Universality of eigenvalue distributions of random tensors

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Mainly based on N. Delporte, G. La Scala, NS, R. Toriumi, in preparation

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§Introduction

Eigenvalue distributions of random matrices play important roles in various applications

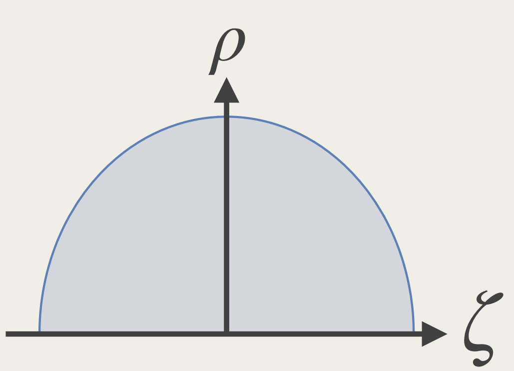
$$M_{ab} v_b = \zeta v_a \qquad \zeta : \text{eigenvalues} \qquad M : \text{Random matrix}$$

An important property of them is the universalities in $N \to \infty$.

There are various types of universalities known.

Ex. Universality of Wigner's semicircle law

Laszlo Erdos, Universality of Wigner random matrices: a Survey of Recent Results, arXiv:1004.0861



The most basic would be

The eigenvalue distributions of real, complex, quartanion Gaussian random matrices (GOE,GUE,GSE) lead to the semi-circle law.

In this talk we want to show a result which may be the corresponding basic statement of the eigenvalue distributions of random tensors

We obtain the eigenvalue distributions of the Gaussian random tensors of various types (real/complex, symmetric/anti-symmetric/no-symmetries, etc.) have the universal form in $N \to \infty$

$$\rho(z) \sim e^{NB\, h_p(z_c^2/z^2)}$$

z : Tensor eigenvalue

$h_p(\,\cdot\,)$: Universal function

NB : Total dimensions of eigenvectors

z_c : Phase transition point of QFT

§ Tensor eigenvalues/vectors

Qi, Lim, 2005, Cartwright-Sturmfels 2013

Terminologies

Tensor : $T_{a_1 a_2 \cdots a_p}$ $\quad a_i = 1, 2, \cdots, N$ $\quad N$: dimension

p : order $\quad$ For simplicity, all a_i have common N

Real tensor : $T_{a_1 a_2 \cdots a_p} \in \mathbb{R}$

Complex tensor : $T_{a_1 a_2 \cdots a_p} \in \mathbb{C}$

Symmetric tensor : $T_{a_1 a_2 \cdots a_p} = T_{\sigma(a_1 a_2 \cdots a_p)}$ $\quad \sigma$: permutations of a_i

Anti-symmetric tensor : $T_{a_1 a_2 \cdots a_p} = \mathrm{sgn}(\sigma)\, T_{\sigma(a_1 a_2 \cdots a_p)}$

Simplest example of eigenvalue/vector equation

Real eigenpair equation of real symmetric tensor of order $p = 3$

$$T_{abc} w_b w_c = z\, w_a \qquad T \in \mathrm{sym}\left(\mathbb{R}^N \otimes \mathbb{R}^N \otimes \mathbb{R}^N\right)$$

$z \in \mathbb{R}$: Eigenvalue

$w \in \mathbb{R}^N,\ |w| = 1$: Eigenvector

Because of non-linearity (unlike for matrix), we may instead consider an eigenvector equation by absorbing z to eigenvector

$$T_{abc} v_b v_c = v_a \qquad v \in \mathbb{R}^N \qquad \left(v = \frac{1}{z} w\right)$$

Eigenvalue $z = |v|^{-1}$

This is more convenient to treat, and practically equivalent.

Examples of applications

Perturbations in gravitational dynamics *O.Evnin, 2104.09797*

$$i\frac{d\alpha_n}{dt} = T_{nmkl}\,\bar{\alpha}_m\alpha_k\alpha_l$$

Susy/D-brane dynamics *Biggs, Maldacena, Narovlansky, 2309.08818*

$$W = T_{a_1a_2\cdots a_p}\phi^{a_1}\phi^{a_2}\cdots\phi^{a_p}$$

Spin glasses (p-spin spherical model) *Crisanti, Sommers, 1992*

$$H = T_{a_1a_2\cdots a_p}w_{a_1}w_{a_2}\cdots w_{a_p}, \quad |w| = 1$$

Quantum information theory *c.f. Book of Qui, Chen, Chen*

$$|\Psi\rangle = T_{a_1a_2\cdots a_p}\,|a_1\rangle_1\otimes|a_2\rangle_2\cdots\otimes|a_p\rangle_p$$: Multi-partite states

Artificial intelligence, Data analysis, ….

Look similar to matrix eigen equations, but properties are largely different

The number of eigenvalues is $\sim e^{const. N} \gg N$

Cartwright-Sturmfels 2013

Computing eigenvalues/vectors of tensors is NP-hard.

Hillar-Lim 2009

Varieties of eigenvalue/vector equations. More distinct than matrices

real/complex, symmetric/anti-symmetric/no-symmetries, etc.

§ Various tensor eigenvector equations

- Real eigenvector of real symmetric tensor

$$T_{a_1 a_2 \cdots a_p} v_{a_2} v_{a_3} \cdots v_{a_p} = v_{a_1} \qquad \text{Eigenvalue} : z = |v|^{2-p}$$

- Real eigenvectors of real tensor with independent indices

  There are p eigenvectors, one for each index.

  Consistency requires $v^{(i)}$ have the same size $|v^{(i)}| = |v^{(j)}|$.

$$T_{a_1 a_2 \cdots a_p} v^{(2)}_{a_2} v^{(3)}_{a_3} \cdots v^{(p)}_{a_p} = v^{(1)}_{a_1}$$

$$T_{a_1 a_2 \cdots a_p} v^{(1)}_{a_1} v^{(3)}_{a_3} \cdots v^{(p)}_{a_p} = v^{(2)}_{a_2} \qquad z = |v^{(i)}|^{2-p}$$

$$\vdots$$

$$T_{a_1 a_2 \cdots a_p} v^{(1)}_{a_1} v^{(2)}_{a_2} \cdots v^{(p-1)}_{a_{p-1}} = v^{(p)}_{a_p}$$

- Real eigenvectors of real anti-symmetric tensor

N. Delporte, G. La Scala, NS, R. Toriumi, in preparation

There exist p real eigenvectors. Consistency requires eigenvectors are orthogonal and have the same size $v^{(i)} \cdot v^{(j)} = 0,\ |v^{(i)}| = |v^{(j)}|$

$$\frac{1}{(p-1)!}\epsilon_{i_1 i_2 \cdots i_p} T_{a_1 a_2 \cdots a_p} v_{a_2}^{(i_2)} v_{a_3}^{(i_3)} \cdots v_{a_p}^{(i_p)} = v_{a_1}^{(i_1)} \qquad z = |v^{(i)}|^{2-p}$$

- Complex eigenvector of complex symmetric tensor

S. Majumder, NS, PTEP 2024 (2024) 9, 093A01, 2408.01030 [hep-th]

$$T_{a_1 a_2 \cdots a_p} v_{a_2} v_{a_3} \cdots v_{a_p} = v_{a_1}^{*}$$

$*$: complex conjugation

$z = |v|^{2-p}$

(There is also a holomorphic version $T_{a_1 a_2 \cdots a_p} v_{a_2} v_{a_3} \cdots v_{a_p} = v_{a_1}$)

- Complex eigenvectors of complex tensor with independent indices

NS, PTEP 2024 (2024) 5, 053A04, arXiv:2404.03385 [hep-th]

There are *p* complex eigenvectors, one for each index.

Consistency requires eigenvectors have the same size $|v^{(i)}| = |v^{(j)}|$

$$T_{a_1a_2\cdots a_p}v^{(2)}_{a_2}v^{(3)}_{a_3}\cdots v^{(p)}_{a_p} = v^{(1)*}_{a_1}$$

$$T_{a_1a_2\cdots a_p}v^{(1)}_{a_1}v^{(3)}_{a_3}\cdots v^{(p)}_{a_p} = v^{(2)*}_{a_2}$$

$$z = |v^{(i)}|^{2-p}$$

$$\vdots$$

$$T_{a_1a_2\cdots a_p}v^{(1)}_{a_1}v^{(2)}_{a_2}\cdots v^{(p-1)}_{a_{p-1}} = v^{(p)*}_{a_p}$$

This case is important for quantum information theory

$|\Psi\rangle = T_{a_1a_2\cdots a_p}|a_1\rangle_1 \otimes |a_2\rangle_2 \cdots \otimes |a_p\rangle_p$: Multipartite states

§ Eigenvector distributions of random tensors

We rewrite distributions of eigenvectors of random tensors into partition functions of quantum field theories

Then we use techniques of quantum field theories to compute distributions

The QFT method is powerful, intuitive, and general, and reveals properties of random tensors which have not been known.

QFT methods

The eigenvector equations are generally

$$f_i(v, T) = 0 \ (i = 1,2,\ldots,m)$$

v : Eigenvectors

T : Tensor

Linear in T

Then the distribution of the eigenvectors of a tensor T is

$$\rho(v, T) = \sum_{\alpha=1}^{\#\mathrm{sol}} \delta\left(v - v^{\alpha}(T)\right)$$

$v^{\alpha}(T)$: Solutions of eigenvector equations for T

$$= |\det J(v, T)| \prod_{i=1}^{m} \delta\left(f_i(v, T)\right)$$

$$J(v, T)_{ai} = \frac{\partial f_i(v, T)}{\partial v_a}$$

: Jacobian for change of variables of the delta functions

Then the distribution for Gaussian random tensor is

$$\rho(v) = \frac{1}{\mathcal{N}} \int dT\, e^{-\alpha\, |T|^2} \rho(v, T) \qquad \mathcal{N} = \int dT\, e^{-\alpha |T|^2}$$

$$= \frac{1}{\mathcal{N}} \int dT\, e^{-\alpha\, |T|^2} |\det J(v, T)| \prod_{i=1}^{m} \delta\big(f_i(v, T)\big)$$

$$\rightarrow \delta(x) = \frac{1}{2\pi} \int d\lambda\, e^{i\lambda x}$$

Distinct treatments of $|\det J(v, T)|$ lead to two kinds of distributions

- $\rho_{\text{signed}}(v)$: Signed distribution. Easier to compute. Not positive def.

$$|\det J(v, T)| \rightarrow \det J(v, T) = \int d\bar{\psi} d\psi\, e^{\,\bar{\psi} J(v,T)\, \psi} \qquad \bar{\psi}, \psi \text{ :Fermions}$$

- $\rho_{\text{genuine}}(v)$: Genuine distribution. Harder but genuine

$$|\det J(v, T)| = \lim_{\epsilon \to +0} \frac{\det(J^T J + \epsilon I)}{\sqrt{\det(J^T J + \epsilon I)}} \qquad \begin{matrix} \longrightarrow \text{Fermions} \\ \longrightarrow \text{Bosons} \end{matrix}$$

QFT expressions

$$\rho_{\text{signed}}(v) = \frac{1}{\mathcal{N}} \int dT d\lambda d\bar{\psi} d\psi \, e^{S_{\text{signed}}}$$

$$S_{\text{signed}} = -\alpha \, |T|^2 + i\,\lambda_j f_j(v,T) + \bar{\psi}_a \frac{\partial f_i(v,T)}{\partial v_a} \psi_i \quad (+S_{GF})$$

$$\rho_{\text{genuine}}(v) = \frac{1}{\mathcal{N}} \int dT d\lambda d\phi d\bar{\psi} d\psi \, e^{S_{\text{genuine}}}$$

$$S_{\text{genuine}} = -\alpha \, |T|^2 + i\,\lambda_j f_j(v,T)$$

$$+\bar{\psi}, \psi, \phi \text{ quadratically coupled with } \frac{\partial f_i(v,T)}{\partial v_a}$$

$$(+S_{GF})$$

(Gauge-fixing S_{GF} needed for complex or anti-symmetric case)

§ Computing QFT expressions

Because the eigenvector equations $f_i(v, T) = 0$ are linear in T, the integration over T, λ are Gaussian, and can be done explicitly.

Then we obtain

$$\rho_{\text{signed}}(|v|) = g(|v|)\, Z_{\text{F}}(|v|)$$

(Maybe overall minus sign)

$$\rho_{\text{genuine}}(|v|) = g(|v|)\, Z_{BF}(|v|)$$

$$Z_{\text{F}}(|v|) = \int d\bar{\psi} d\psi\, e^{S_{\text{F}}}$$

Partition function of QFT of fermions with four-fermi interactions. Coupling $\propto |v|^{2(p-2)}$

$$Z_{\text{BF}}(|v|) = \int d\bar{\psi} d\psi d\phi\, e^{S_{\text{BF}}}$$

Partition function of qft of bosons and fermions with four-body interactions. Coupling $\propto |v|^{2(p-2)}$

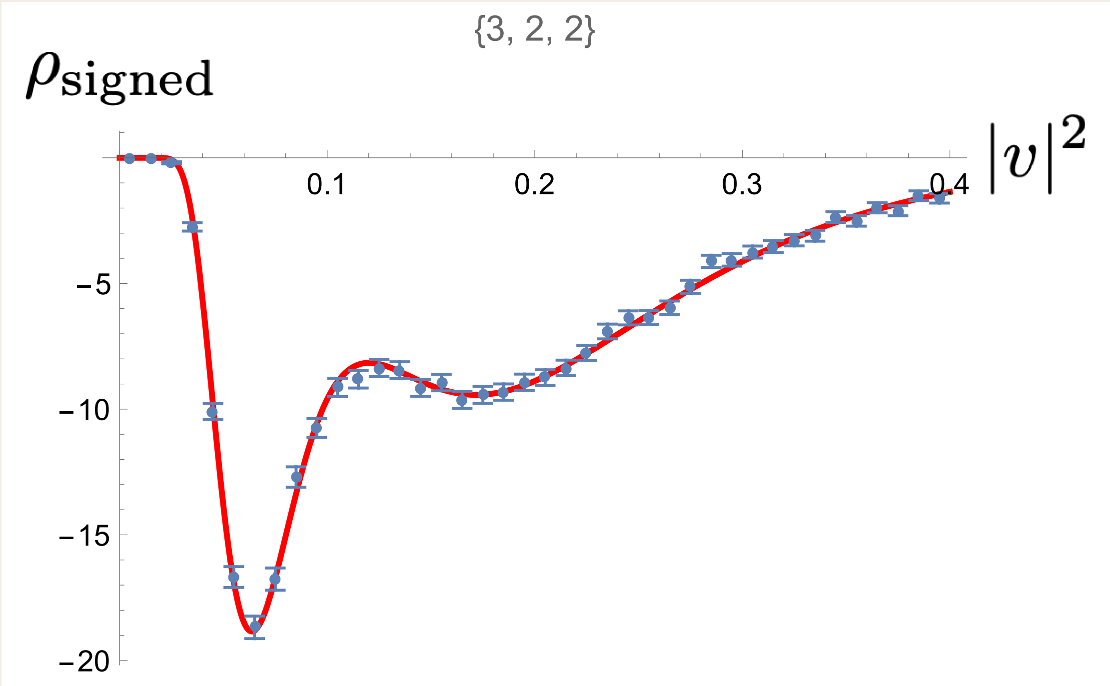
Ex. Complex random tensor of order $p = 3$ and dimension N

$$S_{\mathrm{F}} = \sum_{i=1}^{3} \left(\bar{\psi}_i^1 \cdot \psi_i^1 + \bar{\psi}_i^2 \cdot \psi_i^2\right) + \frac{|v|^2}{\alpha} \sum_{i<j}^{3} \left(\bar{\psi}_i^1 \otimes \psi_j^2 + \bar{\psi}_j^1 \otimes \psi_i^2\right) \cdot \left(\bar{\psi}_i^2 \otimes \psi_j^1 + \bar{\psi}_j^2 \otimes \psi_i^1\right)$$

$\bar{\psi}_j^i, \psi_j^i$: $N-1$ dimensional fermions

By expanding S_{F} in terms of four-fermi interactions, one can always explicitly compute Z_{F} :

$$\rho_{\mathrm{signed}}(|v|) = g(|v|) \cdot (\text{Polynomial fn. of } x = |v|^{2(p-2)} \text{ of finite order})$$



Comparisons with Monte Carlo simulations give good crosschecks

On the other hand, computing $Z_{\rm BF}$ for $\rho_{\rm genuine}$ seems challenging for finite N, because of presence of bosons. (Except real symmetric case)

There is partial susy, but still seems difficult to compute.

In the large-N limit, however, one can rely on some saddle point approximations.

(i) A method based on Schwinger-Dyson equations is powerful.

(ii) Picard-Lefschetz theory gives a rigid method.

(iii) Both $Z_{\rm F}(|v|)$ and $Z_{BF}(|v|)$ have a phase transition point at the same value $|v| = |v|_c$, separating the weak&strong coupling regimes

Illustration of Schwinger-Dyson method

$$S_F = \bar{\psi} \cdot \psi - g(\bar{\psi} \cdot \psi)^2 \qquad g \propto |v|^{2(p-2)}$$

$\langle \bar{\psi}_a \psi_b \rangle = Q\,\delta_{ab}$: working assumption

$$S_{\mathrm{eff}}(Q) = \langle S_{\mathrm{F}} \rangle - N \log Q \sim N\left(Q - \tilde{g}Q^2 - \log Q\right) \qquad \tilde{g} = gN$$

$$\frac{\partial S_{\mathrm{eff}}(Q_*)}{\partial Q_*} = 0 \quad \Rightarrow \quad Z_F \sim e^{N S_{\mathrm{eff}}(Q_*)} \qquad Q_* = \frac{1 - \sqrt{1 - 8g}}{4g}$$

Phase transition at $g = \dfrac{1}{8}$

In $N \to \infty$ we found a universal form across various random tensors (complex/real, symmetric/anti-symmetric/no symmetries)

N. Delporte, G.L. Scala, NS, R. Toriumi, in preparation

$$\rho_{\text{signed}}(z) \sim \text{Re}\left[e^{NB\, h_p(z_c^2/z^2)}\right]$$

$$\rho_{\text{genuine}}(z) \sim e^{NB\, \text{Re}[h_p(z_c^2/z^2)]}$$

z : Eigenvalue $= |v|^{2-p}$

z_c : Phase transition point of QFT of each case

$$h_p(x) = \frac{1}{2}\log(p-1) + \frac{1}{x}\left(-1 + \frac{2}{p} - \sqrt{1-x}\right) + \frac{1}{2}\log x - \log(1-\sqrt{1-x})$$

NB : Total dimensions of eigenvectors

$h_p(\,\cdot\,)$ is universal, only depends on the order p of tensor (#indices)

NB and z_c are not universal, dependent on each tensor eigen problem.

List of non-universal parameters

Types of tensor	NB	z_c^2
Real symmetric	N	$2N(p-1)/(\alpha p)$
Real, indep indices	Np	$2N(p-1)/\alpha$
Real anti-symmetric	Np	$2N/(\alpha p(p-2)!)$
Complex, indep indices	$2Np$	$4N(p-1)/\alpha$
Complex symmetric	$2N$	$4N(p-1)/(\alpha p)$

§Summary and future problems

Eigenvalue distributions of random tensors can explicitly be computed by QFT methods. QFT is powerful and general.

We have found a universal expression of the eigenvalue distributions of Gaussian random tensors across various types of tensors.

This would be a starting point toward showing universalities of random tensor eigenvalue distributions.

A next question would be to study non-Gaussian random tensors.