Variational Loop Vertex Expansion for Cumulants

Vincent Rivasseau

IJCLab, Université Paris-Saclay

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Introduction

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- The LVE is a constructive approach for quartic matrix models, designed to provide bounds that are uniform with respect to the matrix size.
- We review an extension of the LVE known under Loop Vertex Representation and the corresponding work with Krajewski and Sazonov.

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The BKAR Forest Formula

Let f be a smooth function of $n(n-1)/2$ line variables $x_\ell \in [0,1]$, $\ell = (i,j)$, $1 \le i < j \le n$. The forest formula states

$$f(1,\cdots,1) = \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} \Big[\int_0^1 dw_\ell \Big] \Big\} \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial x_\ell} f \Big\} [X^{\mathcal{F}}(w_{\mathcal{F}})], \quad \text{where}$$

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- the "weakening parameter" $X^{\mathcal{F}}_{ij}(w_{\mathcal{F}})]$ is 0 if i and j dont belong to the same connected component of $\mathcal{F}$; otherwise it is the minimum of the $w_{\ell'}$ for ℓ' running over the unique path from i to j in $\mathcal{F}$.

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- Furthermore the real symmetric matrix $X^{\mathcal{F}}_{ij}(w_{\mathcal{F}})]$ (completed by 1 on the diagonal $i = j$) is positive.

Constructive Quantum Field Theory, I

- Cluster expansion = Taylor-Lagrange expansion of the functional integral:

$$F = 1 + H, \quad H = -\lambda \int_0^1 dt \int_{-\infty}^{+\infty} x^4 e^{-\lambda t x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}}$$

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- Mayer expansion: define $H_i = -\lambda \int_0^1 dt \int_{-\infty}^{+\infty} x^4 e^{-\lambda t x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} = H \; \forall i$, $\epsilon_{ij} = 0 \; \forall i, j$ and write

$$F = 1 + H = \sum_{n=0}^{\infty} \prod_{i=1}^{n} H_i(\lambda) \prod_{1 \le i < j \le n} \epsilon_{ij}$$

To prove this, let us define $\eta_{ij} = -1$, $\epsilon_{ij} = 1 + \eta_{ij} = 1 + x_{ij}\eta_{ij}|_{x_{ij}=1}$ and apply the forest formula.

Constructive Quantum Field Theory, II

- $$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \prod_{i=1}^{n} H_i(\lambda) \Big\{ \prod_{\ell \in \mathcal{F}} [\int_0^1 dw_\ell] \eta_\ell \Big\} \prod_{\ell \notin \mathcal{F}} [1 + \eta_\ell x_\ell^{\mathcal{F}}(w)]$$

  The logarithm of the forest formula is simply a tree BKAR formula. Then defining $G = \log F$,

  $$G = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{T}} \prod_{i=1}^{n} H_i(\lambda) \Big\{ \prod_{\ell \in \mathcal{T}} [\int_0^1 dw_\ell] \eta_\ell \Big\} \prod_{\ell \notin \mathcal{T}} [1 + \eta_\ell x_\ell^{\mathcal{F}}(w)]$$

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- Borel summability now easily follows from the Borel summability of H.
- It generalizes well to the case of lattice statistical mechanics ($d > 0$).
- However the link with Feynman graphs is somewhat lost, and furthermore il may be not optimal for curved or random space-time geometries.

Loop Vertex Expansion, I

Intermediate field representation

$$F = \int_{-\infty}^{+\infty} e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i\sqrt{2\lambda}\sigma x^2 - x^2/2 - \sigma^2/2} \frac{dx}{\sqrt{2\pi}} \frac{d\sigma}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\log[1+2i\sqrt{2\lambda}\sigma] - \sigma^2/2} \frac{d\sigma}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \frac{V^n}{n!} d\mu(\sigma)$$

Let us apply the forest formula, but using "replicas" of the intermediate field:

$$V^n(\sigma) \to \prod_{i=1}^{n} V_i(\sigma_i), \quad d\mu(\sigma) \to d\mu_C(\{\sigma_i\}),$$

$C_{ij} = \mathbf{1}_n = x_{ij}|_{x_{ij}=1}$, where $\mathbf{1}_n$ is the $n \times n$ matrix with entries one everywhere.

Loop Vertex Expansion, II

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} [\int_0^1 dw_\ell] \Big\} \int \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^{n} V(\sigma_i) \Big\} d\mu_{C^{\mathcal{F}}}$$

where $C_{ij}^{\mathcal{F}} = x_\ell^{\mathcal{F}}(\{w\})$ if $i < j$, $C_{ii}^{\mathcal{F}} = 1$.

$$G = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{T}} \Big\{ \prod_{\ell \in \mathcal{T}} [\int_0^1 dw_\ell] \Big\} \int \Big\{ \prod_{\ell \in \mathcal{T}} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^{n} V(\sigma_i) \Big\} d\mu_{C^{\mathcal{T}}}$$

where the second sum runs over trees !

Link with Feynman graphs can be recovered, and the conclusion is that the LVE should be better adapted for general background geometries, such as curved geometries, random geometries... $\rightarrow$ in short, to quantum gravity.

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The work of [Gurau-Krajewski,2014]

[Gurau-Krajewski,2014] combines the LVE with cumulants (in the physics language they are usually called connected Dyson-Schwinger functions).

- They *define ordinary cumulants* which are based on ordinary Feynman graphs and amplitudes. Then they define *scalar cumulants* for the topological expansion in the genus of the combinatorial maps. An essential part of their article is devoted to the Weingarten calculus [Collins,2003].

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- They *prove* that any ordinary cumulant is an analytic function inside a cardioid domain in the complex plane. They prove also that any cumulant is Borel summable at the origin.

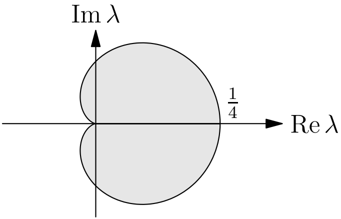

Figure: Analyticity domain in the complex λ plane

The work of [Gurau-Krajewski,2014]

For scalar cumulants regarded as functions of λ with N considered as a parameter, the domain of analyticity is reduced by a factor $1/4$:

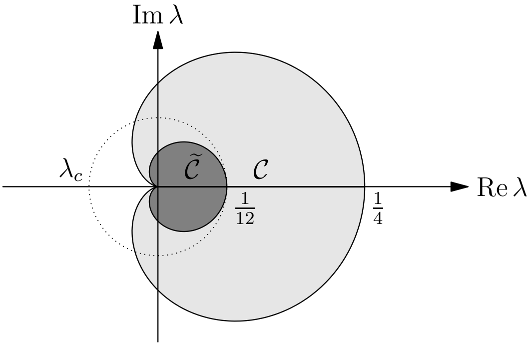

Figure: Analyticity domain of the topological expansion

Their work is essential piece for those seriously interested in the subject of cumulants, even up to today. Presently I work in their footsteps.

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- I find also that Fuss-Catalan functions are shown rather easily to have bounded derivatives of all orders; that is the second feature which allows the LVR to work.
- Fuss-Catalan functions of order p also govern the leading term in the $N \to \infty$ limit of random tensor models of rank p [Bonzom et all, 2011].

The work of [Krajewski-R.-Sazonov,2019]

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- We apply the idea of reparametrization invariance to monomial interactions of arbitrarily high even order.
- The key notion is to use the Fuss-Catalan function T_p defined to be the solution analytic at the origin of the algebraic equation

$$zT_p^p(z) - T_p(z) + 1 = 0 \, .$$

For any square matrix X we define the matrix-valued function

$$A(\lambda, X) \quad := \quad XT_p(-\lambda X^{p-1})$$

and also an N_l by N_l square matrix X_l and an N_r by N_r square matrix X_r through

$$X_l \quad := \quad MM^\dagger, \quad X_r := M^\dagger M.$$

A lemma of [Krajewski-R.-Sazonov,2019]

Lemma

In the sense of formal power series in λ

$$Z(\lambda, N_l, N_r) = \int dMdM^\dagger \exp\{-N_r \mathrm{Tr}_l X_l + \mathcal{S}(X_l, X_r)\}$$

where $\mathcal{S}$, the loop vertex action is

$$\mathcal{S}(X_l, X_r) = -\mathrm{Tr}_{lr} \log \Big[\mathbf{1}_{lr} + \lambda \sum_{k=0}^{p-1} A^k(X_l) \otimes_{lr} A^{p-1-k}(X_r) \Big].$$

The N_l by N_l matrix $A^k(X_l)$ acts on the left index of $\mathcal{H}_{lr}$ and the N_r by N_r matrix $A^{p-1-k}(X_r)$ acts on the right index of $\mathcal{H}_{lr}$, where $\mathcal{H}_l$ is the Hilbert space with $\dim \mathcal{H}_l = N_l$ and $\mathcal{H}_r$ is the Hilbert space with $\dim \mathcal{H}_r = N_r$.

This Lemma is proved by a change of variables $M \to P$ and we define $P(M)$ (up to unitary conjugation) through the implicit function formal power series equation:

$$Y_l := PP^\dagger, \quad Y_r := P^\dagger P, \quad X_l := A(Y_l), \quad X_r := A(Y_r).$$

The key result of [Krajewski-R.-Sazonov,2019]

Theorem

For any $\epsilon > 0$ there exists η small enough such that the expansion defined by $F(\lambda, N) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} A_{\mathcal{T}}$ is absolutely convergent and defines an analytic function of λ, uniformly bounded in N, in the "pacman domain"

$$P(\epsilon, \eta) := \{0 < |\lambda| < \eta, |\arg \lambda| < \pi - \epsilon\},$$

a domain which is uniform in N. Here absolutely convergent and uniformly bounded in N means that for fixed ϵ and η as above there exists a constant K independent of N such that for $\lambda \in P(\epsilon, \eta)$

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} |A_{\mathcal{T}}| \leq K < \infty.$$

where $\mathcal{T}$ is a LVR tree, $A_{\mathcal{T}}$ is the corresponding amplitude and $\mathfrak{T}_n$ is the set of LVR trees with n vertices.

But it applies only to the simplest matrix with interaction $\lambda(\bar{M}M)^p$. We thought that to deduce the case of Hermitian or symmetric matrices would be relatively easy. In fact, it took us two years to understand and write down the corresponding article!

The work of [Krajewski-R.-Sazonov,2021]

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- Since this paper is a sequel to [Krajewski-R.-Sazonov,2019] we would like to stress that the improved method introduced in this paper is both *simpler and more powerful*. The basic formalism is still the LVR, joined to Cauchy holomorphic matrix calculus as in [Krajewski-R.-Sazonov,2019]. But when [Krajewski-R.-Sazonov,2019] used contour integral parameters attached to every *vertex* of the loop representation, this paper introduces more contour integrals, one for each loop vertex *corner*. This results in simpler bounds for the norm of the corner operators.

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- For the Hermitian case it uses the one-to-one change of variables (not singular for λ real positive)

$$K := H\sqrt{1+\lambda^{p-1}H^{2p-2}}, \quad K^2 = H^2 + \lambda^{p-1}H^{2p}.$$

The key result of [Krajewski-R.-Sazonov,2021]

Theorem

For any $\epsilon > 0$ there exists η small enough such that the expansion in LVR trees is absolutely convergent and defines an analytic function of λ, uniformly bounded in N, in the uniform in N in a "pacman domain"

$$P(\epsilon,\eta) := \left\{ 0 < |\lambda| < \eta, |\arg \lambda| < \frac{\pi}{2} + \frac{\pi}{p-1} - \epsilon \right\},$$

More precisely, for fixed ϵ and η as above there exists a constant $O(1)$, independent of N such that for $\lambda \in P(\epsilon,\eta)$

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} |A_{\mathcal{T}}| \le O(1) < \infty\,.$$

where $\mathcal{T}$ is again a LVR tree, $A_{\mathcal{T}}$ is the corresponding amplitude, and $\mathfrak{T}_n$ is the set of LVR trees with n vertices.

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- So I followed the referee's advice. My present aim is to publish first that $\log Z(\lambda, N, J)$, the generated function of the cumulants, is an analytic function inside a cardioid domain in the complex plane, In addition I want to prove the Borel-LeRoy summability at the origin of the coupling constant uniformly in the appropriate norm of $|JJ^\dagger|$.

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- In a second time, I aim at constructing scalar cumulants by Weingarten calculus and obtain explicit and convergent expansions for these scalar cumulants, so as to prove their analyticity and Borel summability, again in the appropriate norm of $|JJ^\dagger|$ *uniformly when* $N \to \infty$.

Variational LVE for Cumulants

I actually work with Vasily Sazonov to combines the LVE, the calculus of variation of a very simple form (in the form of a mass renormalisation) and the idea the cumulants.

- Applying the idea of choosing the initial approximation depending on the coupling constant, we construct the analytic continuation of the cumulants of the quartic matrix model beyond the standard LVE cardioid over the branch cut and for arbitrary large couplings.

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- It is non-trivial extension because of the sources $\bar{J}, J$ created some difficulties. It maybe perhaps solved by using different colors, as in the tensors.
- We want to extend the simple model to Hermitian matrices, the group being $U(N)$, symmetric matrices $O(N)$, symplectic matrices $Sp(N)$, or to the ten different Gaussian random-matrix ensembles of [Altland-Zirnbauer,2001].

In the more distant future

For the more distant future I want to concentrate in hard problems and to do a list of some seven or eight unsolved problems in constructive field theory or constructive condensed matter...

Thanks for your attention!