Geometric Deep Learning meets Quantum Groups

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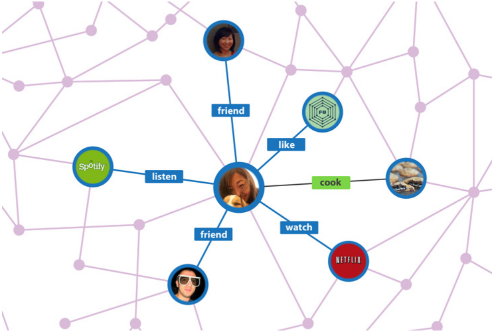


- Machine Learning, Geometric Deep Learning need a theory of differential operators on graphs (meshes).
- Quantum Geometry shows that quantum differential calculus is the right framework to write geometry on graphs.
- Sheaf Neural Networks show greater “expressibility” because of the use of rings of functions versus just function values.

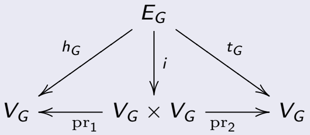
Directed Graphs as Semisimplicial sets

- Δ_+: category with objects the ordered sets $[n] = \{0 < \cdots < n\}$, $n \in \mathbb{N}$, and arrows the injective order preserving maps between them.
- $\Delta_{n,+}$: category with objects the ordered sets $[n] = \{0 < \cdots < n\}$, n fixed.
- $\mathrm{ssSets} := \mathrm{Fun}(\Delta_+^{\mathrm{op}}, \mathrm{Sets})$: category of semisimplicial sets.

Example

diGraphs$= \mathrm{Fun}(\Delta_{1,+}^{\mathrm{op}}, \mathrm{Sets})$*,* $\Delta_{1,+}$ *has objects* $[0] = \{0\}$ *and* $[1] = \{0 < 1\}$*.*

$$
\begin{array}{cccc}
G: & \Delta_{1,+}^{\mathrm{op}} & \longrightarrow & \mathrm{Sets} \\
 & [0] & \mapsto & G([0]) = V_G, \\
 & [1] & \mapsto & G([1]) = E_G, \\
 & [0] \hookrightarrow [1] & \mapsto & h : E_G \to V_G, \quad t : E_G \to V_G
\end{array}
$$



(1)

Attention: *i is not necessarily injective!*

Topology on Graphs

Definition

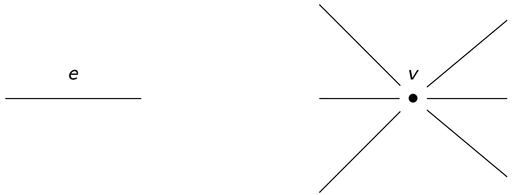
For $G = (V, E)$ we define the poset (partially ordered set) structure:

$$x \leq y \text{ if and only if } x = y \text{ or } x \text{ is a vertex of the edge } y.$$

We define a topology generated by the base of open sets

- $U_v = \{e \in E \mid v \leq e\}$, *that is the open star of* v, *for each vertex* $v \in V$,
- $U_e = \{e\}$, *i.e. the edge* e, *without its vertices, for each* $e \in E$.

Irreducible open sets:



Irreducibles for the dual topology (open are the closed subsets):

Sheaves on Graphs

Theorem (Key Result)

Let X be a topological space. If X has a basis consisting of irreducible open sets, then there is an equivalence between:

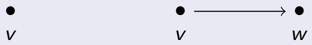
$$\textit{presheaves on irreducible open sets in } X \qquad \Leftrightarrow \qquad \textit{sheaves on } X.$$

Observation

A sheaf of vector spaces on a digraph $G = (E_G, V_G, h_G, t_G)$ for the standard (dual) topology is equivalent to give

- *a vector space $F(v)$ for each vertex $v \in V_G$,*
- *a vector space $F(e)$ for each edge (with its endpoints) $e \in E_G$,*
- *linear maps (restriction maps) $F_{h_G(e) \leq e} : F(e) \to F(h_G(e))$, $F_{t_G(e) \leq e} : F(e) \to F(t_G(e))$ for each edge $e \in E_G$, where, we write $v \leq e$ to mean that v is a vertex of the edge e.*

Observation (Irreducible open sets in the dual standard topology)

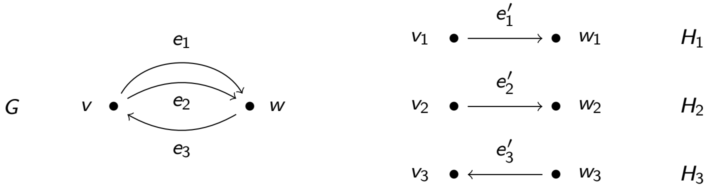


This is the topology and the sheaf definition used in Geometric Deep Learning.

Definition

Let $G \in \mathrm{diGraphs}$. We say that the surjective map $\phi : H \longrightarrow G$ is an **étale directed cover** *if*

1. *H is a disjoint union of graphs in $\mathrm{diGraphs}_{\leq 1}$.*
2. *The arrow $\phi_E : E_H \longrightarrow E_G$ induced by ϕ is bijective when restricted to non self-loops.*

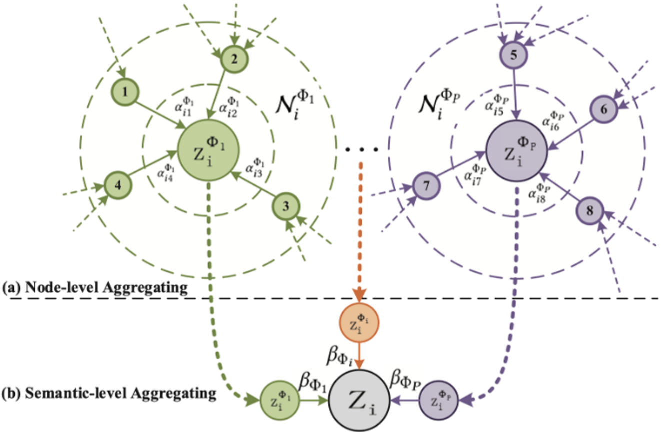


Clearly, given G, such H and ϕ are not unique, but they always exists.

Remark

It is possible to define Grothendieck topologies and étale coverings more general on semisimplicial sets together with their sheaves (sites and topos).

Heterogeneous Graph Attention Network



Ref: "Heterogeneous Graph Attention Network",
https://arxiv.org/pdf/1903.07293

First Order Differential Calculus

Definition

A **first order differential calculus** *(FODC) on an algebra A is* (Γ, d), *where*

i.) Γ *is an A-bimodule.*

ii.) $\mathrm{d}\colon A \to \Gamma$ *is a* **k***-linear map satisfying the Leibniz rule*

$$\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b)$$

for all $a, b \in A$.

iii.) $A \otimes A \to \Gamma$, $a^i \otimes b^i \mapsto a^i \mathrm{d}(b^i)$ *is a (left A-linear and) surjective map.*

Example (Kahler differential, exterior derivative)

Take $A = C^\infty(M)$, *M differentiable manifold,* $\Gamma = \Omega^1(M)$.

$$\mathrm{d} : C^\infty(M) \longrightarrow \Omega^1(M), \qquad f \mapsto df$$

In local coordinates:

$$df = \sum \partial_i f^i dx_i$$

First Order Differential Calculus on diGraphs

Let $G = (V, E) \in \mathrm{diGraphs}_{\leq 1}$ (G directed with at most one edge per direction).

$$A := \mathbf{k}[V] = \mathrm{span}\{\delta_x \mid x \in V\}, \tag{2}$$

where $\delta_x(y) = 1$ if $x = y$ and zero otherwise.

Definition

We define a FODC (Γ^1, d)*, on* $A = \mathbf{k}[V]$

$$\Gamma^1 := \mathbf{k}[E] = \mathrm{span}\{\omega_{x\to y} \,|\, (x, y) \in E\}$$

The A-bimodule structure is given by:

$$f\omega_{x\to y} = f(x)\omega_{x\to y}, \quad \omega_{x\to y}f = \omega_{x\to y}f(y), \quad \mathrm{d}f = \sum_{x\to y\in E} (f(y) - f(x))\omega_{x\to y}$$

We define $\mathrm{d} : A \longrightarrow \Gamma^1$ *on generators as:*

$$\mathrm{d}\delta_x = \sum_{y:y\to x} \omega_{y\to x} - \sum_{y:x\to y} \omega_{x\to y}, \quad \delta_x \mathrm{d}\delta_y = \begin{cases} -\sum_{z:x\to z} \omega_{x\to z} & x = y \\ \omega_{x\to y} & x \to y \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

This FODC is **inner**, i.e. $\mathrm{d}a = [\theta, a]$ for all $a \in A$, where

$$\theta := \sum_{x\to y\in E} \omega_{x\to y}$$

FODC on multi-edge graphs via Étale coverings

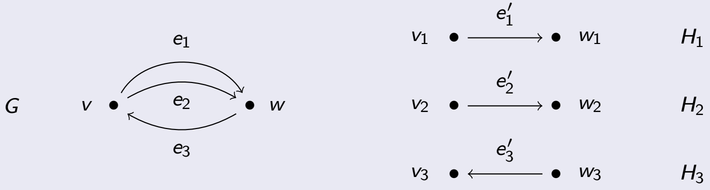
We can extend the theory of FODC to the case of multi-edge graphs.
We illustrate it by an example.

Example

Consider the graph G and its étale covering

$$f : H = H_1 \amalg H_2 \amalg H_3 \longrightarrow G$$

with $G, H \in \mathrm{diGraphs}$ (self-loops are not depicted)



We have that $\Gamma^1_G := \mathrm{span}\{\omega_{v_1 \to w_1}, \omega_{v_2 \to w_2}, \omega_{w_3 \to v_3}\}$, *and*

$$\mathrm{d}(a) = \mathrm{d}_1(f^*(a)_{|V_{H_1}}) + \mathrm{d}_2(f^*(a)_{|V_{H_2}}) + \mathrm{d}_3(f^*(a)_{|V_{H_3}}) \in \Gamma^1 = \Gamma^1_{H_1} \oplus \Gamma^1_{H_2} \oplus \Gamma^1_{H_3}$$

So, for example, if $a = \delta_v$, $\mathrm{d}(\delta_v) = \mathrm{d}_1(\delta_{v_1}) + \mathrm{d}_2(\delta_{v_2}) + \mathrm{d}_3(\delta_{v_3})$.

Definition (Braune et al. 2017)

A **vector bundle** $\mathcal{F}$ *of rank* n *on a set* V *is an assignment:*

$$v \longrightarrow \mathcal{F}_u, \qquad v \in V$$

where $\mathcal{F}_v$ *is a vector space of dimension* n*. We define the* **frame bundle** Fr*, an assignment:*

$$V \ni v \mapsto \{e_i^v\} \subset \mathcal{F}_v$$

where $\{e_i^v\}$ *is a basis for* $\mathcal{F}_v$*. Moreover we denote with* $1\!\!1_{u,v} : \mathcal{F}_u \longrightarrow \mathcal{F}_v$ *the linear map* $1\!\!1_{u,v}(e_i^u) = e_i^v$*.*

Definition

Let $\mathcal{F}$ *be a vector bundle on* V *and let* $G = (V, E) \in \mathrm{diGraphs}$*.*

- *We define a* **weak parallel transport** *a collection of linear maps* $\mathcal{R}_{e,u\to v} : \mathcal{F}_v \longrightarrow \mathcal{F}_u$*, where* e *is an edge between* u *and* v*.*
- *If* $G \in \mathrm{diGraphs}_{\leq 1}$ *is bidirected, we say that a weak parallel transport is a* **parallel transport** *if each* $\mathcal{R}_{e,u\to v}$ *is invertible and* $\mathcal{R}_{e,u\to v} = \mathcal{R}^{-1}_{e',v\to u}$*.*

Definition (Braune et al. 2017)

We define a **connection** *on a digraph G as a collection of linear maps $\Theta_{e,u\to v} := \mathcal{R}_{e,u\to v} - \mathbb{1}_{v,u}$, on all edges $e \in E$, with $\{\mathcal{R}_{e,u\to v}\}$ a weak parallel transport.*

Once a frame bundle is given, we can write:

$$\mathcal{R}_{e,u\to v} : \mathcal{F}_v \longrightarrow \mathcal{F}_u, \qquad e_i^v \mapsto \mathcal{R}^j_{e,u\to v,i} e_j^v$$

Observation

In the differentiable setting the parallel transport for a vector bundle $E \longrightarrow M$ on a differentiable manifold M is a collection of maps:

$$\Gamma(\gamma)_s^t : E_{\gamma(s)} \to E_{\gamma(t)}$$

It allows us to take the derivative of a section V along a curve γ:

$$\nabla_{\dot\gamma} V = \lim_{h\to 0} \frac{\Gamma(\gamma)_h^0 V_{\gamma(h)} - V_{\gamma(0)}}{h} = \frac{d}{dt}\Gamma(\gamma)_t^0 V_{\gamma(t)}\Big|_{t=0}.$$

Rewrite replacing the curve γ with an edge e between vertices u and v (taking the places of $\gamma(0)$ and $\gamma(h)$) of the graph $G \in \mathrm{Graphs}$.

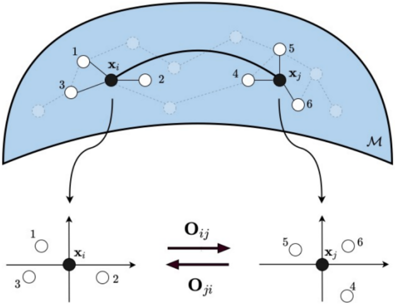
$$V_u \mapsto \mathcal{R}_{e,u\to v} V_v - V_u$$

Observation

Classically there is a correspondence:

Locally constant sheaves $\leftrightarrow$ *vector bundles with a flat connection*

In machine learning the invertibility assumption on parallel transport $\mathcal{R}_{x\to y} = \mathcal{R}_{y\to x}^{-1}$ *severely restricts the connection, making it a flat one!*



Barbero et al. https://arxiv.org/pdf/2206.08702

Metric and Laplacians

Definition

Let $(\Gamma^1_G, \mathrm{d}_G)$ *be a FODC on* $\mathbf{k}[V]$ *associated to* $G = (V, E) \in \mathrm{diGraphs}$*. We define*

1. *a* **quantum metric** *on* Γ^1*, a bimodule map*

$$(,) : \Gamma^1_G \otimes_{A_G} \Gamma^1_G \longrightarrow A_G$$

2. *A* $\mathbf{k}$*-linear map* $\Delta : A_G \to A_G$ *is a* **second order Laplacian** *if*

$$\Delta(ab) = (\Delta a)b + a\Delta b + 2(\mathrm{d}a, \mathrm{d}b)$$

3. **Graph laplacians** *associated to the metric* $(,)$ *are given by:*

$$\Delta_\theta(a) := 2(\theta, \mathrm{d}a) \quad , \quad {}_\theta\Delta(a) := -2(\mathrm{d}a, \theta)$$

   where $\theta = \sum \omega_{x \longrightarrow y}$.

Proposition

If we fix the basis $\{\delta_x\}_{x \in V_G}$ *for* $\mathbf{k}[V_G]$*, we identify* $\mathbf{k}[V_G] \cong \mathbf{k}^{|V_G|}$*,* L *is a linear operator and one can readily check:*

$$L = D - A = (1/2)\Delta_\theta, \qquad \text{for} \quad \lambda_{v \to w, w \to v} = \lambda_{w \to v, v \to w} = 1$$

where D *is the degree matrix (diagonal matrix with the degree of vertices on the diagonal) and* A *is the adjacency matrix of* G*.*

We can extend the definition ${}_\theta\Delta$ when a right connection is given on a vector bundle.

Definition

Assume we have:

- $G \in \mathrm{diGraphs}_{\leq 1}$ *and a FODC* Γ^1,
- M *a free rank* n *right* A_G*-bimodule,*
- ∇ *a right connection*
- $(,)$ *a generalized quantum metric on* Γ^1_G.

Let η *be the left* A_G*-module map* $M \otimes_A \Gamma^1_G \to M \otimes_A \Gamma^1_G \otimes_A \Gamma^1_G$:

$$\eta(m \otimes \omega_{x \to y}) = m \otimes \omega_{x \to y} \otimes \theta$$

Define the **connection Laplacian**

$${}_\theta\Delta^M := -2(\mathbb{1} \otimes (,)) \circ \eta \circ \nabla : M \to M$$

Observation

We have:

$${}_\theta\Delta^M(e_i f^i) = -2 \sum_{x \to y} \lambda_{x \to y \to x}(\mathcal{R}^j_{x \to y, i} f^i(y) - f^i(x)) e_j \delta_x$$

Sheaf Laplacian in Geometric Deep Learning

Definition (Bodnar et al. 2022)

Let $G = (V, E)$ be a directed graph, $f \in \mathbf{k}[V]$. Let $\mathcal{F}$ be a sheaf of vector spaces. We define **sheaf Laplacian**

$$L_F(f)_x := \sum_{y, x \leq x \to y} F^{-1}_{x \leq x \to y}(F_{x \leq x \to y} f_x - F_{y \leq x \leftrightarrow y} f_y) \qquad (4)$$

Recall that a sheaf of vector spaces on a digraph $G = (E_G, V_G, h_G, t_G)$ for the standard (dual) topology is equivalent to give

- a vector space $F(v)$ for each vertex $v \in V_G$,
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- linear maps (restriction maps) $F_{x \leq x \longrightarrow y} : F(x \longrightarrow y) \to F(x)$, $F_{y \leq x \longrightarrow y} : F(x \longrightarrow y) \to F(y)$ for each edge $x \longrightarrow y \in E_G$, where, we write $v \leq e$ to mean that v is a vertex of the edge e.

where the irreducible open sets in the dual standard topology are

Observation

Vector bundles are locally free sheaves (as in ordinary geometry).

Theorem (F.-Simonetti-Zanchetta 2025)

Assume

- $G \in \mathrm{diGraphs}_{\leq 1}$ *is a bidirected graph with* (Γ^1, d), *differential calculus*
- $\mathcal{F}$ *a vector bundle i.e. a sheaf of vector spaces of rank n on G*
- ∇ *connection with* $\mathcal{R}$ *weak parallel transport,*
- M *the free right* A_G*-module associated to the vector bundle* $\mathcal{F}$.

1. *If* $\mathcal{R}^F$ *is a parallel transport, then* ${}_\theta\Delta^M = -L_F$.
2. *If* $\mathcal{F}$ *is a sheaf of inner product spaces and* $F^*_{v\leq e} = F^{-1}_{v\leq e}$ *(i.e. F in an* $\mathrm{O}(n)$*-bundle), then*

$$\nabla^*\nabla = L_F$$

where we fix isomorphisms $M \cong M^*$ *and* $\Gamma^1 \cong (\Gamma^1)^*$.

Remark

This theorem can be proven more generally in the context of semisimplicial sets and their homology/cohomology.

References (Graph Theory and Graph Neural Networks)

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Theorem (Dimakis 1994, Majid 2013)

We have a fully faithful contravariant functor

$$F : \mathrm{diGraphs}_{\leq 1} \longrightarrow (\mathrm{FODC}), \qquad G = (V, E) \mapsto (\Gamma^1, \mathrm{d})$$

realizing an antiequivalence of categories between $\mathrm{diGraphs}_{\leq 1}$ *and the category of FODC* (Γ^1, d) *on* $\mathbf{k}$*-algebras* $A = \mathbf{k}[V]$*, with* V *finite set.*

We can extend the definition of FODC from $\mathrm{diGraphs}_{\leq 1}$ to $\mathrm{diGraphs}$, obtaining still an equivalence of categories.

Theorem (F., Simonetti, Zanchetta 2025)

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Definition

For a given $G \in \mathrm{diGraphs}_{\leq 1}$, we define Γ^2_G as the vector space freely generated by the triangular cliques:

$$\Gamma^2_G := \mathrm{span}\{\omega_{x\to y\to z} \,|\, x \to y, y \to z \in V,\ x \neq y, y \neq z\} \tag{5}$$

Γ^2_G is an A_G-bimodule:

$$f\omega_{x\to y\to z} = f(x)\omega_{x\to y\to z}, \quad \omega_{x\to y\to z}f = \omega_{x\to y\to z}f(z), \quad f \in A_G$$

In analogy to the continuous setting, we refer to Γ^2_G as the space of 2-forms on A_G.

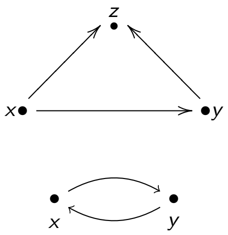

Figure: Triangular cliques

Definition

Define Ω_V^2 to be the **k***-vector space freely generated by the triangular cliques of fully connected G on V:*

$$\Omega_V^2 := \mathrm{span}\{\omega_{x\to y\to z} \,|\, x, y, z \in V,\ x \neq y, y \neq z\}$$

We define the **exterior product** *as the* **k***-linear map:*

$$\Omega_V^1 \times \Omega_V^1 \longrightarrow \Omega_V^2, \quad (\omega_{x\to y}, \omega_{w\to z}) \mapsto \omega_{x\to y} \wedge \omega_{w\to z} := \delta_{y,w}\omega_{x\to y\to z}$$

where $(\Omega_V^1, \mathrm{d}_V^0)$ is the FODC associated with the fully connected graph G.

$$\mathrm{d}_V^1 : \Omega_V^1 \longrightarrow \Omega_V^2 \qquad \mathrm{d}_V^1\omega_{x\to y} := \mathrm{d}_V^0\delta_x \wedge \mathrm{d}_V^0\delta_y \tag{6}$$

The map d_V^1 satisfies the Leibniz rule:

$$\mathrm{d}_V^1(f\omega_{x\longrightarrow y}) = \mathrm{d}_V^0 f \wedge \omega_{x\longrightarrow y} + f\mathrm{d}_V^1\omega_{x\longrightarrow y},\ \mathrm{d}_V^1(\omega_{x\longrightarrow y}f) = \mathrm{d}_V^1\omega_{x\longrightarrow y}f - \omega_{x\longrightarrow y}\wedge \mathrm{d}_V^0 f$$

and $\mathrm{d}_V^1 \circ \mathrm{d}_V^0 = 0$. We also have the explicit expression:

$$\mathrm{d}\omega_{x\to y} = \sum_{u\in V} (\omega_{u\to x\to y} - \omega_{x\to u\to y} + \omega_{x\to y\to u}) \tag{7}$$

General Second Order Differential Calculus on $G = (V, E)$

Let $G = (V, E) \in \mathrm{diGraphs}_{\leq 1}$, we can write the FODC Γ^1_G as a quotient of a **universal calculus** Ω^1 corresponding to the fully connected graph with vertices V:

$$\Gamma^1_G = \Omega^1_V / I, \qquad I = \mathrm{span}\{\omega_{x \to y} \,|\, x \to y \notin E\}$$

The previous proposition, along with the definition of the wedge product, allows us to see $\Omega^\bullet_V := \oplus^2_{i=0} \Omega^i_V$ (where $\Omega^0_V := \mathbf{k}[V]$) as a differential graded algebra $(\Omega^\bullet_V, d^\bullet_V)$ (DGA)

Proposition

Let $G \in \mathrm{diGraphs}_{\leq 1}$. The graded A_G-bimodule $\Gamma^\bullet_G := \oplus^2_{i=0} \Gamma^i_G$, where

$$\Gamma^0_G := A_G, \qquad \Gamma^1_G = \Omega^1_V / I \qquad \Gamma^2_G \cong \Omega^2_V / \mathrm{d}^1_V(I)$$

has a well defined DGA structure induced by the one of $(\Omega^\bullet_V, \mathrm{d}^\bullet_V)$, the bimodule structure being the same.

We notice that any quotient of Γ^2_G by the span of a subset of the triangular cliques will give a well defined differential.

Definition

Let $G = (V, E) \in \mathrm{diGraphs}_{\leq 1}$ and S a subset of its triangular cliques. We define the pair $(\Gamma^\bullet_S, d^\bullet_S)$ with:

$$\Gamma^\bullet_S := \Gamma^\bullet_G / \langle S \rangle, \qquad \mathrm{d}^\bullet_S : \Gamma^\bullet_S \longrightarrow \Gamma^\bullet_S \tag{8}$$

a **second order differential calculus** *on $A = \mathbf{k}[V]$, where $\langle S \rangle$ is the A_G-bimodule generated by S and d^1_S is obtained from d^1_G, by taking the quotient of Γ^2_G by $\langle S \rangle$.*

Note that if $S = \emptyset$ we get that $(\Gamma^\bullet_S, d^\bullet_S) = (\Gamma^\bullet_G, d^\bullet_G)$. In addition, notice that d^1_S satisfies the Leibnitz rule and $\mathrm{d}^1_S \circ \mathrm{d}^0_S = 0$, where $\mathrm{d}^0_S = \mathrm{d}^0_G$ and $\Gamma^i_S := \Gamma^i_G$ for $i = 0, 1$.

Remark

Let V be a finite set. Our approach could be extended to obtain all differential graded algebras on A as quotients of the universal one $\Omega_V := \oplus_n \Omega^n_V$. Moreover one could also extend our results to comprehend the case of étale directed covers,

Theorem (Dimakis 1994, Majid 2013)

We have a fully faithful contravariant functor

$$F : \mathrm{diGraphs}_{\leq 1} \longrightarrow (\mathrm{FODC}), \qquad G = (V, E) \mapsto (\Gamma^1, \mathrm{d})$$

realizing an antiequivalence of categories between $\mathrm{diGraphs}_{\leq 1}$ *and the category of FODC* (Γ^1, d) *on* $\mathbf{k}$*-algebras* $A = \mathbf{k}[V]$*, with* V *finite set.*

We can extend the definition of FODC from $\mathrm{diGraphs}_{\leq 1}$ to $\mathrm{diGraphs}$, obtaining still an equivalence of categories.

Theorem (F., Simonetti, Zanchetta 2025)

We have a fully faithful contravariant functor

$$\mathcal{F} : \mathrm{diGraphs} \longrightarrow (\mathrm{FODC})_{\mathrm{e}}, \quad G \mapsto (\Gamma^1_G, \mathrm{d})$$

giving an antiequivalence of categories, where $(\mathrm{FODC})_{\mathrm{e}}$ *consists of all the FODC* (Γ^1, d) *coming from an étale cover of a given graph.*

Definition

Let $G = (V, E) \in \mathrm{diGraphs}$, $A = \mathbf{k}[V]$ and (Γ^1, d) the FODC on A. Let M be a free rank n left A-module. We define a **left noncommutative connection** ∇ *on M as a map*

$$\nabla : M \longrightarrow \Gamma^1 \otimes M$$

satisfying the Leibniz identity, i.e:

$$\nabla(fm) = \mathrm{d}f \otimes m + f\nabla m, \qquad f \in A, \quad m \in M$$

Analogously, given a free rank n right A-module M, one can define a **right noncommutative connection** ∇ *on M as a map*

$$\nabla : M \longrightarrow M \otimes \Gamma^1$$

satisfying the Leibniz identity:

$$\nabla(mf) = m \otimes \mathrm{d}f + (\nabla m)f, \qquad f \in A, \quad m \in M$$

Once a basis $\{e_i\}_{i=1}^n$ for the free A-module M is chosen, a non commutative right connection amounts to give a map:

$$e_i f^i \mapsto e_i \otimes df^i + e_j \otimes \omega_i^j f^i$$

where ω_i^j is a matrix of 1 forms, i.e. elements of Γ^1

Observation

There is a bijective correspondence between the two notions:

1. *A noncommutative right connection on M, a right A-module of rank n, with respect to the FODC given via G on A.*
2. *A connection on a digraph.*

(2) $\to$ (1). In fact, consider a vector bundle $\mathcal{F}$ of rank n on V, a frame bundle

$$V \ni v \mapsto \{e_i^v\}$$

and a free rank n right A-module n with the choice of a basis $\{e_i\}_{i=1}^n$.
Then given a connection $\Theta_{e,u\to v} := \mathcal{R}_{e,u\to v} - 1\!\!1_{v,u}$ we get:

$$\omega_i^j = \sum_{e,x\to y} [\mathcal{R}^j_{e,x\to y,i} - \delta_{i,j}]\omega_{e,x\to y} \tag{9}$$

$$M \ni e^i f_i \mapsto \sum_{e,x\to y} e_j \otimes \left[f^i(y)\mathcal{R}^j_{e,x\to y,i} - f^i(x)\delta_{ij} \right] \omega_{e,x\to y} \in M \otimes \Gamma^1_G \tag{10}$$

(1) $\to$ (2). Conversely given a right connection

$$e_i f^i \mapsto e_i \otimes df^i + \sum_{j=1}^n e_j \otimes \omega_i^j f^i$$

where $\omega_i^j = \sum_{e,x\to y} a^j_{e,x\to y,i}\omega_{e,x\to y}$ (using the basis $\{\omega_{e,x\to y}\}$ of Γ^1_G as a **k**-vector space), by setting $R^j_{e,x\to y,i} := a^j_{e,x\to y,i} + \delta_{ij}$ we get a connection.

Definition

Assume we have:

- $G \in \mathrm{diGraphs}_{\leq 1}$, (Γ^1_G, d) *a FODC on* $A = \mathbf{k}[G]$
- S *a subset of its triangular cliques*
- M *a free* A *bimodule of rank* n *with basis* $\{e_i\}_{i=1}^n$, $A = \mathbf{k}[G]$.
- $\nabla : M \to \Gamma^1_G \otimes M$ *a noncommutative right connection*

We define:

- *the* **curvature** *of* ∇ *as* $R_\nabla : M \to M \otimes \Gamma^2_G$ *as the right* A *module map defined on the basis* $\{e_i\}_{i=1}^n$ *as follows*

$$R_\nabla(e_i) = e_j \otimes d\omega_i^j + e_j \otimes \omega_k^j \wedge \omega_i^k$$

- *the* **curvature outside of** S *of* ∇ *as:*

$$R^S_\nabla := (\mathbb{1} \otimes \pi_S) \circ R_\nabla : M \to M \otimes \Gamma^2_S$$

where $\pi_S : \Gamma^2_G \to \Gamma^2_S = \Gamma^2_G / \langle S \rangle$ *is the projection morphism.*

We say that ∇ *is* **flat outside** *of* S *if* $R^S_\nabla = 0$. *We say that* ∇ *is* **flat** *if* $R_\nabla = 0$.

Observation

We can rewrite R_∇ in terms the weak parallel transport associated with ∇ as

$$R_\nabla(e_i) = \sum_{x\to y\to z\in \mathrm{tri}(G)} (\mathcal{R}^j_{x\to y,k}\mathcal{R}^k_{y\to z,i} - \mathcal{R}^j_{x\to z,i})e_j \otimes \omega_{x\to y\to z} \tag{11}$$

$\mathrm{tri}(G)$ *is the set of all triangular cliques of* G.

Proposition

Let be G, M and ∇ as above. Then:

1. *If ∇ is flat then $\mathcal{R}_{x\to z} = \mathcal{R}_{x\to y}\mathcal{R}_{y\to z}$ for each triangular clique. In particular, we have that $\mathcal{R}_{x\to y} = \mathcal{R}^{-1}_{y\to x}$ for all edges $x \to y \in E_G$ that are part of a triangular clique of the form $x \to y \to x$.*
2. *Assume G to be bidirected. Consider the set of triangular cliques S consisting of all triangular cliques of the form $x \to y \to z$ having $x, y, z \in V_G$ all distinct. Then ∇ is flat outside of S if and only if the weak parallel transport associated to ∇ is a parallel transport i.e.:*

$$\mathcal{R}_{x\to y} = \mathcal{R}^{-1}_{y\to x} \text{ for all edges } x \to y \in E_G$$

Definition

Let $(\Gamma^1_G, \mathrm{d}_G)$ *be a FODC on* $\mathbf{k}[V]$ *associated to* $G = (V, E) \in \mathrm{diGraphs}$. *We define*

1. *a* **quantum metric** *on* Γ^1, *a bimodule map*

$$(,) : \Gamma^1_G \otimes_{A_G} \Gamma^1_G \longrightarrow A_G$$

2. *A* $\mathbf{k}$*-linear map* $\Delta : A_G \to A_G$ *is a* **second order Laplacian** *if*

$$\Delta(ab) = (\Delta a)b + a\Delta b + 2(\mathrm{d}a, \mathrm{d}b)$$

3. **Graph laplacians** *associated to the metric* $(,)$ *are given by:*

$$\Delta_\theta(a) := 2(\theta, \mathrm{d}a) \quad , \quad {}_\theta\Delta(a) := -2(\mathrm{d}a, \theta)$$

where $\theta = \sum \omega_{x \longrightarrow y}$.

Proposition

If we fix the basis $\{\delta_x\}_{x \in V_G}$ *for* $\mathbf{k}[V_G]$, *we identify* $\mathbf{k}[V_G] \cong \mathbf{k}^{|V_G|}$, *L is a linear operator and one can readily check:*

$$L = D - A = (1/2)\Delta_\theta, \qquad \textit{for} \quad \lambda_{v\to w, w\to v} = \lambda_{w \to v, v \to w} = 1$$

where D is the degree matrix (diagonal matrix with the degree of vertices on the diagonal) and A is the adjacency matrix of G.

Observation

The equality $L = D - A = 2\Delta_\theta$ is obtained from the comparison of the expressions of L and Δ_θ:

$$(La)(x) = \sum_{y,(x,y)\in E_G} (a(x) - a(y)), \qquad \Delta_\theta a(x) = 2 \sum_{y,x\to y\in E_G} \lambda_{x\to y,y\to x}(a(x) - a(y))$$

We can extend the definition ${}_\theta\Delta$ when a right connection is given on a vector bundle.

Definition

Assume we have:

- $G \in \mathrm{diGraphs}_{\leq 1}$ *and a FODC* Γ^1,
- M *a free rank* n *right* A_G*-bimodule,*
- ∇ *a right connection*
- $(,)$ *a generalized quantum metric on* Γ^1_G.

Let η *be the left* A_G*-module map* $M \otimes_A \Gamma^1_G \to M \otimes_A \Gamma^1_G \otimes_A \Gamma^1_G$*:*

$$\eta(m \otimes \omega_{x\to y}) = m \otimes \omega_{x\to y} \otimes \theta$$

Define the **connection Laplacian**

$${}_\theta\Delta^M := -2(\mathbb{1} \otimes (,)) \circ \eta \circ \nabla : M \to M$$

Observation

Assume we have:

- $G \in \mathrm{diGraphs}_{\leq 1}$ *and a FODC* (Γ^1, d),
- M *a free rank* n *right* A_G*-bimodule, with basis* $\{e_i\}_{i=1}^n$
- ∇ *a right connection*
- $(,)$ *a generalized quantum metric on* Γ^1_G.

Then

1. $$_\theta\Delta^M(e_i f^i) = -2 \sum_{x\to y} \lambda_{x\to y\to x}(\mathcal{R}^j_{x\to y,i} f^i(y) - f^i(x)) e_j \delta_x$$
   where in the summation only the edges $x \to y$ *of the maximal bidirected subgraph of* G *appear.*
2. *If* $M = A_G$*, we recover the Laplacian* $_\theta\Delta$.

Definition (Bodnar et al. 2022)

Let $G = (V, E)$ *be a directed graph,* $f \in \mathbf{k}[V]$. *Let* $\mathcal{F}$ *be a sheaf of vector spaces. We define* **sheaf Laplacian**

$$L_F(f)_x := \sum_{y, x \leq x \to y} F^{-1}_{x \leq x \to y} (F_{x \leq x \to y} f_x - F_{y \leq x \leftrightarrow y} f_y) \tag{12}$$

Recall that a sheaf of vector spaces on a digraph $G = (E_G, V_G, h_G, t_G)$ for the standard (dual) topology is equivalent to give

- a vector space $F(v)$ for each vertex $v \in V_G$,
- a vector space $F(e)$ for each edge (with its endpoints) $e \in E_G$,
- linear maps (restriction maps) $F_{x \leq x \longrightarrow y} : F(x \longrightarrow y) \to F(x)$, $F_{y \leq x \longrightarrow y} : F(x \longrightarrow y) \to F(y)$ for each edge $x \longrightarrow y \in E_G$, where, we write $v \leq e$ to mean that v is a vertex of the edge e.

where the irreducible open sets in the dual standard topology are

Observation

Vector bundles are locally free sheaves (as in ordinary geometry).

Theorem (F.-Simonetti-Zanchetta 2025)

Assume

- $G \in \mathrm{diGraphs}_{\leq 1}$ *is a bidirected graph with* (Γ^1, d)*, differential calculus*
- $\mathcal{F}$ *a vector bundle i.e. a sheaf of vector spaces of rank n on G*
- ∇ *connection with* $\mathcal{R}$ *weak parallel transport,*
- *M the free right* A_G*-module associated to the vector bundle* $\mathcal{F}$*.*

1. *If* $\mathcal{R}^F$ *is a parallel transport, then* ${}_\theta\Delta^M = -L_F$*.*
2. *If* $\mathcal{F}$ *is a sheaf of inner product spaces and* $F^*_{v\leq e} = F^{-1}_{v\leq e}$ *(i.e. F in an* $\mathrm{O}(n)$*-bundle), then*
$$\nabla^*\nabla = L_F$$
where we fix isomorphisms $M \cong M^*$ *and* $\Gamma^1 \cong (\Gamma^1)^*$*.*

Remark

This theorem can be proven more generally in the context of semisimplicial sets and their homology/cohomology.