

Supersymmetric Poisson Sigma Models: Geometry and Quantization

Axel Hrelja

Ruđer Bošković Institute, Zagreb

Based on work in progress with Thomas Basile, Athanasios Chatzistavrakidis
and Sylvain Lavau

Corfu, 19th September 2026



Plan of the talk

- Recap of the PSM, its BV quantization and the star product
- Motivation for the Super-PSM and its BV quantization
- First-order product of differential forms
- Outlook: star products on different vector bundles

Poisson sigma model

The Poisson sigma model is a two-dimensional topological field theory whose interaction and gauge symmetry are determined by a Poisson bivector Π on M .

$$X : \Sigma \rightarrow M, \quad A \in \Omega^1(\Sigma, X^*T^*M), \quad \Pi \in \Gamma(\wedge^2 TM), \quad [\Pi, \Pi]_{\text{SN}} = 0$$

The action is given by

$$S[X, A] = \int_{\Sigma} A_{\mu} \wedge dX^{\mu} + \frac{1}{2} \Pi^{\mu\nu}(X) A_{\mu} \wedge A_{\nu}$$

Topological: no metric on Σ enters the action.

Its perturbative quantization on a disk gives the star product of functions on M .

[Ikeda, 1993] [Cattaneo, Felder, 1999]

Gauge symmetry and equations of motion of the PSM

Gauge transformations

$$\delta_\varepsilon X^\mu = \varepsilon_\nu \Pi^{\nu\mu},$$

$$\delta_\varepsilon A_\mu = d\varepsilon_\mu + A_\nu \partial_\mu \Pi^{\nu\rho} \varepsilon_\rho.$$

Equations of motion

$$E^\mu := dX^\mu + \Pi^{\mu\nu} A_\nu = 0,$$

$$dA_\mu + \frac{1}{2} \partial_\mu \Pi^{\nu\rho} A_\nu \wedge A_\rho = 0.$$

$$[\delta_1, \delta_2] A_\mu = \delta_{\varepsilon_{12}} A_\mu - \partial_\mu \partial_\lambda \Pi^{\nu\rho} \varepsilon_{1,\nu} \varepsilon_{2,\rho} E^\lambda$$

where $\varepsilon_{12,\mu} = \partial_\mu \Pi^{\nu\rho} \varepsilon_{1,\nu} \varepsilon_{2,\rho}$

The commutator closes up to equations of motion, the gauge algebra is generally open

[Ikeda, 1993] [Cattaneo, Felder, 1999]

The BV formalism

Introduce ghosts c_μ associated with the gauge parameters, and an antifield conjugate to each field and ghost

$$\text{gh}(\Phi^\dagger) = -1 - \text{gh}(\Phi), \quad \text{deg}_\Sigma \Phi^\dagger = 2 - \text{deg}_\Sigma \Phi.$$

	X	A	c	$X^\dagger$	$A^\dagger$	$c^\dagger$
Form degree	0	1	0	2	1	2
Ghost number	0	0	1	-1	-1	-2

BV action

The BV action is given by

$$S_{\text{BV}} = \int_{T[1]\Sigma} (\mathbf{A}_\mu d\mathbf{X}^\mu + \frac{1}{2} \Pi^{\mu\nu}(\mathbf{X}) \mathbf{A}_\mu \mathbf{A}_\nu)$$

The superfields are given by

$$\mathbf{X}^\mu = X^\mu + A^{\dagger\mu} - c^{\dagger\mu}$$

$$\mathbf{A}_\mu = c_\mu + A_\mu + X_\mu^\dagger$$

Functions on $T[1]\Sigma$ represent differential forms on Σ . The superfields combine components of fixed total degree

The BV action satisfies the classical master equation,

$$(S_{\text{BV}}, S_{\text{BV}})_{\text{BV}} = 0$$

Gauge fixing

We fix the gauge to construct the perturbative propagators

To impose the Lorenz gauge $d^\dagger A = 0$, where $d^\dagger = - * d *$,

We add an antighost $\bar{c}$ and a two-form Lagrange multiplier λ^μ to our field content

$$S_{\text{g.f.}} = \left(S_{\text{BV}} + \int_{\Sigma} \lambda^\mu * \bar{c}_\mu^\dagger \right) \Big|_{\Phi^\dagger = \delta \Psi / \delta \Phi}, \quad \Psi = \int_{\Sigma} \bar{c}^\mu * d^\dagger A_\mu$$

On this gauge-fixing submanifold,

$$\mathbf{X}^\mu = X^\mu - * d \bar{c}^\mu, \quad \mathbf{A}_\mu = c_\mu + A_\mu$$

Next we expand $X = x + \xi$

Gauge fixed action and propagator

$$S_0 = \int_{\Sigma} (A_{\mu} \wedge d\xi^{\mu} + \lambda^{\mu} d^{\dagger} A_{\mu} - \bar{c}^{\mu} * \Delta c_{\mu}), \quad \Delta = dd^{\dagger} + d^{\dagger}d$$

$$S_{\text{int}} = \frac{1}{2} \int_{T[1]\Sigma} \Pi^{\mu\nu}(x + \xi) \mathbf{A}_{\mu} \mathbf{A}_{\nu}.$$

To construct the star product, we take the worldsheet Σ to be a disk, which we conformally transform to the upper half-plane.

We can now find the propagator which is used to compute the correlation functions.

$$\langle \xi^{\mu}(w) A_{\nu}(z) \rangle_0 = \frac{\int \mathcal{D}\xi \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\lambda \xi^{\mu}(w) A_{\nu}(z) e^{iS_0/\hbar}}{\int \mathcal{D}\xi \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\lambda e^{iS_0/\hbar}} = \frac{i\hbar}{2\pi} \delta_{\nu}^{\mu} d_z G(z, w).$$

$$G(z, w) = \frac{1}{2i} \log \frac{(z - w)(z - \bar{w})}{(\bar{z} - \bar{w})(\bar{z} - w)}.$$

The star product

We impose the following boundary conditions

$$A|_{\partial\Sigma} = 0, \quad X(\infty) = x, \quad \mathbf{X} = x + \boldsymbol{\xi}.$$

The normalized perturbative correlation function is

$$(f \star g)(x) \equiv \langle f(x + \boldsymbol{\xi}(1))g(x + \boldsymbol{\xi}(0)) \rangle = \frac{\langle f(x + \boldsymbol{\xi}(1))g(x + \boldsymbol{\xi}(0))e^{iS_{\text{int}}/\hbar} \rangle_0}{\langle e^{iS_{\text{int}}/\hbar} \rangle_0}.$$

We use Wick's theorem to contract the fields using the propagators to obtain the tensor structure of a term in the expansion, while the integrals of propagators give the numerical prefactors.

This correlator satisfies the conditions of being a deformation quantization, so this correlator is the star product

[Cattaneo, Felder, 1999] [Kontsevich, 1997]

Kontsevich star product

For a Poisson manifold (M, Π) , the expansion gives the perturbatively expanded Kontsevich star product

$$\begin{aligned}(f \star g)(x) = & fg + \frac{i\hbar}{2} \Pi^{\mu\nu} \partial_\mu f \partial_\nu g - \frac{\hbar^2}{8} \Pi^{\mu\nu} \Pi^{\rho\sigma} \partial_\mu \partial_\rho f \partial_\nu \partial_\sigma g \\ & - \frac{\hbar^2}{12} \Pi^{\mu\nu} \partial_\nu \Pi^{\rho\sigma} (\partial_\mu \partial_\rho f \partial_\sigma g - \partial_\rho f \partial_\mu \partial_\sigma g) + \mathcal{O}(\hbar^3)\end{aligned}$$

The first correction is the Poisson bracket. At second order, derivatives also act on the Poisson bivector

$$(f \star g) \star h = f \star (g \star h), \quad f \star g - g \star f = i\hbar \{f, g\} + \mathcal{O}(\hbar^3)$$

[Cattaneo, Felder, 1999] [Kontsevich, 1997]

Why pass to target spaces with a super-Poisson bivector?

Differential forms

Arias, Boulanger, Sundell and Torres-Gomez proposed a differential PSM as a route to deforming the algebra of forms. [Arias, Boulanger, Sundell, Torres-Gomez, 2015]

McCurdy and Zumino constructed a covariant star product for forms under suitable geometric conditions. [McCurdy, Zumino, 2009]

Want to connect these approaches through path-integral quantization

Two-dimensional supergravity

Super-PSMs also describe supersymmetric gravity models. JT supergravity provides a concrete example associated with $\mathfrak{osp}(1|2, \mathbb{R})$ [Ikeda, 1993],

[AH, Basile, Chatzistavrakidis, Lavau, 2025]

Poisson supermanifolds

Consider a supermanifold with even coordinates x^μ and odd coordinates θ^a :

$$\mathcal{M} \cong \Pi E, \quad C^\infty(\Pi E) \cong \Gamma(\wedge^\bullet E^*).$$

If its algebra of functions is equipped with an $\mathbb{R}$ -bilinear graded antisymmetric even bracket

$$\{-, -\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$$

which obeys the Leibniz rule and Jacobi identity, then $(\mathcal{M}, \{-, -\})$ is called a Poisson supermanifold.

Equivalently, it means that $\mathcal{M}$ is equipped with a Poisson bivector

$$\mathcal{P} \in \Gamma(\wedge^2 T\mathcal{M}) \quad \text{such that} \quad \{f, g\} = \mathcal{P}(\mathrm{d}f, \mathrm{d}g), \quad \forall f, g \in C^\infty(\mathcal{M})$$

Components of the super-Poisson bivector

$$\{x^\mu, x^\nu\} = \mathcal{P}^{\mu\nu}(x, \theta), \quad \{x^\mu, \theta^a\} = \mathcal{P}^{\mu a}(x, \theta), \quad \{\theta^a, \theta^b\} = \mathcal{P}^{ab}(x, \theta)$$

$$\mathcal{P}^{\mu\nu} = \Pi^{\mu\nu}(x) + \frac{1}{2}\theta^a\theta^b\mathcal{P}_{ab}{}^{\mu\nu}(x) + \cdots,$$

$$\mathcal{P}^{\mu a} = \theta^b\Gamma_b^{\mu a}(x) + \cdots,$$

$$\mathcal{P}^{ab} = g^{ab}(x) + \frac{1}{2}\theta^c\theta^d R_{cd}{}^{ab}(x) + \cdots.$$

Here $\mathcal{P}^{\mu\nu} = -\mathcal{P}^{\nu\mu}$, $\mathcal{P}^{\mu a} = -\mathcal{P}^{a\mu}$ and $\mathcal{P}^{ab} = \mathcal{P}^{ba}$.

The leading mixed coefficient is a contravariant connection on E^* , which we assume is induced by an affine connection on E

$$\Gamma_b^{\mu a} = -\Pi^{\mu\nu}\Gamma_{\nu b}^a.$$

[Basile, Chatzistavrakidis, Lavau, 2025]

Super-Poisson sigma-model: Fields and gradings

The super-Poisson sigma model is a topological field theory in two dimensions, whose classical fields are degree-preserving maps of $\mathbb{Z} \times \mathbb{Z}_2$ -graded manifolds,

$$T[1, 0]\Sigma \longrightarrow T^*[1, 0]\mathcal{M}, \quad \mathcal{M} \cong E[0, 1].$$

Field	X^μ	A_μ	θ^a	χ_a
Worldsheet form degree	0	1	0	1
Grassmann parity	0	0	1	1

Σ smooth manifold concentrated in the bidegree $[0, 0]$, and E some vector bundle.

The pair (X^μ, θ^a) originates from coordinates on $\mathcal{M} \cong E[0, 1]$, and (A_μ, χ_a) are their momenta

[Basile, Chatzistavrakidis, Lavau, 2025]

The super-Poisson sigma-model action

The action is given by

$$S[X, A, \theta, \chi] = \int_{\Sigma} \left[A_{\mu} \wedge dX^{\mu} + \chi_a \wedge d\theta^a + \frac{1}{2} P^{\mu\nu}(X, \theta) A_{\mu} \wedge A_{\nu} \right. \\ \left. + P^{\mu a}(X, \theta) A_{\mu} \wedge \chi_a + \frac{1}{2} P^{ab}(X, \theta) \chi_a \wedge \chi_b \right].$$

There are two scalar gauge parameters: ε_{μ} , ε_a , which are respectively even and odd.

The gauge algebra is generally open, just as in the bosonic model.

[Ikeda, 1993], [Basile, Chatzistavrakidis, Lavau, 2025]

Tangent bundle: differential forms as superfunctions

For $E = TM$,

$$\theta^\mu \longleftrightarrow dx^\mu, \quad C^\infty(T[0, 1]M) \cong \Omega^\bullet(M).$$

In particular,

$$\alpha = \alpha_\mu(x) dx^\mu \longleftrightarrow \alpha_\mu(x) \theta^\mu.$$

The wedge product becomes multiplication of superfunctions. The de Rham differential becomes

$$q_S = \theta^\mu \frac{\partial}{\partial x^\mu}, \quad q_S^2 = 0.$$

When $\mathcal{L}_{q_S} \mathcal{P} = 0$, the de Rham differential generates a global supersymmetry of the model

[Arias, Boulanger, Sundell, Torres-Gomez, 2015] [Basile, Chatzistavrakidis, Lavau, 2025]

Tangent bundle: Poisson bivector components

Compatibility of the super-Poisson bracket with q_S requires

$$\tilde{\nabla}\Pi = 0, \quad R_{\rho\sigma}{}^{\mu\nu} = -\Pi^{(\mu|\tau}\tilde{R}^{|\nu)}{}_{\tau\rho\sigma} + \Pi^{\kappa\lambda}\Gamma_{\kappa\rho}^{\mu}\Gamma_{\lambda\sigma}^{\nu}.$$

Taking all of this into account, the super-Poisson bivector has the following form

$$\mathcal{P}^{\mu\nu} \equiv \{x^{\mu}, x^{\nu}\} = \Pi^{\mu\nu},$$

$$\mathcal{C}^{\mu\nu} \equiv \{x^{\mu}, \theta^{\nu}\} = -\Pi^{\mu\kappa}\Gamma_{\kappa\lambda}^{\nu}\theta^{\lambda},$$

$$\mathcal{R}^{\mu\nu} \equiv \{\theta^{\mu}, \theta^{\nu}\} = -\left(\frac{1}{2}\Pi^{(\mu|\tau}\tilde{R}^{|\nu)}{}_{\tau\rho\sigma} - \Pi^{\kappa\lambda}\Gamma_{\kappa\rho}^{\mu}\Gamma_{\lambda\sigma}^{\nu}\right)\theta^{\rho}\theta^{\sigma}.$$

[Arias, Boulanger, Sundell, Torres-Gomez, 2015] [AH, Basile, Chatzistavrakidis, Lavau, 2025]

Gauge fixed action

$$\mathbf{X}^\mu = X^\mu - *d\bar{c}^\mu, \quad \mathbf{A}_\mu = c_\mu + A_\mu, \quad \boldsymbol{\theta}^a = \theta^a + *\bar{\beta}^a, \quad \boldsymbol{\chi}_a = \beta_a + \chi_a$$

Expanding $\mathbf{X} = x + \boldsymbol{\xi}$, $\boldsymbol{\theta} = \theta + \boldsymbol{\psi}$

$$S_{\text{g.f.}} = \overbrace{\int_{\Sigma} A_\mu \wedge d\xi^\mu + \lambda^\mu d^\dagger A_\mu - \bar{c}^\mu * \Delta c_\mu + \chi_a \wedge d\psi^a + \ell^a d^\dagger \chi_a - \bar{\beta}^a * \Delta \beta_a}^{S_0} \\ + \underbrace{\int_{T[1]\Sigma} \frac{1}{2} \mathcal{P}^{\mu\nu}(\mathbf{X}, \boldsymbol{\theta}) \mathbf{A}_\mu \mathbf{A}_\nu + \mathcal{P}^{\mu a}(\mathbf{X}, \boldsymbol{\theta}) \mathbf{A}_\mu \boldsymbol{\chi}_a + \frac{1}{2} \mathcal{P}^{ab}(\mathbf{X}, \boldsymbol{\theta}) \boldsymbol{\chi}_a \boldsymbol{\chi}_a}_{S_{\text{int}}}$$

the kinetic terms give the propagators

$$\xi^\mu \longleftrightarrow A_\mu, \quad \psi^\mu \longleftrightarrow \chi_\mu.$$

The correlation function for superfunctions

We proceed in the same manner as in the bosonic case

Using the boundary conditions

$$A|_{\partial\Sigma} = 0 = \chi|_{\partial\Sigma} \quad (X, \theta)(\infty) = (x, \theta),$$

For $\tilde{f}, \tilde{g} \in C^\infty(E[0, 1])$,

$$(\tilde{f} \star \tilde{g})(x, \theta) = \frac{\left\langle \tilde{f}(x + \boldsymbol{\xi}(1), \theta + \boldsymbol{\psi}(1)) \tilde{g}(x + \boldsymbol{\xi}(0), \theta + \boldsymbol{\psi}(0)) e^{iS_{\text{int}}/\hbar} \right\rangle_0}{\left\langle e^{iS_{\text{int}}/\hbar} \right\rangle_0}.$$

[Cattaneo, Felder, 1999]

Generalized star product

After performing this calculation, we expect to obtain the generalization of the Kontsevich formula for two functions of x and θ , $\tilde{f}(x, \theta)$ and $\tilde{g}(x, \theta)$

$$\begin{aligned}(\tilde{f} \star \tilde{g})(x, \theta) = & \tilde{f}(x, \theta) \tilde{g}(x, \theta) + \frac{i\hbar}{2} \left(\tilde{f}(x, \theta) \overleftarrow{\partial}_\alpha \right) P^{\alpha\beta} \left(\overrightarrow{\partial}_\beta \tilde{g}(x, \theta) \right) \\ & - \frac{\hbar^2}{8} \left(\tilde{f}(x, \theta) \overleftarrow{\partial}_\alpha \overleftarrow{\partial}_\gamma \right) P^{\alpha\beta} P^{\gamma\delta} \left(\overrightarrow{\partial}_\delta \overrightarrow{\partial}_\beta \tilde{g}(x, \theta) \right) \\ & - \frac{\hbar^2}{12} \left(\tilde{f}(x, \theta) \overleftarrow{\partial}_\alpha \overleftarrow{\partial}_\gamma \right) P^{\alpha\beta} \left(\overrightarrow{\partial}_\beta P^{\gamma\delta} \right) \left(\overrightarrow{\partial}_\delta \tilde{g}(x, \theta) \right) \\ & + \frac{\hbar^2}{12} \left(\tilde{f}(x, \theta) \overleftarrow{\partial}_\gamma \right) P^{\alpha\beta} \left(\overrightarrow{\partial}_\beta P^{\gamma\delta} \right) \left(\overrightarrow{\partial}_\delta \overrightarrow{\partial}_\alpha \tilde{g}(x, \theta) \right) + \mathcal{O}(\hbar^3).\end{aligned}$$

The Poisson bracket of one-forms

For $E = TM$, taking

$$\tilde{f}(x, \theta) = \alpha_\kappa(x) \theta^\kappa \equiv \alpha$$

$$\tilde{g}(x, \theta) = \beta_\lambda(x) \theta^\lambda \equiv \beta$$

We obtain the star product between forms

$$\alpha \star \beta = \alpha \wedge \beta + \frac{i\hbar}{2} \{\alpha, \beta\} + O(\hbar^2).$$

$$\{\alpha, \beta\} = \theta^\kappa \theta^\lambda \left[\Pi^{\mu\nu} (\nabla_\mu \alpha)_\kappa (\nabla_\nu \beta)_\lambda - \frac{1}{2} \Pi^{(\rho|\tau} \tilde{R}^{|\sigma)}_{\tau\kappa\lambda} \alpha_\rho \beta_\sigma \right]$$

Same bracket used by McCurdy and Zumino.

[McCurdy, Zumino, 2009]

Outlook: Different bundles, different geometries

The algebra to deform is $C^\infty(\Pi E) \cong \Gamma(\wedge^\bullet E^*)$.

Class	Geometric data	Product to study
Tangent model	$E = TM$	Forms
Cotangent model	$E = T^*M$	Polyvector fields
Degree -2 model	$P^{ab} = g^{ab}(x)$, other blocks zero	
Higher θ degree	$\mathcal{P}_{ab}{}^{\mu\nu}$ and further coefficients	Products that can mix exterior degrees

Thank you!

[Basile, Chatzistavrakidis, Lavau, 2025]

[AH, Basile, Chatzistavrakidis, Lavau, 2025]