Double Scaling Limit of the BMN Matrix Model

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Based on Y.A., Goro Ishiki, Shinji Shimasaki [arXiv: 2607.25116]

Introduction

Matrix Models can be
non-perturbative formulation of string/M theory

c=1 matrix model, BFSS model, IKKT model,…

↓

BMN matrix model (a.k.a. plane-wave matrix model)

$$S = \frac{1}{g^2}\int dt\, \mathrm{tr}\left(\frac{1}{2}(D_t X^i)^2 - \frac{1}{4}\left(-\frac{\mu}{3}\varepsilon^{abc}X_c + i[X^a, X^b]\right)^2 + \frac{1}{2}[X^a, X^m]^2 + \frac{1}{4}[X^m, X^n]^2 - \frac{1}{2}\left(\frac{\mu}{6}\right)^2 X_m X^m + \text{fermions}\right)$$

$$\left(\begin{array}{l} i=1,\ldots,8,9, \\ a=1,2,3, \\ m=4,\ldots,8,9 \end{array}\right)$$

… the mass-deformation of the BFSS
w/ maximal supersymmetry [Berenstein, Maldacena, Nastase '02]

The double scaling limit
is an important concept in various matrix models in many contexts,
e.g. emergent geometry from a matrix model

Introduction

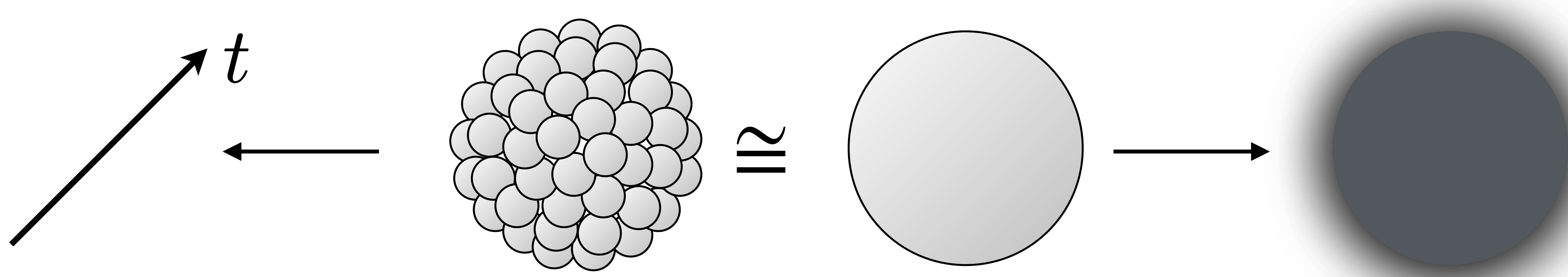
Gauge/gravity duality

[Lin, Lunin, Maldacena '04; Lin, Maldacena '05]

Vacuum
BMN matrix model

$\Leftrightarrow$

Gravity solution
IIA/11D SUGRA



matrix model

blown-up D0s/gravitons

p-branes

gravitational field

$$X^a = \frac{\mu}{3} L_a$$

L_a: $SU(2)$ generators
($a = 1,2,3$)

Many degenerate vacua

As many "plane-wave" solutions as brane configs

Each matrix-model vacuum should describe the corresponding brane config. and thus its gravitational field.

Introduction

The vacua of the BMN model

The BMN model has many degenerate vacua, protected by susy.

[Dasgupta, Sheikh-Jabbari, Raamsdonk '02]

$$X^a = \frac{\mu}{3}\begin{pmatrix} \mathbf{1}_{N_1} \otimes L_a^{[n_1]} & & \\ & \mathbf{1}_{N_2} \otimes L_a^{[n_2]} & \\ & & \ddots \end{pmatrix}$$

$(a = 1,2,3)$

$L_a^{[n_s]}$: $SU(2)$ generator in the n_s dimensional irrep.

N_s: multiplicity of this irrep.

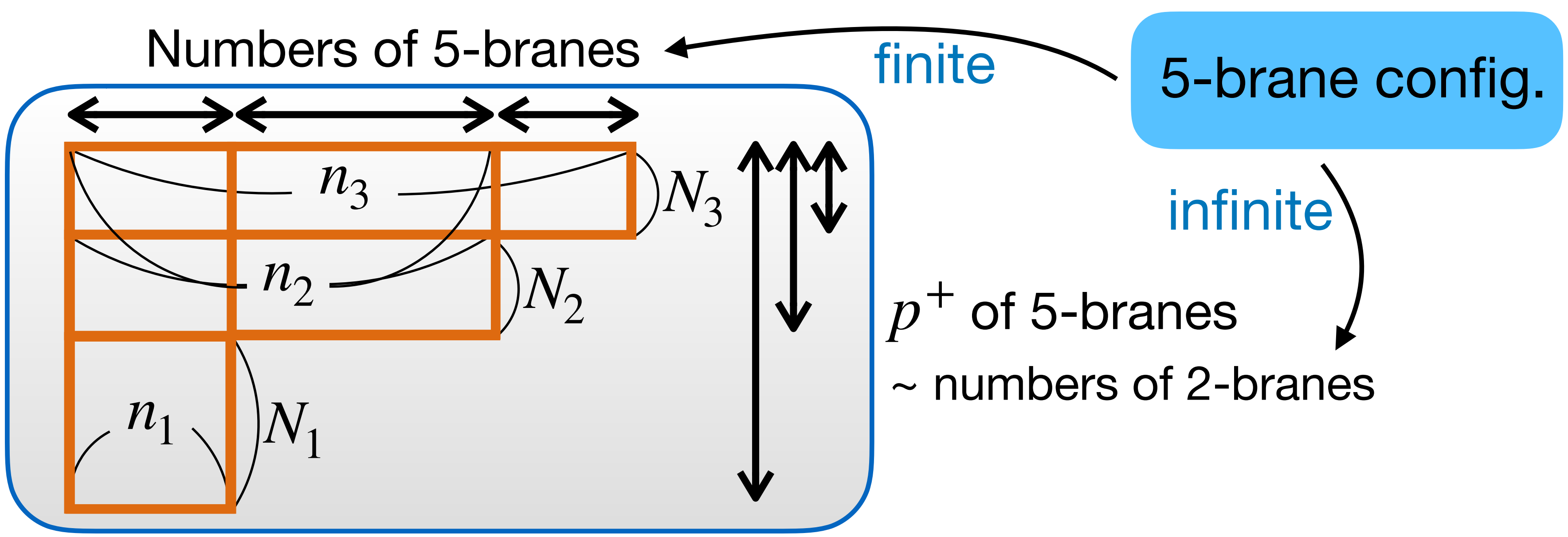
$$X^m = 0$$

$(m = 4,\dots,9)$

The vacuum is parameterised by $\{N_s, n_s\}_{s=1,2,\cdots}$

Partition of N

$$N = \sum_s N_s n_s$$



[Maldacena, Sheikh-Jabbari, Raamsdonk '02]

Introduction

Double scaling limit for 5-branes

BMN vacuum

- On the gravity side, the plane-wave solution (with branes) and the 5-brane solution were constructed.

  [Lin, Lunin, Maldacena '04; Lin, Maldacena '05]

- On the gravity side, a double scaling limit to reproduce the simplest 5-brane solution from a plane-wave solution was identified.

  [Ling, Mohazab, Shieh, Anders, Raamsdonk '06]

- The double scaling limit was generalised to the case for general 5-brane solutions on both sides, and the existence of the double scaling limit in the gauge theory was numerically checked.

  [Y.A., Ishiki, Okada, Shimasaki '14; Y.A., Ishiki, Matsumoto, Shimasaki, Watanabe '22]

The double scaling limit was analytically derived solely on the gauge-theory side without referring to any info on the gravity side.

[Y.A., Ishiki, Shimasaki, '26]

Introduction

Overview of my talk

- Focus on a 1/4-BPS sector of the BMN matrix model

  We consider the BMN model around a vacuum for 5-branes.
  The path integral is reduced to a simpler matrix integral
  by the localisation technique.

- Separate the matrix eigenvalues into two parts
  - The part that reproduces the 5-brane solution
  - The part that acts as a probe and
    describes dynamics on the 5-branes

- Find the limit that keeps the probe part non-trivial

  … the double scaling limit that describes the 5-brane system

Focus on a 1/4-BPS sector

Vacuum for 5-branes

forming 5-branes

$$X^a = 2\begin{pmatrix} \mathbf{1}_{N_1} \otimes L_a^{[n_1]} & \\ & \mathbf{1}_{N_2} \otimes L_a^{[n_2]} \end{pmatrix}$$

probe

$(a = 1,2,3)$

※ $\mu = 6$ for simplicity

※ We take $n_2 < n_1$

A 5-brane limit: n_1: fixed, $N_1 \to \infty$ and $\lambda = g^2 N_1 \to \infty$

How fast do we take λ to infinity?

Quarter-BPS sector (4 supercharges)

[Y.A., Ishiki, Okada, Shimasaki '12]

The BPS sector in which

$$\phi = X^3 + X^8 \sinh\tau - iX^9 \cosh\tau$$ is invariant

$$= \bar{\epsilon}\gamma^\mu\epsilon\, X_\mu \qquad \begin{cases} \gamma^{4567}\epsilon(\tau) = \epsilon(\tau) \\ \gamma^{03}\epsilon(\tau) = \epsilon(-\tau) \end{cases}$$

(after a double Wick rotation $t = -i\tau,\ X^9 = iX^0$)

Focus on a 1/4-BPS sector

Localising locus in the localisation computation [Y.A., Ishiki, Okada, Shimasaki '12]

same as the vacuum "moduli"

$$\phi \to 2\begin{pmatrix} \mathbf{1}_{N_1}\otimes L_3^{[n_1]} & \\ & \mathbf{1}_{N_2}\otimes L_3^{[n_2]} \end{pmatrix} + i\begin{pmatrix} \mathscr{X}\otimes \mathbf{1}_{n_1} & \\ & \mathscr{Y}\otimes \mathbf{1}_{n_2} \end{pmatrix}$$

x_i : eigenvalue of $\mathscr{X}$

y_i : eigenvalue of $\mathscr{Y}$

Partition fn. in the 1/4-BPS sector

$$Z = \int DX\, D\Psi\, e^{-S} = \int [dx][dy]\, e^{-S_{\mathrm{eff}}}$$

Exact result
(up to instanton effect)

$$S_{\mathrm{eff}} = \frac{2}{g^2}\sum_{i=1}^{N_1} n_1 x_i^2 - \sum_{J=0}^{n_1-1}\sum_{i,j\neq i}^{N_1} \ln\left[\frac{\{(2J+2)^2+(x_i-x_j)^2\}\{(2J)^2+(x_i-x_j)^2\}}{\{(2J+1)^2+(x_i-x_j)^2\}^2}\right]^{\frac{1}{2}}$$

$$+\frac{2}{g^2}\sum_{i=1}^{N_2} n_2 y_i^2 - \sum_{J=0}^{n_2-1}\sum_{i,j\neq i}^{N_2} \ln\left[\frac{\{(2J+2)^2+(y_i-y_j)^2\}\{(2J)^2+(y_i-y_j)^2\}}{\{(2J+1)^2+(y_i-y_j)^2\}^2}\right]^{\frac{1}{2}}$$

$$-\sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \ln\left[\frac{\{(2J+2)^2+(x_i-y_j)^2\}\{(2J)^2+(x_i-y_j)^2\}}{\{(2J+1)^2+(x_i-y_j)^2\}^2}\right]$$

Separate the matrix eigenvalues

$$Z = \int [dx][dy]\, e^{-S_{\text{eff}}}$$

$$= \int [dy]\, e^{-s_2(y)} \underbrace{\int [dx]\, e^{-s_1(x)} \prod_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[\frac{\{(2J+2)^2+(x_i-y_j)^2\}\{(2J)^2+(x_i-y_j)^2\}}{\{(2J+1)^2+(x_i-y_j)^2\}^2} \right]}_{Z_1(y)}$$

$$s_s(x) = \frac{2n_s}{g^2} \sum_{i=1}^{N_s} x_i^2 - \sum_{J=0}^{n_s-1} \sum_{i,j\neq i}^{N_s} \ln \left[\frac{\{(2J+2)^2+(x_i-x_j)^2\}\{(2J)^2+(x_i-x_j)^2\}}{\{(2J+1)^2+(x_i-x_j)^2\}^2} \right]^{\frac{1}{2}}$$

For the time being, we focus on $Z_1(y)$ from now on while leaving y_i unintegrated:

$$\langle \mathcal{O}(x) \rangle_1 = \frac{1}{Z_1(y)} \int [dx]\, \mathcal{O}(x)\, e^{-s_1(x)} \prod_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[\frac{\{(2J+2)^2+(x_i-y_j)^2\}\{(2J)^2+(x_i-y_j)^2\}}{\{(2J+1)^2+(x_i-y_j)^2\}^2} \right]$$

Separate the matrix eigenvalues

Eigenval. density: $\rho^{(1)}(x) = \sum_{i=1}^{N_1} \delta(x - x_i)$

Resolvents: $\begin{cases} R^{(1)}(z) = \sum_{i=1}^{N_1} \dfrac{1}{z - x_i} \\ R^{(2)}(z) = \sum_{i=1}^{N_2} \dfrac{1}{z - y_i} \end{cases}$

(i) Loop equation for x_i

$$0 = \int [dx] \sum_{k=1}^{N_1} \frac{\partial}{\partial x_k} \left(\frac{1}{z - x_k} e^{-s_1(x)} \prod_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[\frac{\{(2J+2)^2 + (x_i - y_j)^2\}\{(2J)^2 + (x_i - y_j)^2\}}{\{(2J+1)^2 + (x_i - y_j)^2\}^2} \right] \right)$$

$(\lambda = g^2 N_1)$

$$\begin{aligned} 0 = &\left\langle R^{(1)}(z)^2 \right\rangle + \int dx \frac{1}{z - x} \Bigg\langle \left(R^{(1)}(x + 2i) - 2R^{(1)}(x + i) + \mathrm{c.c.} \right) \rho^{(1)}(x) \\ &\quad + \sum_{J=1}^{n_1 - 1} \left(R^{(1)}(x + (2J+2)i) + R^{(1)}(x + (2J)i) - 2R^{(1)}(x + (2J+1)i) + \mathrm{c.c.} \right) \rho^{(1)}(x) \Bigg\rangle_1 \\ &+ \int dx \frac{1}{z - x} \sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \left(R^{(2)}(x + (2J+2)i) + R^{(2)}(x + (2J)i) - 2R^{(2)}(x + (2J+1)i) + \mathrm{c.c.} \right) \left\langle \rho^{(1)}(x) \right\rangle_1 \\ &- \frac{4 n_1 N_1}{\lambda} \int dx \frac{x}{z - x} \langle \rho^{(1)}(x) \rangle_1 \end{aligned}$$

Separate the matrix eigenvalues

Eigenval. density: $\rho^{(1)}(x) = \sum_{i=1}^{N_1} \delta(x - x_i)$

Resolvents: $\begin{cases} R^{(1)}(z) = \sum_{i=1}^{N_1} \frac{1}{z - x_i} \\ R^{(2)}(z) = \sum_{i=1}^{N_2} \frac{1}{z - y_i} \end{cases}$

(ii) Derivative of free energy

$$\ln Z_1(y) = \ln \int [dx] \sum_{k=1}^{N_1} e^{-s_1(x)} \prod_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[\frac{\{(2J+2)^2 + (x_i - y_j)^2\}\{(2J)^2 + (x_i - y_j)^2\}}{\{(2J+1)^2 + (x_i - y_j)^2\}^2} \right]$$

$$\boxed{\frac{\partial}{\partial y_j} \ln Z_1(y) = \left\langle \sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \Big(R^{(1)}(y_j + (2J+2)i) + R^{(1)}(y_j + (2J)i) - 2R^{(1)}(y_j + (2J+1)i) + \mathrm{c.c.} \Big) \right\rangle_1}$$

A similar structure appears in the loop eq.

Find the double scaling limit

1/N$_1$ expansion

$$\ln Z_1(y) = \sum_{g=0}^{\infty}\sum_{h=0}^{\infty} N_1^{2-2g-h} N_2^h f_{g,h}(y) \qquad \langle R^{(1)}(z)\rangle_1 = \sum_{g=0}^{\infty}\sum_{h=0}^{\infty} N_1^{1-2g-h} N_2^h\, r_{g,h}^{(1)}(y)$$

Loop eq. to leading order

$$0 = \sum_{J=0}^{n_1-1}\Big(r_{00}^{(1)}(x+(2J+2)i) + r_{00}^{(1)}(x+(2J)i) - 2r_{00}^{(1)}(x+(2J+1)i) + \mathrm{c.c.}\Big) - \frac{4n_1}{\lambda}x$$

when x is inside the support $[-x_m, x_m]$ of $\rho^{(1)}(x)$

One can explicitly solve this equation for large x_m/n_1.

$$r_{00}^{(1)}(z) = \frac{8}{x_m^4}\left(-\frac{1}{3}z^3 + \frac{x_m^2}{2}z + \frac{1}{3}(z^2 - x_m^2)^{\frac{3}{2}} + O\Big(\frac{n_1}{x_m}\Big)\right)$$

$$x_m = (8\lambda)^{1/4} + O(\lambda^0)$$

[Y.A., Ishiki, Shimasaki, '26]

Find the double scaling limit

Derivative of free energy to non-trivial leading order

Free energy $$\ln Z_1(y) = N_1^2\left(\underbrace{f_{00}}_{\text{constant in } y} + \frac{N_2}{N_1}\underbrace{f_{01}(y)}_{\text{non-trivial leading order}} + \frac{N_2^2}{N_1^2}f_{02}(y) + \cdots\right) + \cdots$$

The relation of the free energy gives

$$\frac{\partial}{\partial y_j}f_{01}(y) = \frac{1}{N_2}\sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1}\left(r_{00}^{(1)}(y_j+(2J+2)i) + r_{00}^{(1)}(y_j+(2J)i) - 2r_{00}^{(1)}(y_j+(2J+1)i) + \mathrm{c.c.}\right)$$

Plugging the solution $r_{00}^{(1)}$ to it, we finally obtain

$$f_{01}(y) = \frac{2n_2}{N_2\lambda}\sum_{i=1}^{N_2}y_i^2 - \frac{n_1 A(n_1)e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi N_2\lambda^{5/8}}\sin\frac{\pi n_2}{n_1}\sum_{i=1}^{N_2}\cosh\frac{\pi y_i}{n_1} + O\left(e^{-\frac{2\pi}{n_1}(8\lambda)^{1/4}}\right)$$

Find the double scaling limit

Loop eq. to next-to-leading order

$$0 = \sum_{J=0}^{n_1-1}\left(r_{01}^{(1)}(x+(2J+2)i) + r_{01}^{(1)}(x+(2J)i) - 2r_{01}^{(1)}(x+(2J+1)i) + \mathrm{c.c.}\right)$$
$$+\frac{1}{N_2}\sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1}\left(R^{(2)}(x+(2J+2)i) + R^{(2)}(x+(2J)i) - 2R^{(2)}(x+(2J+1)i) + \mathrm{c.c.}\right)$$

when x is inside the support $[-x_m, x_m]$ of $\rho^{(1)}(x)$

Derivative of free energy to non-trivial next-to-leading order

Using the relation of the free energy, we obtain

$$f_{02}(y) = \frac{1}{2N_2^2}\sum_{i,j\neq i}\sum_{k=1}^{\infty}\left(\sum_{J=kn_1}^{kn_1+n_2-1} - \sum_{J=kn_1-n_2}^{kn_1-1}\right)\ln\frac{\left\{(2J+2)^2+(y_i-y_j)^2\right\}\left\{(2J)^2+(y_i-y_j)^2\right\}}{\left\{(2J+1)^2+(y_i-y_j)^2\right\}^2}$$
$$+O\left(e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}\right)$$

We now have obtained the free energy as

$$\ln Z_1(y) = N_1^2\left(f_{00} + \frac{N_2}{N_1}f_{01}(y) + \frac{N_2^2}{N_1^2}f_{02}(y) + \cdots\right) + \cdots$$

Find the double scaling limit

Let's come back to the probe part y_i

$$Z = \int [dy]\, e^{-s_2(y)} Z_1(y)$$

$$s_2(y) = \cancel{\frac{2n_2}{g^2}\sum_{i=1}^{N_2} y_i^2} - \underbrace{\frac{1}{2}\sum_{J=0}^{n_2-1}\sum_{i,j\neq i}^{N_2} \ln \frac{\{(2J+2)^2+(y_i-y_j)^2\}\{(2J)^2+(y_i-y_j)^2\}}{\{(2J+1)^2+(y_i-y_j)^2\}^2}}$$

cancel

combined

$$-\ln Z_1(y) = -N_1^2 f_{00} - \cancel{\frac{2n_2}{g^2}\sum_{i=1}^{N_2} y_i^2} + \boxed{\frac{n_1 N_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi \lambda^{5/8}} \sin\frac{\pi n_2}{n_1} \sum_{i=1}^{N_2} \cosh\frac{\pi y_i}{n_1}}$$

$$\underbrace{-\frac{1}{2}\sum_{k=1}^{\infty}\left(\sum_{J=kn_1}^{kn_1+n_2-1} - \sum_{J=kn_1-n_2}^{kn_1-1}\right)\sum_{i,j\neq i}^{N_2} \ln \frac{\{(2J+2)^2+(y_i-y_j)^2\}\{(2J)^2+(y_i-y_j)^2\}}{\{(2J+1)^2+(y_i-y_j)^2\}^2}} + \cdots$$

cosh potential term

Find the double scaling limit

The large-N_2 saddle point eq. for y_i

$$\rho_0^{(2)}(y) = N_2 \lim_{N_2\to\infty} \sum_{i=1}^{N_2} \frac{1}{N_2} \langle \delta(y - y_i) \rangle$$

$$0 = \underset{\text{chem. pot.}}{\text{const.}} + \boxed{\frac{n_1 N_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi^2 \lambda^{5/8}}} \sin\frac{\pi n_2}{n_1} \sum_{i=1}^{N_2} \cosh\frac{\pi y_i}{n_1} + \rho_0^{(2)}(y)$$

$$- \sum_{k=-\infty}^{\infty} \frac{1}{\pi} \int_{-y_m}^{y_m} dy' \left(\frac{|2kn_1 + 2n_2|}{(y-y')^2 + |2kn_1 + 2n_2|^2} - \frac{|2kn_1|}{(y-y')^2 + |2kn_1|^2} \right) \rho_0^{(2)}(y') + \underset{\to 0}{\underline{\cdots}}$$

$$\frac{4}{\pi^2 g_0}$$

The double scaling limit

[Y.A., Ishiki, Shimasaki, '26]

n_1: fixed, $N_1 \to \infty$ and $\lambda = g^2 N_1 \to \infty$

with $\boxed{g_0 = \frac{4\lambda^{5/8}}{n_1 N_1 A(n_1)} e^{\frac{\pi}{n_1}(8\lambda)^{1/4}} \text{ fixed}}$

This saddle point eq. is equivalent to the equation that determines the NS5-brane solution in the gravity dual.

[Lin, Maldacena '05; Ling, Mohazab, Shieh, Anders, Raamsdonk '06; Y.A., Ishiki, Matsumoto, Shimasaki, Watanabe '22]

Summary and discussion

- We have derived the double scaling limit by keeping the probe part finite and non-trivial solely on the gauge theory side.
  The saddle pt. eq. and the double scaling limit are exactly the ones obtained on the gravity side.

- It is straightforward to generalise the discussion of this double scaling limit to general 5-brane solutions.
  [Y.A., Ishiki, Shimasaki, '26]

- The obtained double-scaled matrix integral is considered to be the 1/4-BPS sector of type IIA Little String Theory.
  g_0 should correspond to the effective string coupling.
  [Lin, Maldacena '05; Y.A., Ishiki, Matsumoto, Shimasaki, Watanabe '22]

- The procedure should be applicable to other matrix models.