Quantum Variational Calculus on Quantum Spacetime

Shahn Majid (QMUL)

Corfu/Calista 25

- Joint work with Francisco Simao arXiv:2508.02628

Problem: To formulate physics on quantum spacetimes we need variational calculus

- to connect the path integral to Hamilton quantisation
- at classical level: derive Euler-Lagrance eqm and Noether charges

Solved Today: for a scalar field on a lattice / discrete Abelian group as spacetime

- Exactly conserved charges, the lattice is not merely an approx but a discrete NCG could even be finite
- For example on $\mathbb{Z}$ if ϕ obeys the discrete wave equation $(\Delta_{\mathbb{Z}} + m^2)\phi = 0$ then $E[\phi] = -\frac{1}{2}(\partial_+\phi)(\partial_-\phi) + \frac{m^2}{2}\phi^2$ is constant

I Graphs as quantum differential geometry

Propn: X a set, $A = \mathbb{C}(X)$, $\Omega^1, *$ ⟷ bidirected graph with vertices X

$$\Omega^1 = \mathrm{span}_{\mathbb{C}}\{e_{x\to y}\} \qquad f.e_{x\to y} = f(x)e_{x\to y}, \quad e_{x\to y}.f = e_{x\to y}f(y)$$

$$\mathrm{d}f = \sum_{x\to y}(f(y)-f(x))e_{x\to y} \qquad e^*_{x\to y} = -e_{y\to x}$$

- metric $g = \sum_{x\to y} g_{x\to y}e_{x\to y}\otimes e_{y\to x} \in \Omega^1\otimes_A\Omega^1;\quad g_{x\to y}\in\mathbb{R}\setminus\{0\}$

edge symmetric if $g_{x\to y} = g_{y\to x}$ ➡ real \`square-length' on each edge

$T_A\Omega^1$ = Path algebra, in degree *i* $\Omega^{1\otimes_A i} = \{e_{x_1\to x_2}\otimes e_{x_2\to x_3}\cdots\otimes e_{x_{i-1}\to x_i}\}$

- $\Omega_{min} = T_A\Omega^1/$ relations $\sum_{y:p\to y\to q} e_{p\to y}\wedge e_{y\to q} = 0 \quad \forall p,q$

may take further relations eg for Cayley graph on a discrete group

- Cayley graph on discrete group X=G generated by $\mathcal{C} \subseteq G\backslash\{e\}$

Arrows = $\{x \to xa,\ a \in \mathcal{C}\}$ ➡ $\Omega^1 = \mathbb{C}(X)\{e^a\}$ for basic 1-forms (`tetrad')

$$e^a = \sum_{x\to xa} \delta_x \mathrm{d}\delta_{xa}, \qquad e^a f = R_a(f)e^a, \quad R_a(f)(x) = f(xa)$$
$$\mathrm{d}f = \partial_a f)e^a, \quad \partial_a = R_a - \mathrm{id}$$

➡ (a) $\Omega = \mathbb{C}(G)\Lambda$

$\Lambda =$ Grassmann algebra on $\{e^a\}$ if G abelian

(b) $g = \sum_a g_a e^a \otimes e^{a^{-1}}$

metric tensor forced to be `antidiagonal'

- For example on $\mathbb{Z}$, $\Omega^1 = \mathbb{C}(\mathbb{Z})\{e^\pm\}$,

$$e^\pm f = (R_\pm f)e^\pm$$
$$(R_\pm f)(i) = f(i \pm 1)$$
$$(\partial_\pm f)(i) = f(i \pm 1) - f(i))$$

Flat metric $g = e^+ \otimes e^- + e^- \otimes e^+$

$\nabla e^\pm = 0$ Laplacian $\Delta_{\mathbb{Z}} = -\frac{1}{2}(\ ,\ )\nabla \mathrm{d} = (\partial_+ + \partial_-)f$

$$(\Delta_{\mathbb{Z}} f)(i) = f(i+1) + f(i-1) - 2f(i)$$

QRG fully worked out for any edge-symmetric metric on $\mathbb{Z}$, zero curvature iff $g_+(i) = \alpha^i$ a geometric sequence [SM Class. Quantum Grav 36 (2019)]

II Recap of jet bundles and classical varcalc on M

Space of fields $\phi \in F = C^\infty(M)$, $j_\infty : F \to \mathscr{J}^\infty$ sections of jet bundle

$$J^\infty = M \times \mathbb{R}^{\mathbb{N}} = \{(x, u, u_i, u_{ij}, \cdots)\},$$

$$j_\infty(\phi)(x) = (x, \phi(x), \partial_i\phi(x), \partial_i\partial_j\phi(x), \cdots) = \{(x, \partial_I\phi)\} \qquad u_I(j_\infty(\phi)(x)) = (\partial_I\phi)(x)$$

➡ $$e_\infty : M \times F \to J^\infty, \quad (x,\phi) \mapsto j_\infty(\phi)(x)$$

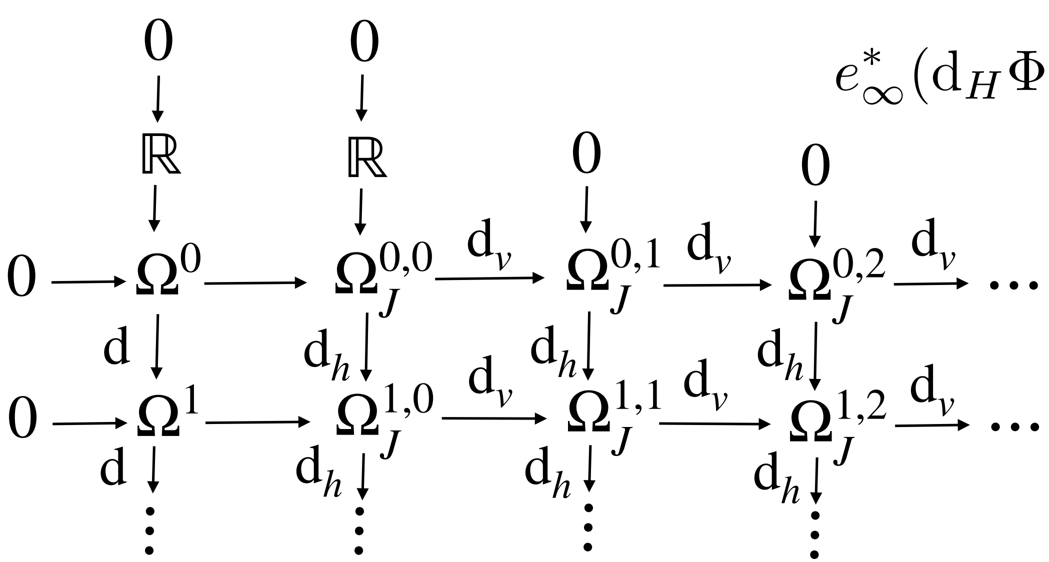
➡ $$e_\infty^* : \Omega(J^\infty) \hookrightarrow \Omega(M)\underline{\otimes}\Omega(F)$$

➡

e.g. $\Omega^1(J^\infty) = C^\infty(J^\infty)\{\mathrm{d}x^i, \mathrm{d}u_I\}$

$$e_\infty^*(\mathrm{d}_H\Phi) = \mathrm{d}_M e_\infty^*(\Phi), \qquad e_\infty^*(\mathrm{d}_V\Phi) = \mathrm{d}_F e_\infty^*(\Phi)$$

$$\Phi \in C^\infty(J^\infty)$$



Anderson variational bicomplex

➡ $$\mathrm{d}_H\Phi = (D_i\Phi)\mathrm{d}x^i$$

$$D_i\Phi := \partial_i\Phi + \sum_I \frac{\partial\Phi}{\partial u_I} u_{iI}.$$

$$\mathrm{d}_V\Phi = \sum_I \frac{\partial\Phi}{\partial u_I}(\mathrm{d}u_I - u_{iI}\mathrm{d}x^i)$$

Lagrangian $L\mathrm{Vol} \in \Omega^{\mathrm{top},0}(J^\infty)$ $L = L(u, u_i) \in C^\infty(J^\infty)$

Action $S[\phi] := \int_M e_\infty^*(L\mathrm{Vol})(x,\phi)$

→ $\mathrm{d}_F S[\phi] = \int_M e_\infty^*(\mathrm{d}_V L \wedge \mathrm{Vol})(x,\phi)$ $\mathrm{d}_V L \wedge \mathrm{Vol} = EL - \mathrm{d}_h \Theta$

EL-form $EL \in \Omega^{\mathrm{top},1}(J^\infty)$

zero var ↔ $EL = 0$

boundary form $\Theta \in \Omega^{\mathrm{top}-1,1}(J^\infty)$

$$EL = \left(\frac{\partial L}{\partial u} - D_i\left(\frac{\partial L}{\partial u_i}\right)\right)\mathrm{d}_V u \wedge \mathrm{Vol}, \qquad \Theta = \frac{\partial L}{\partial u_i}\mathrm{d}_V u \wedge \mathrm{Vol}_i \qquad \mathrm{Vol}_i := \iota_{\partial_i}\mathrm{Vol}$$

Any vector field $X_E = X^i\partial_i + X\frac{\partial}{\partial u}$ on $E = M \times \mathbb{R} \to M$ extends to

$X_\infty = X^i\partial_i + X^I\frac{\partial}{\partial u_I} = X_H + X_V$ on J^∞

Symmetry iff $\mathcal{L}_{X_\infty}(L\mathrm{Vol}) = \mathrm{d}_H \sigma_X$ for some $\sigma_X \in \Omega^{\mathrm{top}-1,0}(J^\infty)$

→ Noether thm $\mathrm{d}_H j_X = \iota_{X_V} EL$ i.e. conserved on shell

where $j_X := \sigma_X - \iota_{X_H}(L\mathrm{Vol}) - \iota_{X_V}\Theta \in \Omega^{\mathrm{top}-1,0}(J^\infty)$

III varcalc on $M=\mathbb{Z}$, EL eqm and Noether theorem

$$J^\infty = \mathbb{Z}\times\mathbb{R}^\infty = \mathbb{Z}\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^3\times\dots$$ coordinates (i, u_I) $\qquad e^I \in Sym(\{e^\pm\})$

$$j_\infty(\phi) = \phi + \partial_a\phi e^a + \cdots = \sum_I \partial_I\phi\, e^I$$ $\qquad 1,\ e^\pm,\ e^+e^+,\ e^+e^- + e^-e^+,\ e^-e^-, \cdots$

- $e_\infty : \mathbb{Z}\times F \to J^\infty \qquad e_\infty(\phi)(i) = (i, (\partial_I\phi)(i))$

- $e_\infty^* : \Omega(J^\infty) \subset \Omega(\mathbb{Z})\underline{\otimes}\Omega(F)$

- **Thm** noncommutative double complex $\Omega^{\cdot,\cdot}(J^\infty)$ with

$$\mathrm{d}_H\Phi = [\theta,\Phi], \qquad \Phi\in C(J^\infty) = C(\mathbb{Z})[u_I] \qquad \theta := e^+ + e^-$$

$$\mathrm{d}_V\Phi = \sum_I \frac{\partial\Phi}{\partial u_I}\mathrm{d}_V u_I, \quad \mathrm{d}_V u_I = \mathrm{d}u_I - [\theta, u_I]$$

$$[e^a, u_I] = u_{aI}e^a, \qquad [\mathrm{d}u_I, \phi] = \sum_a u_{aI}(\partial_a\phi)e^a, \quad [\mathrm{d}u_I, u_J] = \sum_a u_{aI}u_{aJ}e^a$$

$$\{e^a, \mathrm{d}u_I\} + \mathrm{d}u_{aI}\wedge e^a = 0, \quad \{\mathrm{d}u_I, \mathrm{d}u_J\} + \sum_a (u_{aI}\mathrm{d}u_{aJ} + (\mathrm{d}u_{aI})u_{aJ})\wedge e^a = 0.$$

- For $\omega\in\Omega(J^\infty)$ $\qquad e^a\omega = (-1)^{|\omega|}R_a(\omega)e^a, \qquad \mathrm{d}_H\omega = (-1)^{|\omega|}D_a(\omega)e^a \qquad D_a = R_a - \mathrm{id}$

  defines D_a

$$L = L(u, u_a) \qquad \mathrm{Vol} = e^+ \wedge e^- \qquad \mathrm{Vol}_+ = e^-,\ \mathrm{Vol}_- = -e^+)$$

Thm $EL = \left(\frac{\partial L}{\partial u} + D_{a^{-1}}\left(\frac{\partial L}{\partial u_a}\right)\right) \mathrm{d}_V u \wedge \mathrm{Vol}, \qquad \Theta = R_{a^{-1}}\left(\frac{\partial L}{\partial u_a}\right) \mathrm{d}_V u \wedge \mathrm{Vol}_a.$

obeys $\mathrm{d}_V(L\mathrm{Vol}) = EL - \mathrm{d}_H \Theta$

Noether theorem

interior product $\iota_H + \iota_V$ on $\Omega(J^\infty)$ $\qquad \iota_H e^a = \epsilon^a, \qquad \iota_V \mathrm{d}_V u_I = -\sum_a \epsilon^a u_{aI}$

$j_0 = \sigma - \iota_H(L\mathrm{Vol}) - \iota_V \Theta$ naive Noether current almost works

Propn for free particle Lagrangian $L = \frac{1}{4}\sum_{a\in\mathcal{C}} u_a^2 - \frac{1}{2}m^2u^2$

$$j = -\sum_{a,b\in\mathcal{C}} \epsilon^b \left(\frac{1}{2}\left(u_{a^{-1}} + \frac{1}{2}u_{ba^{-1}}\right)u_b + \delta_b^a L\right)\mathrm{Vol}_a$$

is conserved on shell
$\mathrm{d}_H j = 0$

extra term

→ $T_{ab} = -\left(\frac{1}{2}u_a u_b + \frac{1}{4}u_{ab}u_b + \delta_b^{a^{-1}} L\right)$ is conserved on shell $\sum_a D_{a^{-1}} T_{ab} = 0$

IV Generalisation to $M = G$ an Abelian group

$\Omega(G) = C(G)\Lambda_G$ $\quad \Lambda_G$ Grassmann algebra on basic 1-forms $e^a,\ a \in G\backslash\{e\}$

$\mathrm{d} = [\theta, \ \}$ for $\theta = \sum_a e^a$ $\quad e^I =$ symmetric products of these $\quad j_\infty(\phi) = \partial_I(\phi)e^I$

$\mathrm{Vol} = e^1 \wedge \cdots \wedge e^{|\mathcal{C}|} = e^a \wedge \mathrm{Vol}_a$ (no sum). and then same formulae as in Thm

Lattice case $\quad G = \mathbb{Z}^m$

$$\mathcal{C} = \{\pm v^i = \pm(0,\ldots,0,1,0,\ldots,0) \in \mathbb{Z}^m | i = 1,\ldots,m\}$$

$$\mathrm{Vol}_{\mathbb{Z}^m} = e^{1+} \wedge e^{1-} \wedge \cdots \wedge e^{m+} \wedge e^{m-}$$

$$S[\phi] = \frac{1}{2}\int_{\mathbb{Z}^m}\left(\frac{1}{2}\sum_{a\in\mathcal{C}}(\partial_a\phi)^2 - m^2\phi^2\right)\mathrm{Vol}_{\mathbb{Z}^m} \qquad L = \frac{1}{4}\sum_{a\in\mathcal{C}} u_a^2 - \frac{1}{2}m^2u^2$$

$$EL = \left(-m^2u + \frac{1}{2}\sum_{a\in\mathcal{C}} u_{aa^{-1}}\right) d_V u \wedge \mathrm{Vol}, \qquad \Theta = -\frac{1}{2}\sum_{a\in\mathcal{C}} u_{a^{-1}} d_V u \wedge \mathrm{Vol}_a$$

$\Rightarrow$ $\left(\sum_{a\in\mathcal{C}}\partial_a + m^2\right)\phi = 0$ $\quad$ etc. as before

$$E = -\frac{1}{2}\int_{\mathbb{Z}^{m-1}}\left((\partial_t\phi)(\partial_{t^{-1}}\phi) + \frac{1}{2}\sum_{b\in\mathcal{C}_{m-1}}(\partial_b\phi)^2 - m^2\right)\mathrm{Vol}_{\mathbb{Z}^{m-1}}$$

$$P_b = -\frac{1}{2}\int_{\mathbb{Z}^{m-1}}(\partial_{t^{-1}}\phi)\,(\partial_b\phi - \partial_{b^{-1}}\phi)\,\mathrm{Vol}_{\mathbb{Z}^{m-1}}.$$

on shell conserved energy & momentum

VI Next steps and related work

- varcalc on any graph: doable but can't solve for a QLC in general to write down actions

- varcalc on noncommutative spacetimes like fuzzy sphere, κ-Mink

$\mathcal{J}^\infty$ (section of jet bundle) not a problem, what is analogue of $\Omega(J^\infty)$?

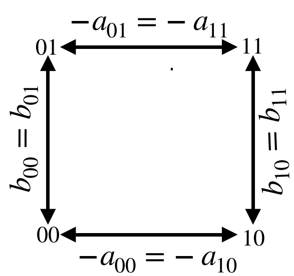
SM & FS Lett. Math. Phys.(2023)

Flood, Mantegaza, Winther, Selecta (2025)

X. Han & SM arXiv: 2507.02848

- conserved charges in lattice gauge theory, lattice gravity?

- Physical applications (i) quantum gravity on one placquette



$\Delta\,|R_{ave}|^2$ → potential solution to problem of cosmological constant

Blitz and SM, Class. Quantum Grav. (2025)

(ii) Lorentzian quantum gravity on fuzzy sphere *A as fibre* → Kaluza-Klein structure of gravity+YM+Liouville on spacetime

Thank you

Liu & SM J. High Energ. Phys. (2024) , arXiv: 2507.21861