

Double field theory and deformation quantisation

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From closed strings to generalised geometry

Low-energy limit: The massless NS-NS sector of the closed string is described by supergravity.

Standard formulation:

$$S_{\text{NS-NS}} = \int d^D x \sqrt{-g} e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - \frac{1}{12} H_3^2 \right)$$

- $g_{\mu\nu}$: spacetime metric
- $B_{\mu\nu}$: Kalb-Ramond 2-form ($H_3 = dB_2$)
- Φ : dilaton

The stringy problem:

- String theory has **T-duality** ($R \leftrightarrow \alpha'/R$).
- T-duality acts via highly non-linear Buscher rules, mixing g and B .
- **Goal:** Formulate the theory so that the $O(D, D)$ symmetry is *manifest*.

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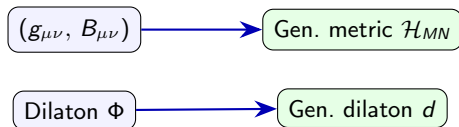
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Generalised geometry/double field theory (DFT) solution: Repackage into $O(D, D)$ covariant objects.



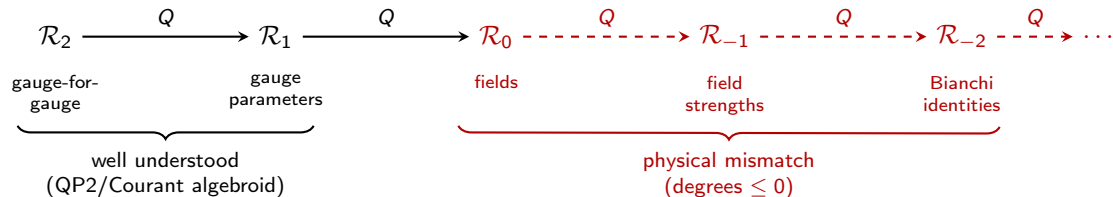
Motivation

BV-BRST structure: fields, ghosts, ghosts-for-ghosts, field strengths, Bianchi identities, ... organised in a cochain complex.

$$\begin{array}{ccc} \mathcal{R}_2 & \xrightarrow{Q} & \mathcal{R}_1 \\ \text{gauge-for-} & & \text{gauge} \\ \text{gauge} & & \text{parameters} \\ \underbrace{\hspace{10em}} & & \\ \text{well understood} & & \\ \text{(QP2/Courant algebroid)} & & \end{array}$$

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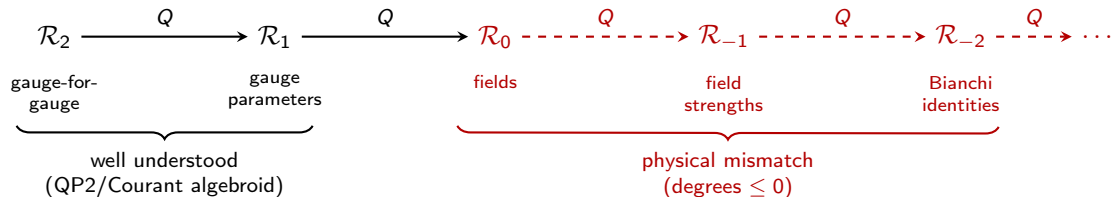


Three gaps in the QP2 story

- 1 **Negative degrees** not well understood.
- 2 **Dilaton** d is missing — no $O(D, D)$ scalar in $\mathcal{R}_0$.
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$\Rightarrow$ **Our solution: deformation quantisation**

Graded symplectic manifold (Roytenberg '99):

$$\mathcal{M} = T^*[2]T[1]M, \quad \omega = dx^i \wedge dp_i + \frac{1}{2} \eta_{IJ} d\xi^I \wedge d\xi^J$$

Coordinate	Degree	Role
x^i	0	base
$\xi^I = (\xi^i, \xi_i)$	1	fibres of $E = TM \oplus T^*M$
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The catch: degree-2 functions give diffeos + frame rotations, but **no $O(D, D)$ singlet** (no dilaton).

Noncommutative deformation

Deformation quantisation: $\mathcal{M} \longrightarrow \mathcal{M}_{\hbar} = \mathcal{M} \times \mathbb{R}[2]$

Graded Moyal–Weyl $\star$ -product (normal-ordered):

$$f \star g = f \exp\left(\hbar\left(2\overleftarrow{\partial}_x \overrightarrow{\partial}_p + \overleftarrow{\partial}_\xi \eta \overrightarrow{\partial}_\xi\right)\right) g$$

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- Exterior algebra $\Lambda(E^*) \longrightarrow$ **Clifford algebra** $Cl(E^*, \eta)$
- Classical Poisson bracket $\longrightarrow$ **graded star-commutator:**

$$\{f, g\}_\star = \frac{1}{2\hbar}(f \star g - (-1)^{|f||g|} g \star f)$$

Where the dilaton hides

Generic degree-2 function on $\mathcal{M}_{\hbar}$ (canonical transformations):

$$\alpha = v^i(x) p_i + \frac{1}{2} T_{IJ}(x) \xi^I \star \xi^J$$

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The noncommutative effect:

- $\hbar = 0$: Yields only $\mathfrak{o}(D, D)$.
- $\hbar \neq 0$: The trace survives! $\implies$ **generalised dilaton** $d = \frac{1}{2} T_I{}^I$.

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$\mathcal{R}_0 \simeq \mathfrak{o}(D, D) \oplus \mathbb{R} \quad (\text{frame} + \text{dilaton})$

Deformed master equation

Degree-3 Hamiltonian separates:

$$S = E_I^j(x) p_j \xi^I + \underbrace{\frac{1}{3!} F_{IJK}(x) \xi^I \xi^J \xi^K}_{\text{3-form flux}} + \underbrace{F_I(x) \hbar \xi^I}_{\text{1-form flux}}$$

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Deformed master equation $\{S, S\}_\star = 0 \implies$

- **Section condition:** $\eta^{IJ} p_I p_J = 0$
- **Fluxes:** Fixes F_{IJK} and F_I in terms of the frame and dilaton d .
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Gauge-inequivalent deformations are classified by $H_{\text{dR}}^3(M) \oplus H_{\text{dR}}^1(M)$.

The complete tensor hierarchy

Degree	Rep.	Local element	Interpretation
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$\Rightarrow$ The $\hbar$ -tower packages the dilaton and its descendants.

Link to the Dirac generating operator

The differential on the Clifford bundle $Cl(E, \eta)$ is **second order**:

$$\bar{Q}(f) = \xi^I \star \partial_I f = \underbrace{\xi^I \partial_I f}_{\text{exterior deriv.}} + \underbrace{\hbar \eta^{IJ} \partial_{\xi^I} \partial_J f}_{\text{codifferential}}$$

This is the **Dirac generating operator** (DGO) of Carow-Watamura et al. '20.

$\implies$ Deformation quantisation provides the **missing link** between the QP-manifold and the Clifford/Dirac approach to DFT.

Ingredients:

- 1 **Generalised metric** $\mathcal{H} \implies$ chiral operator $\hat{\mathcal{H}} = \mathcal{H}_A{}^B \xi^A \partial_{\xi^B}$ splits the Hamiltonian into eigenspaces:

$$S = S_3 + S_1 + S_{-1} + S_{-3}, \quad \hat{\mathcal{H}}(S_m) = m S_m$$

- 2 **Trace:** Berezin integral + Hodge dual on $Cl(E^*, \eta)$, extracts the $O(D, D)$ scalar.
- 3 **Symmetries:** Local $O(D)_L \times O(D)_R$ invariance + $\mathbb{Z}_2$ (odd under $\hat{\mathcal{H}} \rightarrow -\hat{\mathcal{H}}$).

Dynamics: the action

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What we find: These requirements uniquely fix the action (up to normalisation):

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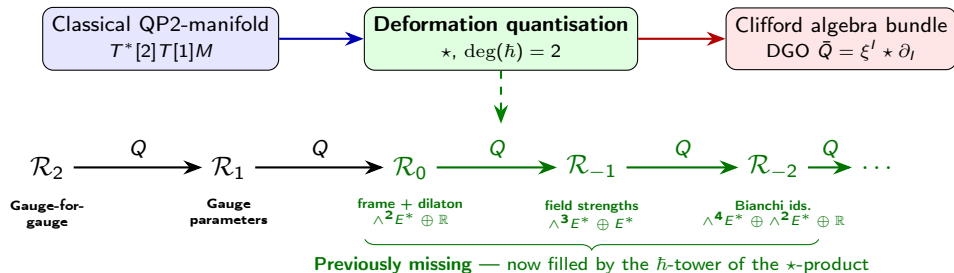
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$\implies$ Reproduces full NS-NS supergravity in T-duality covariant form (Geissbühler, Marqués, Núñez, Penas '13).

The big picture



Take-home message

- ➊ Deformation quantisation **natively generates the generalised dilaton**.
- ➋ It **completes the tensor hierarchy**, filling the negative degrees via the $\hbar$ -tower.
- ➌ It **yields the DFT action**, linking the QP-manifold to the Clifford/Dirac approach.

- **Generalised Cartan geometry:** Systematic construction of torsion and curvature tensors for generalised Cartan geometry (Hassler, Hulik, Osten '24)
- **Ramond–Ramond fields:** Inclusion of RR fields as $O(D, D)$ spinors?
- **Exceptional field theory:** Can we deform exceptional Q-manifolds (Arvanitakis '18) to capture the negative-degree sector of U-duality covariant formulations?
- **AKSZ-contact theory:** Is there an AKSZ-type sigma model for $\mathcal{M}_{\hbar} = \mathcal{M} \times \mathbb{R}[2]$ interpreted as a graded contact manifold, connecting to Contreras, Alba, Mehta '25?

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Thank you!

Backup: Why normal ordering?

Two natural quantisations of the same graded Poisson structure:

	Normal-ordered (ours)	Symmetric / Weyl
$\star$ -product	$e^{\hbar(2\overrightarrow{\partial_x}\overrightarrow{\partial_p}+\overrightarrow{\partial_\xi}\eta\overrightarrow{\partial_\xi})}$	$e^{\hbar(\overrightarrow{\partial_x}\overrightarrow{\partial_p}-\overrightarrow{\partial_p}\overrightarrow{\partial_x}+\overrightarrow{\partial_\xi}\eta\overrightarrow{\partial_\xi})}$
Hamiltonian S	classical form	deformed by $\hbar$ -terms
projected $\bar{Q}$	2nd order: $\xi^I \star \partial_I$	1st order: $\xi^I \partial_I$
physics	DGO, Clifford, dilaton	ordinary de Rham differential

With the symmetric product, the two orderings average back to the classical differential:

$$\bar{Q}f = \xi^I \star \partial_I f - \{\xi^I, \partial_I f\}_\star = \frac{1}{2}(\xi^I \star \partial_I f + (-1)^{|f|} \partial_I f \star \xi^I) = \xi^I \partial_I f$$

$\implies$ no codifferential, no Clifford structure, no dilaton.

They are equivalent quantisations — related by a gauge transformation

$f \star g = T^{-1}(Tf \star_W Tg)$ with $T = e^{-\frac{\hbar}{2}\partial_x\partial_p}$. Under T the $\hbar$ -corrections **move between S and $\bar{Q}$** : Weyl keeps the differential classical and deforms the Hamiltonian (frame divergence enters S); normal ordering keeps S classical and deforms the differential into the DGO.

Backup: Why can we drop the p 's?

For the flat Hamiltonian $S_0 = \xi^i p_i$, the differential acts as

$$Q_0(\xi_i) = p_i, \quad Q_0(p_i) = 0, \quad Q_0(x^i) = Q_0(\xi^i) = 0,$$

so (ξ_i, p_i) form a **contractible pair**:

- contracting homotopy: $\rho = \xi_i \partial / \partial p_i$,
- counting operator: $N = Q_0 \rho + \rho Q_0$, with $N(\xi_i) = \xi_i$, $N(p_i) = p_i$, $N(x^i) = N(\xi^i) = 0$.

Any Q_0 -closed F_k with $N(F_k) = kF_k$, $k \geq 1$, is exact:

$$F_k = Q_0\left(\frac{1}{k} \rho F_k\right) \implies H^\bullet(C^\infty(\mathcal{M}), Q) \cong H_{\text{dR}}^\bullet(M)$$

(Roytenberg '02; extends to any Q related by canonical transformation.)

- The momenta carry **no cohomological information** — projecting to $p = 0$ is harmless for the complex.
- What it *does* destroy is the DGLA structure ($S \notin \bar{C}^\infty$, Leibniz anomaly).
- We keep the doubled $\xi^I = (\xi^i, \xi_i)$ to retain manifest $O(D, D)$ covariance.

Backup: extending Ševera's classification

In the coordinate frame ($E_I{}^J = \delta_I^J$) with $S = p_I \xi^I + \frac{1}{3!} F_{IJK}(x) \xi^I \xi^J \xi^K + \hbar F_I(x) \xi^I$, the master equation gives

$$\{S, S\}_\star = 0 \iff \partial_{[I} F_{JKL]} = 0, \quad \partial_{[I} F_{J]} = 0.$$

Poincaré lemma: $H_{ijk} = 3 \partial_{[i} B_{jk]}$ and $F_i = 2 \partial_i d$ — defining the B -field and the dilaton.

Shifting F_3 , F_1 by exact forms does not change the homological structure, so gauge-inequivalent deformations are classified by

$$\text{deformations of } \mathcal{M}_\hbar \cong H_{\text{dR}}^1(M) \oplus H_{\text{dR}}^3(M).$$

- H^3 : the classical Ševera class of exact Courant algebroids (H -flux; magnetic sources = NS5-branes).
- H^1 : **new** — the dilaton-flux class (closed 1-form; magnetic sources = codimension-2 exotic “NS7”-branes).

The H^1 class has no analogue for the undeformed QP2-manifold: it is a genuine topological consequence of the deformation.

Backup: the measure e^{-2d} from $\langle \bar{Q}f \rangle = 0$

Compatibility condition (imposed for degree-1 test functions only): for all $f = V_A(x)\xi^A$,

$$\langle \bar{Q}f \rangle = \int_M d^D x \, \mu \, \text{Tr}(\bar{Q}f) = 0.$$

Projecting $p \rightarrow 0$ and using $\text{Tr}(\xi^A \star \xi^B) = \eta^{AB}$:

$$\bar{Q}f = -D_A V_B \xi^A \star \xi^B + \hbar F_A V^A + \frac{1}{2} F_{ABC} V^A \xi^B \xi^C \implies \text{Tr}(\bar{Q}f) = -D_A V^A + F_A V^A.$$

Rewrite as a total derivative, insert the master equation input $F_A + \partial_M E_A^M = 2D_A d$, and integrate by parts:

$$\begin{aligned} 0 &= \int \mu \left(-\partial_M V^M + (F_A + \partial_M E_A^M) V^A \right) = \int \left(D_A \mu + 2\mu D_A d \right) V^A \quad \forall V \\ &\implies \boxed{\mu = e^{-2d}} \quad \text{and} \quad \int e^{-2d} D_A V^A = \int e^{-2d} F_A V^A. \end{aligned}$$

Backup: explicit Bianchi identities

The deformed master equation $\{S, S\}_\star = 0$ yields, beyond the section condition and flux definitions, the following Bianchi identities:

$$E_{[K}{}^I \partial_I F_{|LMN]} - \frac{3}{4} F^P{}_{[KL} F_{P|MN]} = 0 \quad (F_3 \text{ Bianchi})$$

$$E_L{}^I \partial_I F^L{}_{JK} - 2E_{[J}{}^I \partial_I F_{|K]} + F_I F^I{}_{JK} = 0 \quad (F_1 \text{ Bianchi})$$

$$E_I{}^M \partial_M F^I - \frac{1}{2} F_I F^I + \frac{1}{12} F_{IJK} F^{IJK} = 0 \quad (\text{Scalar Bianchi})$$

These reproduce the DFT Bianchi identities of Geissbühler et al. '13.

Backup: the chiral splitting of S

Generalised metric $\mathcal{H} \Rightarrow$ chiral Euler field $\hat{\mathcal{H}} = \mathcal{H}_A{}^B \xi^A \partial_{\xi^B}$; $S = \sum_m S_m$ with $\hat{\mathcal{H}}(S_m) = m S_m$:

$$S_3 = \frac{1}{3!} F_{abc}(x) \xi^a \xi^b \xi^c,$$

$$S_{-3} = \frac{1}{3!} F_{\bar{a}\bar{b}\bar{c}}(x) \xi^{\bar{a}} \xi^{\bar{b}} \xi^{\bar{c}},$$

$$S_1 = \xi^a E_a{}^M(x) p_M + \frac{1}{2} F_{ab\bar{c}}(x) \xi^a \xi^b \xi^{\bar{c}} + F_a(x) \xi^a \hbar,$$

$$S_{-1} = \xi^{\bar{a}} E_{\bar{a}}{}^M(x) p_M + \frac{1}{2} F_{\bar{a}\bar{b}\bar{c}}(x) \xi^{\bar{a}} \xi^{\bar{b}} \xi^{\bar{c}} + F_{\bar{a}}(x) \xi^{\bar{a}} \hbar.$$

$\hat{Z}(S_m) = \text{sgn}(m) S_m$, realised on degree-3 functions by $\hat{Z} = \frac{1}{12} (13\hat{\mathcal{H}} - \hat{\mathcal{H}}^3)$.

Double Lorentz variations $\delta_\Lambda S = \{\Lambda_L + \Lambda_R, S\}_\star$ give the flux transformation rules

$$\delta_\Lambda F_{ABC} = 3 D_C \Lambda_{AB} + 3 \Lambda_A{}^D F_{DBC} \quad \delta_\Lambda F_A = -D^B \Lambda_{BA} + \Lambda_A{}^B F_B$$

(up to the antisymmetrisation in $\frac{1}{3!} \delta_\Lambda F_{ABC} \xi^A \xi^B \xi^C$), used to fix the action via the $\wedge^2 C_+$ Bianchi identity $-2D_a F_b = (D^C - F^C) F_{abC}$.