Kinematical Lie algebras and contact sub-Riemannian symmetric spaces

Pierre Bieliavsky (joint work with Nicolas Boulanger)

Corfu, 19/09/2025

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implement thermodynamical element?

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[Figueroa-O'Farril 2018]: classification of $\mathfrak{g}(D)$'s.

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$$\mathcal{D} \quad \text{is spanned by} \quad e := \partial_y \quad \text{and} \quad f := y\partial_x - \partial_z \, .$$

$$e\,,\ f \text{ and } [e, f] = \partial_x \quad \text{span a Heisenberg Lie algebra}$$

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"Symmetric" C-C spaces are (generalized) Kinematical Lie algebras.

Symmetric spaces [O. Loos, 1969]

A **symmetric space** is a pair (M, s) where M is a smooth manifold and $s : M \times M \to M : (x, y) \mapsto s_x(y)$ is a smooth map, such that

1. $s_x^2 = \mathtt{Id}_M$.
2. x is an isolated fixed point of s_x.
3. $s_x \, s_y \, s_x \; = \; s_{s_x(y)}$.

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Proposition: The expression

$$(\nabla_X Y)_x \;:=\; \frac{1}{2}\,[X\,,\;Y \,+\, s_{x\star}Y]_x$$

defines a linear connection in $T(M)$.
It is the unique linear connection **invariant under the symmetries**: $\{s_x\}_{x\in M} \subset \mathrm{Aff}(\nabla)$.
Moreover:

$$T^\nabla \;\equiv\; 0 \quad \text{and} \quad \nabla R^\nabla \;\equiv\; 0\,.$$

Symplectic symmetric spaces [B.-Cahen-Gutt, 1994]

A **symplectic symmetric space** is a triple (M, s, ω) where (M, s) is a symmetric space and ω is a **non-degenerate differential 2-form** on M which is invariant under the symmetries:

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Example: Massive (coadjoint) orbits of the Poincaré group are SSS. Their symplectic connection ∇ is *never* metric (i.e. Levi-Civita).

A differential one-form θ on a connected smooth manifold $\mathcal{M}$ is a **contact form** if

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The vector field ξ on $\mathcal{M}$ characterized by $\iota_\xi \mathrm{d}\,\theta = 0$ and $\langle \theta\,,\,\xi \rangle = 1$ is called the **Reeb field**.

sub-Riemannian contact manifolds [Strichartz 1986]

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$$g \in \underline{\mathcal{D}^\star \otimes \mathcal{D}^\star}\;,$$

the triple $(\mathcal{M}, \mathcal{D}, g)$ is called a **contact sub-Riemannian manifold.**

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the triple $(\mathcal{M}, \mathcal{D}, g)$ is called a **contact sub-Riemannian manifold**. An **sub-isometry** is a diffeomorphism of $\mathcal{M}$ preserving the distribution and the partial metric.

The adapted connection [Falbel-Gorodski 1994]

Theorem: Let $(\mathcal{M}, \mathcal{D}, g)$ be a contact sub-Riemannian manifold. Then there exists a unique pair (∇, τ) where ∇ is a linear connection in $T(\mathcal{M})$ and $\tau \in \underline{\mathtt{End}(\mathcal{D})}$ **(sub-torsion)** such that

1. $\nabla : \underline{T(M)} \times \underline{\mathcal{D}} \to \underline{\mathcal{D}}$
2. $\nabla\xi \; = \; 0$
3. $\nabla g \; = \; 0$
4. $T^{\nabla}(X, Y) \; = \; \mathrm{d}\,\theta(X, Y)\,\xi$
5. $T^{\nabla}(\xi, X) \; = \; \tau(X).$

Proposition [B.-Falbel-Gorodski-Tausk, 1995] Every two points in $\mathcal{M}$ can be joined by a horizontal broken ∇-geodesic.

sub-Riemannian contact symmetric spaces [Strichartz 1986]

A **sub-Riemannian contact symmetric space** is a quadruple $(\mathcal{M}, \mathcal{D}, g, \psi)$ where $\psi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a smooth map such that, for every $x \in M$, the map $\psi_x : \mathcal{M} \to \mathcal{M} : y \mapsto \psi(x, y)$ is a **sub-isometry** that fixes x and such that

$$\psi_{x_{\star x}}|_{\mathcal{D}_x} \;=\; -\mathtt{Id}_{\mathcal{D}_x} \;.$$

Classification of sub-Riemannian contact symmetric spaces [B.-Falbel-Gorodski 1995]

Theorem: The Reeb field of a sub-Riemannian contact symmetric space $(\mathcal{M}, \mathcal{D}, g, \psi)$ induces a principal fibration of $\mathcal{M}$ over a symplectic symmetric space M ($\dim(\mathcal{M}) = \dim(M) + 1$).

Classification of sub-Riemannian contact symmetric spaces [B.-Falbel-Gorodski 1995]

type			*examples*	*sub-torsion*	*holonomy*
solvable			H^{2n+1}	zero	trivial
semisimple	Hermitean		S^1-fibration over Hermitean Riemannian symmetric space	zero	irreducible if symmetric space is irreducible
	non-Hermitean	simple	$SO(n+2)/SO(n)$ $(n \geq 3)$	nonzero	irreducible
			$SO(n,2)/SO(n)$ $(n \geq 3)$	nonzero	irreducible
			$SO(n+1,1)/SO(n)$	nonzero	irreducible if $n \geq 3$
		non-simple	$SO(4)/SO(2)$	nonzero	not irreducible
			$SO(2,2)/SO(2)$	nonzero	not irreducible
else			$SO(n+1) \bowtie R^{n+1}/SO(n)$	nonzero	irreducible if $n \geq 3$
			$SO(n,1) \bowtie R^{n+1}/SO(n)$	nonzero	irreducible if $n \geq 3$
			twisted product of H^{2n+1} and Hermitean	zero	not irreducible

Table 1. Contact sub-Riemannian symmetric spaces of dimension $2n+1 \geq 5$, $n \geq 2$

Generalized Kinematical Lie algebras [B.-Boulanger, 2025]

$\mathfrak{g} = \mathcal{K} \oplus \mathbb{R} \oplus (V \oplus V)$

is a (finite dimensional real) Lie algebra such that

1. $\mathcal{K}$ is a (compact) Lie sub-algebra.
2. V is a simple faithful $\mathcal{K}$-module equipped with an invariant quadratic form (possibly signed).
3. The isotypical component of V in $\Lambda^2(V)$ is empty.
4. $[\mathbb{R}, \mathcal{K}] = \{0\}$.

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Theorem: Let G be a (connected simply connected) generalized kinematical Lie group.Then:

1. $\mathcal{M} := G/K$ is a sub-Riemannian contact symmetric space, while $M := G/H$ is a symplectic symmetric space.
2. The principal fibration $G/K \to G/H : gK \mapsto gH$ realizes the Reeb fibration of $\mathcal{M}$ over its associated symplectic symmetric space.
3. A sub-Riemannian contact symmetric space comes from such a generalized Kinematical Lie group that way if and only if its horizontal holonomy acts reducibly.