Quantum Borel–Weil Theorems and One-Cross Bundles

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- Representations use q-analogues, but representation theory is the same

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Analogously, the Hopf dual of $U_q(\mathfrak{sl}_2)$ is the *quantum coordinate algebra* $\mathcal{O}_q(SU_2)$, which we think of as the algebra of regular functions on the "quantum group" q-deformation of SU_2.

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$$S(a) = d, \quad S(b) = -\,q^{-1}b,$$

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Hopf fibration in Kampa park, Prague

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In the classical limit, we get a single relation,

$$x^2 + y^2 + z^2 = 1.$$

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- Each summand $\mathcal{E}_k$ is a invertible finitely generated projective module (both left and right) over $\mathcal{O}_q(S^2)$ and q-deforms the module of sections of a line bundle over S^2.

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It plays a **central role** in noncommutative geometry.

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In essence, when considering quantum homogeneous spaces, we "quotient out the worst of the noncommutativity".

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- Noncommutative holomorphic structures on its quantum line bundles.

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$$\Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \Omega^2$$

- Here Ω^1 is an $\mathcal{O}_q(S^2)$-bimodule and $\Omega^0 = \mathcal{O}_q(SU_2) \cong \Omega^2$.

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- Each Ω^i admits a left coaction $\Delta_L : \Omega^i \to \mathcal{O}_q(SU_2) \otimes \Omega^i$.

de Rham complex

Podleś Sphere

$$\begin{array}{c}\Omega^2\\ \mathrm{d}\big\uparrow\\ \Omega^1\\ \mathrm{d}\big\uparrow\\ \Omega^0\end{array}$$

- Here Ω^1 is an $\mathcal{O}_q(S^2)$-bimodule and $\Omega^0 = \mathcal{O}_q(SU_2) \cong \Omega^2$

- Each Ω^i admits a left coaction $\Delta_L : \Omega^i \to \mathcal{O}_q(SU_2) \otimes \Omega^i$.

- The maps $d : \Omega^j \to \Omega^{j+1}$ satisfy $d^2 = 0$, and the Leibniz rule: $\mathrm{d}(\omega \wedge \nu) = \mathrm{d}\omega \wedge \nu + (-1)^{|\omega|}\omega \wedge \mathrm{d}\nu$.

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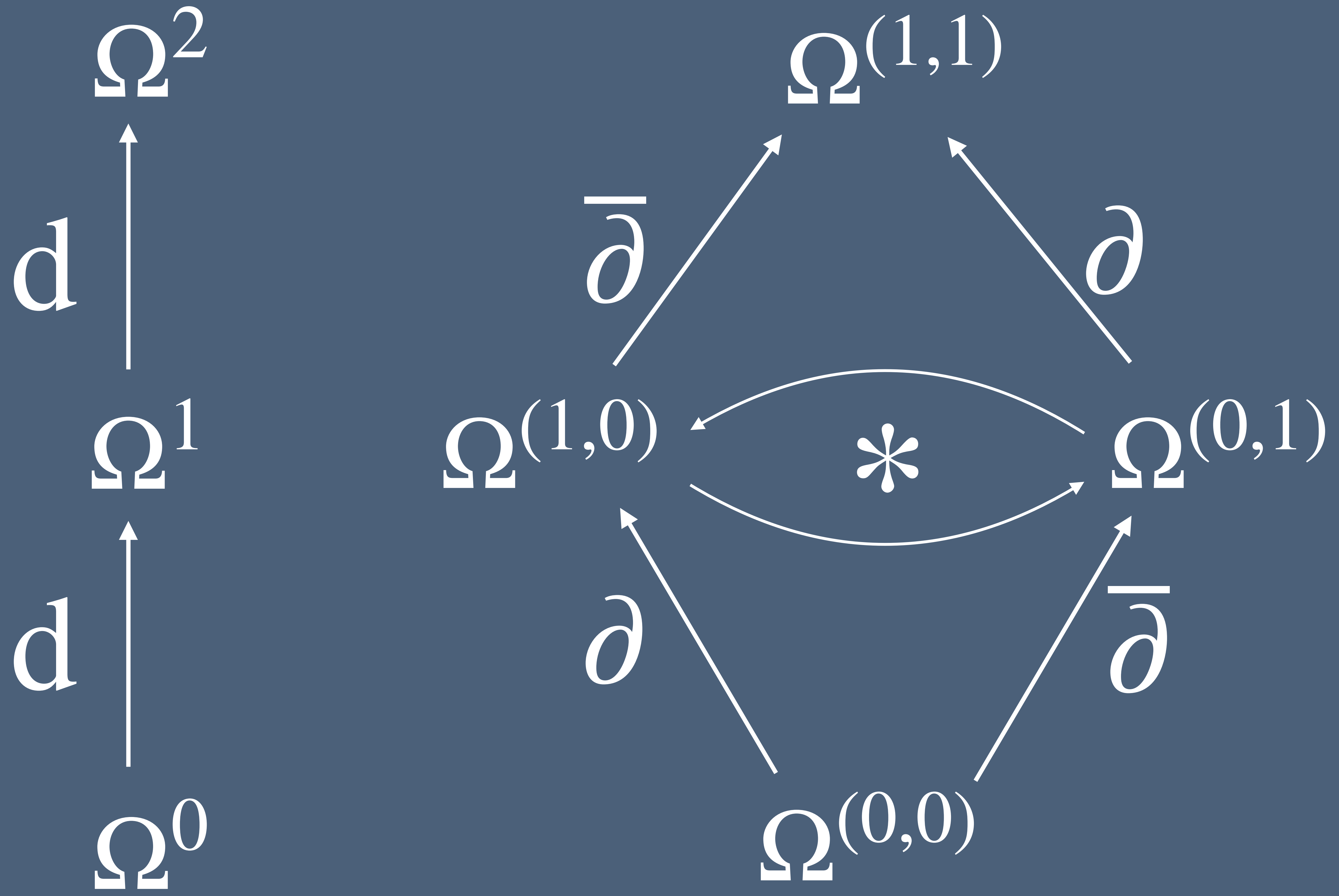
(Formally this means that we have a left $\mathcal{O}_q(SU_2)$-covariant different calculus.)

Hodge diamond: Dolbeaut double complex

Podleś Sphere



The de Rham complex can be refined into a $\mathbb{N}^2$-graded complex called the **Dolbeaut complex**

- $\Omega^1 = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$
- $\mathrm{d} = \partial + \overline{\partial}$
- $(\Omega^{(1,0)})^* = \Omega^{(0,1)}$

Hodge diamond

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Hodge diamond

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Recall that $\mathcal{O}_q(SU_2)$ admits a $\mathbb{Z}$-algebra grading $\mathcal{O}_q(SU_2) \cong \bigoplus_{k \in \mathbb{Z}} \mathcal{E}_k$

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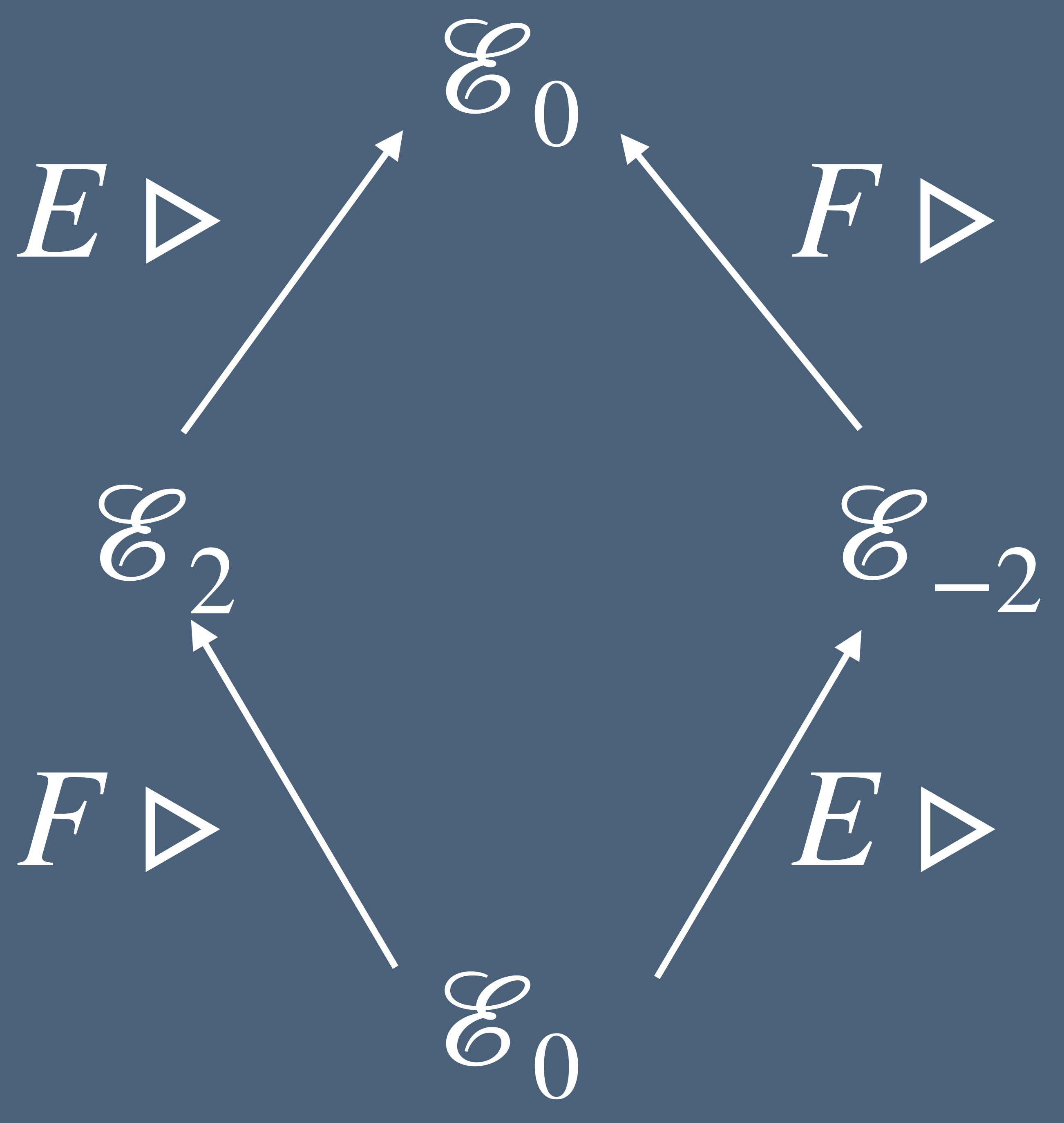

Connections

For finitely generated projective modules

For any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$ and $\mathcal{F}$ a finitely generated projective left module $\mathcal{O}_q(G/L_S)$. A connection is a linear map

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$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla(f)\,.$$

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So what do *noncommutative* holomorphic sections look like?

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Communications in Mathematical Physics

Noncommutative Riemannian and Spin Geometry of the Standard q-Sphere

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Proposition A.1. *The space of holomorphic sections* $\mathcal{E}^{\rm hol}_{-n}$ *of the charge* $-n$ q*-monopole bundle contains the standard* $n+1$*-dimensional corepresentation of* $\mathbb{C}_q[SL_2]$.

JOURNAL ARTICLE

Holomorphic Structures on the Quantum Projective Line

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Masoud Khalkhali, Giovanni Landi, Walter Daniël van Suijlekom

International Mathematics Research Notices, Volume 2011, Issue 4, 2011, Pages 851–884,
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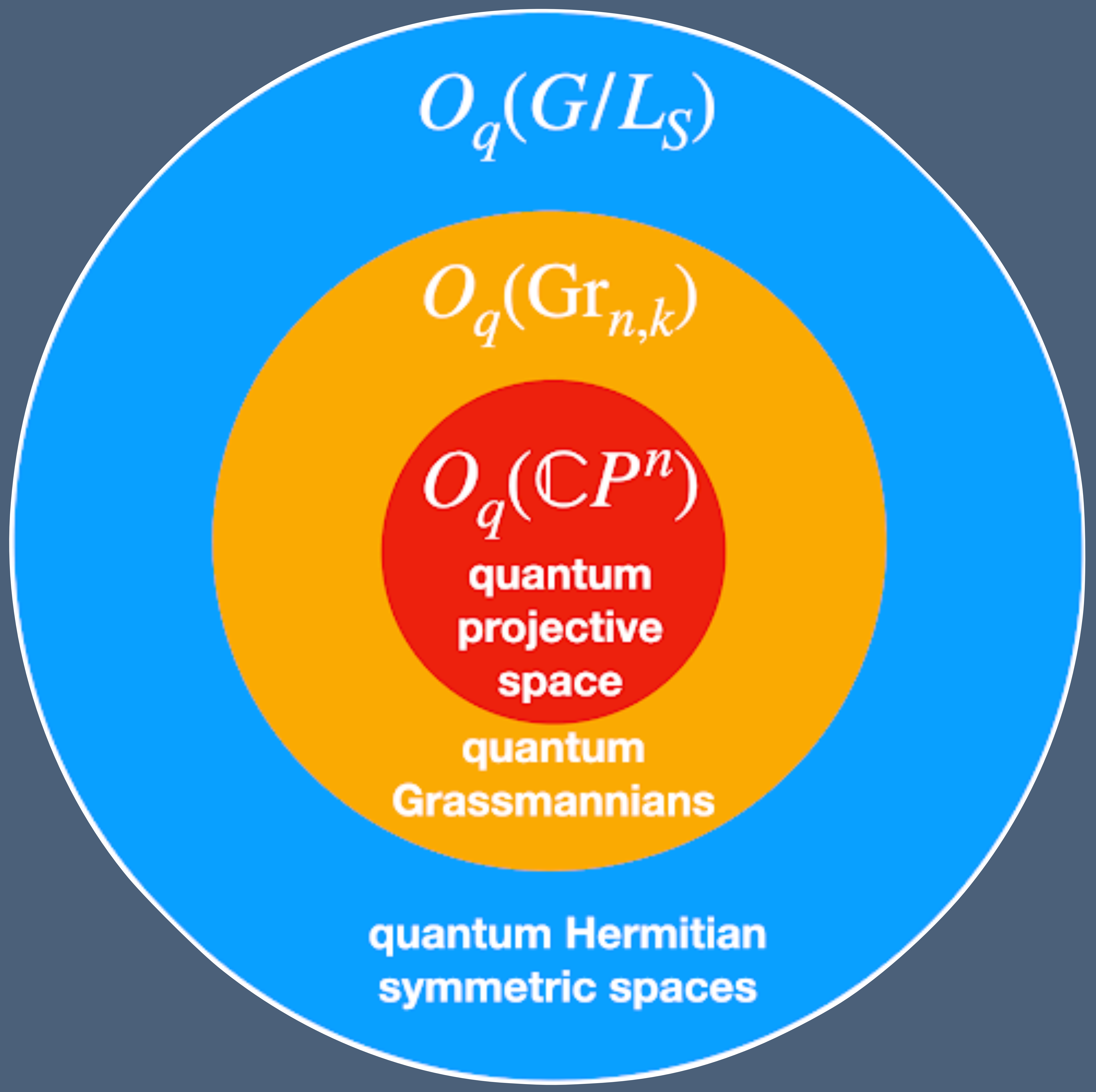

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III. Quantum homogeneous spaces and one-cross bundles

Quantum homogeneous spaces

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- If $A \otimes_B - : {}_B\mathrm{Mod} \to \mathbf{Vect}_{\mathbb{C}}$ is a faithfully flat functor then we say that B is a *quantum homogeneous space.*

Differential graded algebras and differential calculi

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If $\Omega^\bullet$ is generated as an algebra by those elements of degree 1 and 2, then we say that it is a *differential calculus* over B.

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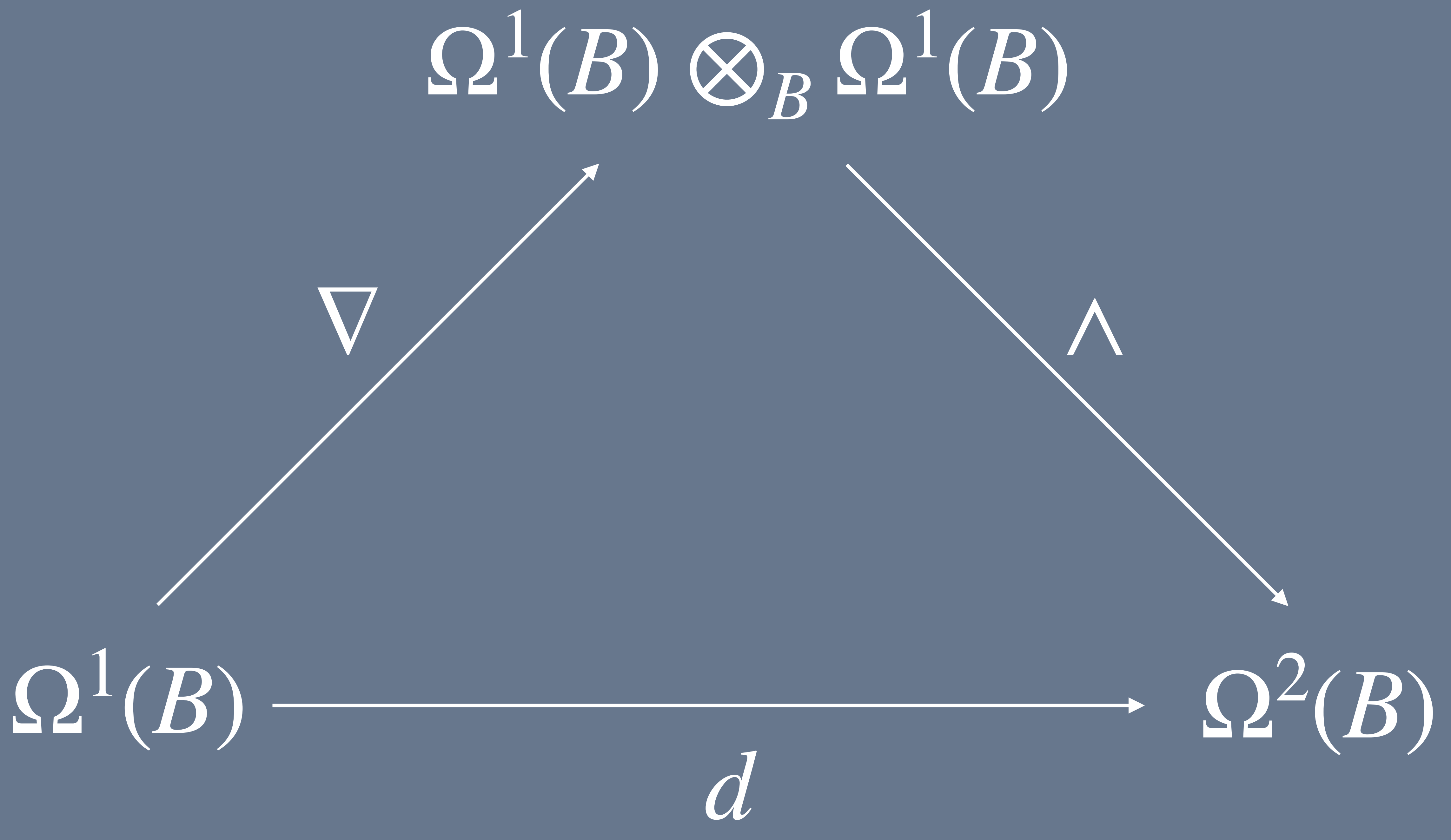

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Proposition: *For any one-cross bundle, there exists a unique covariant torsion-free*

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where $\mathrm{Gen}(B)$ *is a generating set of* B.

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Corollary: *Every relative Hopf module* $\mathcal{F} \in {}^A_B\mathrm{mod}$ *admits a covariant holomorphic structure*

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If $\mathcal{F}$ *is simple then* $\overline{\partial}_{\mathcal{F}}$ *is unique.*

V. Examples

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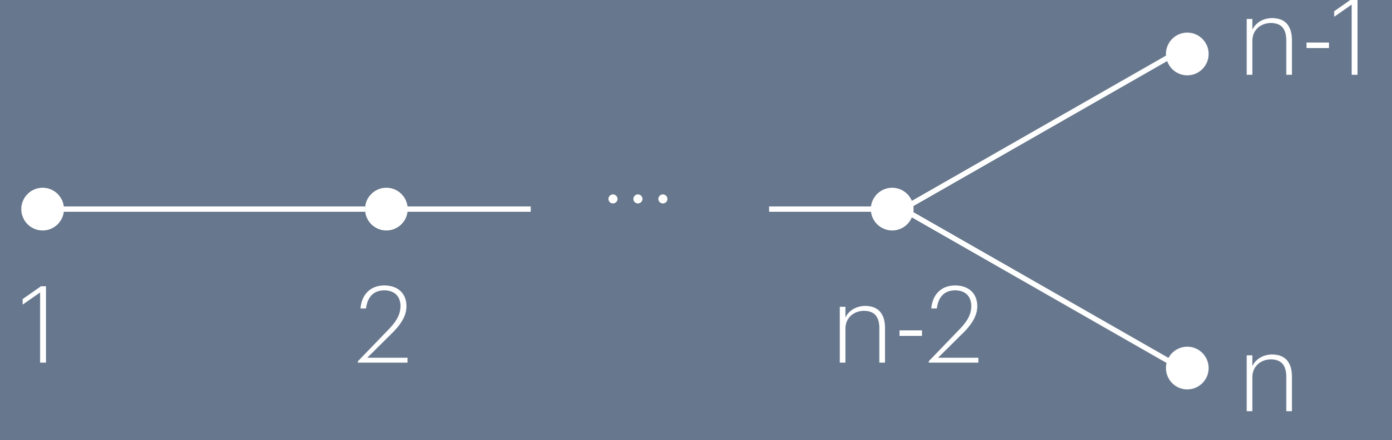
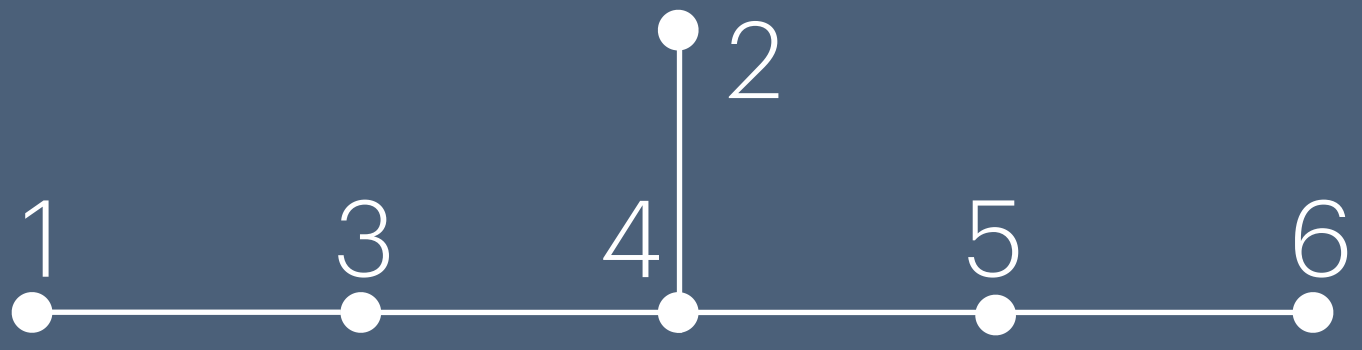
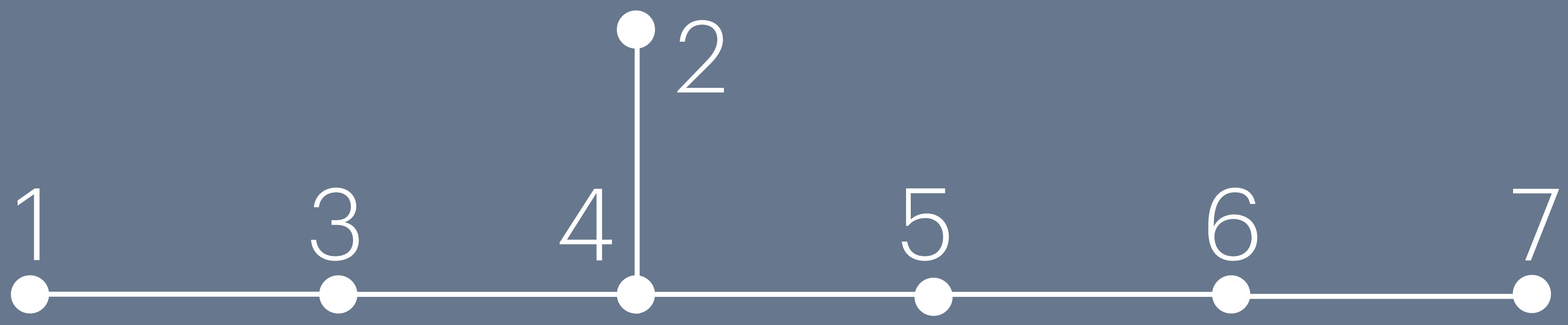
then we get quantum projective space $\mathcal{O}_q(\mathbb{C}P^n)$. If we set $X = E_1$, then we recover the holomorphic structures on $\mathcal{E}_k$, from the work of Khalkhali and Moatadelro.

Root system	Dynkin diagram	Heighest weight
A_n		$\alpha_1 + \cdots + \alpha_n$
B_n		$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$
C_n		$2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n$
D_n		$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$

Root system	Dynkin diagram	Heighest weight
E_6		$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7		$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$ $+3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4$ $+5\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
F_4		$3\alpha_1 + 2\alpha_2$

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cominiscule root, then we recover the celebrated *Heckenberger–Kolb differential calculi* and their holomorphic structures.

Example: For the non-cominiscule case, we recover something totally new . . .

Quantum projective space (Bucharest)

Thank you!

Σας ευχαριστώ!