and differentials

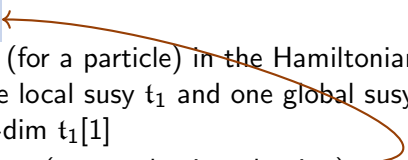
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Observables	$(C^\infty(M) \otimes S^\bullet(V^\vee))[[\gamma]]$	$PV^{*,*}(M)[[\gamma]], \gamma'' = "u$
Diff. operators	$\gamma\theta \cdot p$ $\bar{\psi} \cdot \bar{p}$	$\gamma \operatorname{div} \gamma$ $\bar{\partial}$

Table: The correspondence to the worldline theory.

The equivariant parameter u should rather be the *ghost* γ for the local supersymmetry of the theory.

A BV field theory for the deformation

BV formalism

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That is $\mathcal{O}(\mathrm{dCrit} S) = \mathcal{O}(\Gamma_{\mathrm{d}S}) \otimes_{\mathcal{O}(T^*M)} P$ with S a functional of $PV^{*,*}(M)$ and the differential for the projective resolution P provided by

$$\iota_{\mathrm{d}S} = -\{S, -\} =: Q,$$

with $\{-, -\}$ a 1-shifted Poisson bracket, which is canonically present. [\[Gwilliam\]](#)

Q is a differential on the space $PV^{*,*}(M)$ as long as

$$\{S, S\} = 0 \quad \text{classical master equation}$$

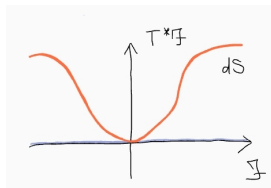


Figure: Critical locus of a functional S of the fields $\mathcal{F}$.

Problems

The Poisson bracket is defined rather on $\ker \operatorname{div}_\Upsilon \subset PV^{*,*}(M)$.

1. Because of the restriction of the fields' space, S is difficult to explicitly write down while preserving *locality*;
2. As a BV theory, Kodaira–Spencer is *degenerate*.

Resolution: theory of potentials

The constraint $\ker \operatorname{div}_\gamma$ can be resolved.

Set M to be a compact CY 3-fold and focus on a subcomplex of $PV^{1,*}(M)[[\gamma]] \oplus PV^{2,*}(M)[[\partial_\gamma]]$ (one among many!).

It can be arranged into a double complex:

$$\begin{array}{ccccccc} PV^{1,0} & \xrightarrow{\bar{\partial}} & PV^{1,1} & \xrightarrow{\bar{\partial}} & PV^{1,2} & \xrightarrow{\bar{\partial}} & PV^{1,3} \\ \delta(\beta)\operatorname{div}\uparrow & & \delta(\beta)\operatorname{div}\uparrow & & \delta(\beta)\operatorname{div}\uparrow & & \delta(\beta)\operatorname{div}\uparrow \\ PV^{2,0}\delta(\gamma) & \xrightarrow{\bar{\partial}} & PV^{2,1}\delta(\gamma) & \xrightarrow{\bar{\partial}} & PV^{2,2}\delta(\gamma) & \xrightarrow{\bar{\partial}} & PV^{2,3}\delta(\gamma) \end{array}$$

Some observations

1. Polyvectors with values in *distributions*: Poincaré dual, crucial for integration;
2. With $\beta \in \mathfrak{t}_1^*[-1]$, $\delta(\beta)$ div is operator for Poincaré duality (*Picture Changing Operator*);

Identities for distributions of Poisson conjugated variables like $\delta(\beta)\delta(\gamma) = 1$ are used throughout.

An action functional

With the pairing on γ -dependent polyvectors

$$(F, G) := \int_{CY \times \mathbb{R}} d\gamma \left((F \wedge G) \lrcorner \Upsilon \right) \wedge \Upsilon$$

there is now an action functional

$$\begin{aligned} S &= (PV^{2,1}\delta(\gamma), \operatorname{div} PV^{2,1}) + 2(PV^{2,1}\delta(\gamma), \bar{\partial} PV^{1,1}) \\ &\quad + (PV^{2,0}\delta(\gamma), \bar{\partial} \operatorname{div} PV^{*2,2}) + (PV^{2,0}\delta(\gamma), \bar{\partial} PV^{*1,2}) \\ &\quad + (PV^{*2,2}\delta(\gamma), \bar{\partial} PV^{1,0}) \\ &= (A, \delta(\beta) \operatorname{div} \circ \bar{\partial} A) + (A, \bar{\partial} A) \end{aligned}$$

(here starred = antifields, i.e. Poisson -or BV- conjugated fields;
and $A := \sum PV^{*,*}$)

Comments

- Instead of using the Hodge decomposition

$$PV^{1,1} = H_{\bar{\partial}} \oplus \text{im div}$$

and write an action functional just for some of the modes
[Barannikov–Kontsevich]

$$S = (\alpha, \bar{\partial} \text{div} \alpha), \quad \text{div} \alpha + \mu_0 = \mu ,$$

we let the decomposition be a consequence of the field equations:

$$\begin{cases} \bar{\partial}(\text{div} PV^{2,1} + PV^{1,1}) = 0 \\ \bar{\partial} PV^{2,1} = 0 \end{cases}$$

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- For a local interaction term, it works to choose:

$$\frac{1}{3!}(A, A \wedge A)$$

Comments

- ▶ At the locus of the field eq. $PV^{1,1} = \text{div}PV^{2,1} + H_{\bar{\partial}}^{1,1}$, we get back Barannikov–Kontsevich's
- ▶ The latter is a BV action functional with respect to a regular, shifted Poisson bracket provided that at the end we project onto div-exact antifields.

Calabi–Yau deformations

We have dismissed CY deformations along the way, to focus on complex structure deformations.

For those, what works is to extend to inverse powers of γ : in string theory, this is the *Large Hilbert Space*

$$\begin{array}{ccccc}
 \gamma^{-2}PV^{3,0} & & \gamma^{-2}PV^{3,1} + \gamma^{-1}PV^{2,0} & & \gamma^{-2}PV^{3,2} + \gamma^{-1}PV^{2,1} + PV^{1,0} \\
 \tilde{A}^{gg} & \longrightarrow & \tilde{A}^g & \longrightarrow & \tilde{A} \\
 \underline{-3} & & \underline{-2} & & \underline{-1} \\
 & & & & \\
 \longrightarrow & \gamma^{-2}PV^{3,3} + \gamma^{-1}PV^{2,2} + PV^{1,1} + \gamma PV^{0,0} & \longrightarrow & \gamma^{-1}PV^{2,3} + PV^{1,2} + \gamma PV^{0,1} \\
 & A & & \tilde{A}^* \\
 & \underline{0} & & \underline{1} \\
 & & & & \\
 & PV^{1,3} + \gamma PV^{0,2} & \longrightarrow & \gamma PV^{0,3} \\
 & \tilde{A}^{*g} & \longrightarrow & \tilde{A}^{*gg} \\
 & \underline{2} & & \underline{3}
 \end{array}$$

- ▶ Manually removing $PV^{3,3}$: codimension 1 coisotropic subspace;
- ▶ For a pairing, we now prescribe that the $d\gamma$ integral is a contour integral that picks up the simple pole at zero;
- ▶ Then we have a reduced action functional

$$\begin{aligned}
 S = & \underbrace{\frac{1}{2}(\tilde{A}, \gamma \bar{\partial} \operatorname{div} \tilde{A}) + \frac{1}{3!}(\gamma \operatorname{div} \tilde{A} - h, (\gamma \operatorname{div} \tilde{A} - h)^{\wedge 2})}_{\text{classical}} \\
 & - \underbrace{(\tilde{A}^*, \bar{\partial} \tilde{A}^g + \operatorname{div} \tilde{A}^g \wedge (\gamma \operatorname{div} \tilde{A} - h))}_{\text{gauge symmetry}} \\
 & - \underbrace{(\tilde{A}^{*g}, \bar{\partial} \tilde{A}^{gg} + \frac{1}{2} \gamma^{-1} (\gamma \operatorname{div} \tilde{A}^g)^{\wedge 2} + \operatorname{div} \tilde{A}^{gg} \wedge (\gamma \operatorname{div} \tilde{A} - h))}_{\text{Lie algebra}} \\
 & - \underbrace{(\tilde{A}^{*gg}, \operatorname{div} \tilde{A}^g \wedge \gamma \operatorname{div} \tilde{A}^{gg})}_{\text{reducibility}}
 \end{aligned}$$

h is element in $H_{\bar{\partial}}$.

Outcomes:

1. The field equations for this theory are Calabi–Yau deformations
2. It is a BV field theory but the naive bracket must be complemented with a projection to the div-exact part of the antifields (still a *degenerate* BV theory)

Conclusions

- ▶ We have discussed a new worldline model for the Kodaira–Spencer dgla
- ▶ We got a BV theory of "potentials" for the complex structure deformations
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Thank you for listening -instead of sunbathing-!

