

T-Poincaré groups at the locally compact level

Corfu - Noncommutative and Generalised Geometries and their
Physical Applications

Arthur Massar

Université Catholique de Louvain

Phd student of P. Bieliavsky

September 19, 2026

Non-commutative spacetime

Impose non-trivial commutation relations on spacetime coordinates

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}(x).$$

→ Not compatible with Poincaré action in general! Need to quantize the Poincaré group $\mathcal{P} = \text{SO}(1, 3) \ltimes \mathbb{R}^4$.

Non-commutative spacetime

Impose non-trivial commutation relations on spacetime coordinates

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}(x).$$

→ Not compatible with Poincaré action in general! Need to quantize the Poincaré group $\mathcal{P} = \text{SO}(1, 3) \ltimes \mathbb{R}^4$.

Lie algebraic spacetime: M^* is a Lie algebra.

$$[\hat{x}^\mu, \hat{x}^\nu] = c_\rho^{\mu\nu} \hat{x}^\rho.$$

Corresponding Lie group should play a role!

Non-commutative spacetime

Impose non-trivial commutation relations on spacetime coordinates

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}(x).$$

→ Not compatible with Poincaré action in general! Need to quantize the Poincaré group $\mathcal{P} = \text{SO}(1, 3) \ltimes \mathbb{R}^4$.

Lie algebraic spacetime: M^* is a Lie algebra.

$$[\hat{x}^\mu, \hat{x}^\nu] = c_\rho^{\mu\nu} \hat{x}^\rho.$$

Corresponding Lie group should play a role!

Most constructions are **algebraic**: deformations of $\mathfrak{p} := \text{Lie}(\mathcal{P})$.
Here: functional analysis and deformations of $C_c^\infty(\mathcal{P})$ with **non-formal** parameter.

κ -Poincaré and T-Poincaré with $\theta = 0$ [Mer24]

κ -Poincaré [LNR92]	T-Poincaré ($\theta = 0$, 12 families)
$r = \frac{1}{\kappa} P_j \wedge M_{j0}$	$r = \frac{-1}{2} \omega^{\mu\rho\sigma} P_\mu \wedge M_{\rho\sigma}$
r not triangular	r triangular
$[\hat{x}^0, \hat{x}^j] = \frac{i}{\kappa} \hat{x}^j$	$[\hat{x}^\mu, \hat{x}^\nu] = i\omega^{[\mu\nu]}_\rho \hat{x}^\rho$

κ -Poincaré and T-Poincaré with $\theta = 0$ [Mer24]

κ -Poincaré [LNR92]	T-Poincaré ($\theta = 0$, 12 families)
$r = \frac{1}{\kappa} P_j \wedge M_{j0}$	$r = \frac{-1}{2} \omega^{\mu\rho\sigma} P_\mu \wedge M_{\rho\sigma}$
r not triangular	r triangular
$[\hat{x}^0, \hat{x}^j] = \frac{i}{\kappa} \hat{x}^j$	$[\hat{x}^\mu, \hat{x}^\nu] = i\omega^{[\mu\nu]}_\rho \hat{x}^\rho$
$J : M^* \hookrightarrow \mathfrak{so}(1, 4)$	$J : M^* \hookrightarrow \mathfrak{p} := \mathfrak{so}(1, 3) \ltimes \mathbb{R}^4$
$J(M^*) = \mathfrak{an}(3)$	$J(x^\mu) = P^\mu + r^\sharp(x^\mu)$
$\mathfrak{so}(1, 4) = \mathfrak{so}(1, 3) + \mathfrak{an}(3)$	$\mathfrak{p} = \mathfrak{so}(1, 3) + J(M^*)$
$\mathfrak{p}_\kappa \cong \mathfrak{so}(1, 3) \ltimes \mathfrak{an}(3)$	$\mathfrak{p}_T \cong \mathfrak{so}(1, 3) \ltimes J(M^*)$
[MR94]	Drinfel'd Twist [Tol07]

κ -Poincaré and T-Poincaré with $\theta = 0$ [Mer24]

κ -Poincaré [LNR92]	T-Poincaré ($\theta = 0$, 12 families)
$r = \frac{1}{\kappa} P_j \wedge M_{j0}$	$r = \frac{-1}{2} \omega^{\mu\rho\sigma} P_\mu \wedge M_{\rho\sigma}$
r not triangular	r triangular
$[\hat{x}^0, \hat{x}^j] = \frac{i}{\kappa} \hat{x}^j$	$[\hat{x}^\mu, \hat{x}^\nu] = i\omega^{[\mu\nu]}_\rho \hat{x}^\rho$
$J : M^* \hookrightarrow \mathfrak{so}(1, 4)$	$J : M^* \hookrightarrow \mathfrak{p} := \mathfrak{so}(1, 3) \ltimes \mathbb{R}^4$
$J(M^*) = \mathfrak{an}(3)$	$J(x^\mu) = P^\mu + r^\sharp(x^\mu)$
$\mathfrak{so}(1, 4) = \mathfrak{so}(1, 3) + \mathfrak{an}(3)$	$\mathfrak{p} = \mathfrak{so}(1, 3) + J(M^*)$
$\mathfrak{p}_\kappa \cong \mathfrak{so}(1, 3) \ltimes \mathfrak{an}(3)$	$\mathfrak{p}_T \cong \mathfrak{so}(1, 3) \ltimes J(M^*)$
[MR94]	Drinfel'd Twist [Tol07]
$\mathrm{SO}(1, 4) = \mathrm{SO}(1, 3) \cdot \mathrm{AN}(3)$	$\mathcal{P} = \mathrm{SO}(1, 3) \cdot H$
[Sta97; KN03; AK22]	with $\mathrm{Lie}(H) = J(M^*)$

Abelian approximation for T-Poincaré

Recall: $\mathcal{P} = \mathrm{SO}(1, 3) \cdot H = \mathbb{R}^4 \cdot \mathrm{SO}(1, 3)$ up to a nullset. Then

$$H = \{x\Lambda(x) : x \in U\} \quad \text{for some } \Lambda : 0 \in U \subseteq \mathbb{R}^4 \rightarrow \mathrm{SO}(1, 3).$$

Abelian approximation for T-Poincaré

Recall: $\mathcal{P} = \mathrm{SO}(1, 3) \cdot H = \mathbb{R}^4 \cdot \mathrm{SO}(1, 3)$ up to a nullset. Then

$$H = \{x\Lambda(x) : x \in U\} \quad \text{for some } \Lambda : 0 \in U \subseteq \mathbb{R}^4 \rightarrow \mathrm{SO}(1, 3).$$

Properties of Λ

1. $\Lambda : U \rightarrow \mathrm{SO}(1, 3)$ is smooth;
2. $H = \{x\Lambda(x) : x \in U\} \subseteq \mathcal{P}$ is closed;
3. $\mathbb{R}^4 \setminus U$ has measure 0;
4. $\Lambda(0) = 1$ and $d_0\Lambda = r^\sharp \eta^b : \mathbb{R}^4 \rightarrow \mathrm{SO}(1, 3)$;
5. If $x, y \in U$, then $x + \Lambda(x)\tilde{y} \in U$ and $-\Lambda(x)^{-1}x \in U$;
6. $\Lambda(x)\Lambda(y) = \Lambda(x + \Lambda(x)y)$.

Left-invariant $\star$ -product

Theorem

Consider $\Lambda : U \subseteq \mathbb{R}^4 \rightarrow \mathrm{SO}(1, 3)$ as above. Then for every $l \neq 0$, we have a left-invariant continuous associative product

$$\begin{aligned}\star_l : C_c^\infty(\mathcal{P}) \times C_c^\infty(\mathcal{P}) &\rightarrow C_c^\infty(\mathcal{P}) \\ (\varphi \star_l \psi)(g_0) &= \frac{1}{l^4} \int_{U \times \mathbb{R}^4} e^{\frac{-2\pi i}{l} \eta(x, y)} \varphi(g_0 \Lambda(x)) \psi(g_0 y) \, dx \, dy.\end{aligned}$$

Left-invariant $\star$ -product

Theorem

Consider $\Lambda : U \subseteq \mathbb{R}^4 \rightarrow \text{SO}(1, 3)$ as above. Then for every $l \neq 0$, we have a left-invariant continuous associative product

$$\star_l : C_c^\infty(\mathcal{P}) \times C_c^\infty(\mathcal{P}) \rightarrow C_c^\infty(\mathcal{P})$$
$$(\varphi \star_l \psi)(g_0) = \frac{1}{l^4} \int_{U \times \mathbb{R}^4} e^{\frac{-2\pi i}{l} \eta(x, y)} \varphi(g_0 \Lambda(x)) \psi(g_0 y) \, dx \, dy.$$

Moreover,

$$\lim_{l \rightarrow 0} \varphi \star_l \psi = \varphi \psi;$$
$$\lim_{l \rightarrow 0} \frac{\varphi \star_l \psi - \psi \star_l \varphi}{l} = \frac{i}{2\pi} \tilde{r}(\varphi \otimes \psi),$$

where $\tilde{r}$ is the left-invariant Poisson bivector.

Theorem

Consider $\Lambda : U \subseteq \mathbb{R}^4 \rightarrow \text{SO}(1, 3)$ as above. Then for every $l \neq 0$, we have continuous associative products

$$\star_l^M : C_c^\infty(\mathbb{R}^4) \times C_c^\infty(\mathbb{R}^4) \rightarrow C_c^\infty(\mathbb{R}^4)$$

$$(\varphi \star_l^M \psi)(x_0) = \frac{1}{l^4} \int_{U \times \mathbb{R}^4} e^{\frac{2\pi i}{l} \eta(x, y)} \varphi(\Lambda(x)^{-1} x_0) \psi(x_0 + y) \, dx \, dy.$$

Theorem

Consider $\Lambda : U \subseteq \mathbb{R}^4 \rightarrow \text{SO}(1, 3)$ as above. Then for every $l \neq 0$, we have continuous associative products

$$\star_l^M : C_c^\infty(\mathbb{R}^4) \times C_c^\infty(\mathbb{R}^4) \rightarrow C_c^\infty(\mathbb{R}^4)$$
$$(\varphi \star_l^M \psi)(x_0) = \frac{1}{l^4} \int_{U \times \mathbb{R}^4} e^{\frac{2\pi i}{l} \eta(x, y)} \varphi(\Lambda(x)^{-1} x_0) \psi(x_0 + y) \, dx \, dy.$$

Moreover,

$$\lim_{l \rightarrow 0} \varphi \star_l^M \psi = \varphi \psi;$$
$$\lim_{l \rightarrow 0} \frac{\varphi \star_l^M \psi - \psi \star_l^M \varphi}{l} = \frac{i}{2\pi} \{\varphi, \psi\}_M,$$

where $\{\cdot, \cdot\}_M$ is characterized by $\{x^\mu, x^\nu\}_M = \omega^{[\mu\nu]}_\rho x^\rho$.

Locally compact quantum groups

Definition ([KV00; KV03])

A *locally compact quantum group* consists of

- a von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$;
- a coassociative coproduct $\Delta : M \rightarrow M \hat{\otimes} M$;
- quantum Haar weights $\varphi^L, \varphi^R : M_+ \rightarrow [0, +\infty]$

+ suitable axioms.

Locally compact quantum groups

Definition ([KV00; KV03])

A *locally compact quantum group* consists of

- a von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$;
 - a coassociative coproduct $\Delta : M \rightarrow M \hat{\otimes} M$;
 - quantum Haar weights $\varphi^L, \varphi^R : M_+ \rightarrow [0, +\infty]$
- + suitable axioms.

Example

G locally compact group. Then $L^\infty(G)$ is a lcqg with

$$\Delta_G : L^\infty(G) \rightarrow L^\infty(G) \hat{\otimes} L^\infty(G) \cong L^\infty(G \times G)$$

$$\Delta_G(f)(x, y) = f(xy).$$

$$\varphi^{L/R}(f) = \int_G f(x) \, dx^{L/R}.$$

Dual unitary 2-cocycle

Theorem

Consider $\Lambda : U \subseteq \mathbb{R}^4 \rightarrow \mathrm{SO}(1,3)$ as before. Then for every $l \neq 0$, the operator $\Omega \in W^*(\mathcal{P} \times \mathcal{P})$ determined by

$$\Omega_l = \frac{1}{l^4} \int_{U \times \mathbb{R}^4} e^{\frac{2\pi i}{l} \eta(x,y)} \lambda_{\Lambda(x)^{-1}} \otimes \lambda_{-y} \, dx \, dy$$

is unitary and satisfies the 2-cocycle equation

$$(\Omega_l \otimes 1)(\hat{\Delta} \otimes \mathrm{id})(\Omega_l) = (1 \otimes \Omega_l)(\mathrm{id} \otimes \hat{\Delta})(\Omega_l).$$

In particular, $(W^*(\mathcal{P}), \Omega_l \hat{\Delta}(\cdot) \Omega_l^*)$ is a locally compact quantum group [De 11], and the family $\{(W^*(\mathcal{P}), \Omega_l \hat{\Delta}(\cdot) \Omega_l^*)\}_{l \neq 0}$ quantizes the Poincaré group.

Here $\hat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g$.

Could also obtain Ω_l from [Bie+24].

The space $U \subseteq \mathbb{R}^4$ is a space of plane waves

$$E_u(x) = e^{-2\pi i \eta(u,x)}$$

In some cases, U splits into two connected components (eg. light-like κ -Poincaré).

Already observed in $(1+1)D$ light-like κ -Poincaré [DM23].

Recall: $\mathrm{SO}(1, 4) = \mathrm{SO}(1, 3) \cdot AN(3)$.

Two strategies:

1. Construct an $\mathrm{SO}(1, 3)$ -equivariant diffeomorphism

$$\gamma : AN(3) \rightarrow \mathbb{R}^4.$$

Recall: $SO(1, 4) = SO(1, 3) \cdot AN(3)$.

Two strategies:

1. Construct an $SO(1, 3)$ -equivariant diffeomorphism

$$\gamma : AN(3) \rightarrow \mathbb{R}^4.$$

2. Use $SO(1, 4) = SO(4)AN(3)$ to construct a κ -euclidean quantum group: Deformation of $SO(4) \ltimes \mathbb{R}^4$. Then twist $SO(4)$ into $SO(1, 3)$, see [Sta19]. But computations are quite involved.

Key takeaways

- Can construct non-commutative spacetime and Poincaré symmetries using operator algebras for T-Poincaré models thanks to the triangularity.
- Explicit $\star$ -products on $C_c^\infty(\mathcal{P})$ and $C_c^\infty(\mathbb{R}^4)$.
- Peculiar global phenomenon, such as splitting of the space of plane waves.
- Strategies to construct κ -Poincaré.

Thank you for your attention!

References

- [AK22] M. Arzano and J. Kowalski-Glikman. “A group theoretic description of the κ -Poincaré Hopf algebra”. In: *Physics Letters B* 835 (2022), p. 137535. ISSN: 0370-2693. DOI: 10.1016/j.physletb.2022.137535.
- [Bie+24] P. Bieliavsky, V. Gayral, S. Neshveyev, and L. Tuset. “Quantization of locally compact groups associated with essentially bijective 1-cocycles”. In: *International Journal of Mathematics* 35.08 (2024), p. 2450027. ISSN: 0129-167X, 1793-6519. DOI: 10.1142/S0129167X24500277.

- [De 11] K. De Commer. “Galois Objects and Cocycle Twisting for Locally Compact Quantum Groups”. In: *Journal of Operator Theory* 66.1 (2011), pp. 59–106. ISSN: 0379-4024.
- [DM23] M. G. Di Luca and F. Mercati. “New class of plane waves for κ -noncommutative quantum field theory”. In: *Physical Review D* 107.10 (2023), p. 105018. ISSN: 2470-0010, 2470-0029. DOI: 10.1103/PhysRevD.107.105018.
- [KN03] J. Kowalski-Glikman and S. Nowak. “Doubly special relativity and de Sitter space”. In: *Classical and Quantum Gravity* 20.22 (2003), pp. 4799–4816. ISSN: 0264-9381, 1361-6382. DOI: 10.1088/0264-9381/20/22/006.

- [KV00] J. Kustermans and S. Vaes. “Locally compact quantum groups”. In: *Annales scientifiques de l'École Normale Supérieure* 33.6 (2000), pp. 837–934. ISSN: 1873-2151. DOI: 10.1016/s0012-9593(00)01055-7.
- [KV03] J. Kustermans and S. Vaes. “Locally Compact Quantum Groups in the Von Neumann Algebraic Setting”. In: *Mathematica Scandinavica* 92.1 (2003), pp. 68–92. ISSN: 0025-5521. DOI: 10.7146/MATH.SCAND.A-14394.
- [LNR92] J. Lukierski, A. Nowicki, and H. Ruegg. “New quantum Poincaré algebra and κ -deformed field theory”. In: *Physics Letters B* 293.3 (1992), pp. 344–352. ISSN: 0370-2693. DOI: 10.1016/0370-2693(92)90894-A.

- [MR94] S. Majid and H. Ruegg. “Bicrossproduct structure of κ -Poincaré group and non-commutative geometry”. In: *Physics Letters B* 334.3-4 (1994), pp. 348–354. ISSN: 03702693. DOI: 10.1016/0370-2693(94)90699-8.
- [Mer24] F. Mercati. “T-Minkowski noncommutative spacetimes I: Poincaré groups, differential calculi and braiding”. In: *Progress of Theoretical and Experimental Physics* 2024.7 (2024), 073B06. ISSN: 2050-3911. DOI: 10.1093/ptep/ptae088.
- [Sta97] P. Stachura. *Double Lie algebras and Manin triples*. 1997. DOI: 10.48550/arXiv.q-alg/9712040. Pre-published.
- [Sta19] P. Stachura. “The κ -Poincaré group on a C^* -level”. In: *Int. J. Math.* 30.04 (2019), p. 1950022. DOI: 10.1142/S0129167X19500228.

- [Tol07] V. N. Tolstoy. *Twisted Quantum Deformations of Lorentz and Poincaré algebras*. 2007. DOI: 10.48550/arXiv.0712.3962. Pre-published.