Proving the

Weak Gravity Conjecture

in perturbative String Theory

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What is Quantum Gravity?

Too hard → we ask a simpler question:
Swampland question: How should a consistent quantum gravity look in the IR, as an effective field theory?

Can we constrain some of these properties?

Outline

We focus on the weak gravity conjecture, arguably the most "rigorous" and expected (maybe also the most useless?)

- Motivation
- Evidence
- Proof
  - Modular Invariance and Spectral Flow
  - Self-repulsiveness proof
  - Self-repulsiveness $\implies$ superextremality

Weak Gravity Conjecture

In a Quantum Gravity Theory with a massless photon there is a charged particle satisfying

Weak Gravity Conjecture

$$\frac{|q|}{m} \geq \left.\frac{|Q|}{M}\right|_{\text{large black hole}}$$

Mild WGC: "There should be a *superextremal* state in the spectrum".

Refinements, such as "tower WGC", "lattice WGC", "sublattice WGC" require infinite number of states satisfying the condition.

Repulsive Force Conjecture

Related to the WGC, but not equivalent in general

Repulsive Force Conjecture

$$F = \frac{\mathcal{F}}{V_{D-2} r^{D-2}} \,, \mathcal{F} = f^{ab} Q_a Q_b - k_N m^2 - G_{ij} \frac{\partial m}{\partial \phi^i} \frac{\partial m}{\partial \phi^j} \geq 0$$

F resultant of electric repulsion, gravitational attraction and interaction with moduli.

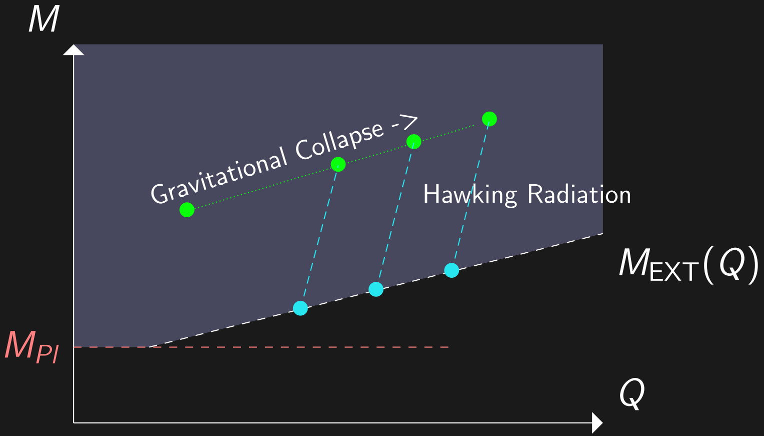
"A pair of identical particles should repel each other".

Similar refinements "lattice RFC", "sublattice RFC", etc.

Motivation

Remnants

By gravitational collapse + Hawking radiation



we would have
stable black hole remnants $\sim$ global symmetry

Black hole decay

To avoid this, we need black holes to be allowed to decay, i.e. there should be superextremal states in the spectrum. .

$M = Q$

Extremal BH

BH decay $\longrightarrow$

Subextremal BH

$M' \geq Q'$

$m \leq q$

Superextremal particle

Evidence

Heterotic on T^k

Heterotic string theory on T^k is a simple example where everything can be computed.
For each Q in the charge lattice, compare lightest states and black hole extremality bound:

$$\frac{\alpha'}{4}m^2 = \frac{1}{2}\, max\left(Q^2 - 2, \tilde{Q}^2\right) \leq \frac{\alpha'}{4}M_{\text{EXT}}^2(Q)$$

$$\text{Two cases: } \begin{cases} Q^2 \leq \tilde{Q}^2 & \text{Exactly Extremal} \\ Q^2 > \tilde{Q}^2 & \text{Stricly superextremeal} \end{cases}$$

In any case, for every Q there is a superextremal particle.

Lattice WGC is satisfied

Heterotic on $T^3/\mathbb{Z}^2$

Heterotic string theory on $T^3/\mathbb{Z}^2$.

$$\text{Spectrum} \qquad m^2 \sim \frac{n_1^2}{R_1^2} + \frac{n_2^2}{R_2^2} + \frac{n_3^2}{R_3^2}$$

Orbifold implies identifications $n_1 \leftrightarrow n_2, R_1 = R_2$

$U(1)^2$, with charges $Q_A = n_1 + n_2, Q_B = n_3$

One can compute

$$m^2 - M_{\text{EXT}}^2(Q) = \frac{(n_1 - n_2)^2}{2R_1^2}$$

If $n_1 \neq n_2$, there is no extremal particle.

Sublattice WGC is satisfied

Proof

Strategy

General idea:

- Exploit structure and properties of the 2d worldsheet CFT to compute the two sides of the WGC bound.

Steps in the proof:

- we identify candidate superextremal states - we need appropriate charged states that exist in any string theory;
- we compute the long-range forces (= contributions to the RFC) between these states;
- we show that if we find self-repulsive states, there are also superextremal states.

1

Identify charged states that exist in any string theory via modular invariance arguments on the worldsheet;

Flavored Partition Function

Given the modular invariant partition function

$$Z(\tau,\overline{\tau}) = \mathrm{Tr}\left[q^{L_0-\frac{c}{24}}\overline{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right] = \sum q^{h-\frac{c}{24}}\overline{q}^{\tilde{h}-\frac{\tilde{c}}{24}}\,, \qquad q = e^{2\pi i\tau}$$

we Introduce the "flavored" partition function

$$Z(\mu,\tau;\tilde{\mu},\overline{\tau}) = \sum q^{h-\frac{c}{24}}y^{Q}\overline{q}^{\tilde{h}-\frac{\tilde{c}}{24}}\tilde{y}^{\tilde{Q}}$$

with $y^Q = e^{2\pi i\mu_a Q^a}, \tilde{y}^{\tilde{Q}} = e^{-2\pi i\tilde{\mu}_{\tilde{a}}\tilde{Q}^{\tilde{a}}}$,
S now acts non trivially

$$Z(\mu,\tau;\tilde{\mu},\overline{\tau}) = \begin{cases} Z(\mu,\tau+1;\tilde{\mu},\overline{\tau}+1) \\ e^{-\frac{\pi i}{\tau}\mu^2+\frac{\pi i}{\overline{\tau}}\tilde{\mu}^2} Z\left(\frac{\mu}{\tau},-\frac{1}{\tau};\frac{\tilde{\mu}}{\overline{\tau}},-\frac{1}{\overline{\tau}}\right) \end{cases}$$

Spectral Flow

Invariance over *period lattice* Γ $((\mu, \tilde{\mu}) \to (\mu + \rho, \tilde{\mu} + \tilde{\rho}))$ and introducing $\widehat{h} = h - \frac{1}{2}Q^2, \widehat{\tilde{h}} = \tilde{h} - \frac{1}{2}\tilde{Q}^2$, the flavored partition function reads

$$Z = \sum q^{\widehat{h} - \frac{c}{24}} \overline{q}^{\widehat{\tilde{h}} - \frac{\tilde{c}}{24}} q^{\frac{1}{2}(Q+\rho)^2} y^{Q+\rho} \overline{q}^{\frac{1}{2}(\tilde{Q}+\tilde{\rho})^2} \tilde{y}^{\tilde{Q}+\tilde{\rho}} \,.$$

which tells us that the spectrum is invariant under

$$(Q, \tilde{Q}) \to (Q, \tilde{Q}) + (\rho, \tilde{\rho}),\ (\rho, \tilde{\rho}) \in \Gamma \text{ , with } \widehat{h},\ \widehat{\tilde{h}} \text{ fixed.}$$

Applying this to neutral $(Q = 0, \tilde{Q} = 0)$ states, we have a massive spectrum

$$\frac{\alpha'}{4}m^2 = \frac{1}{2}\max(h, \tilde{h}) - 1 = \frac{\alpha'}{4}m^2 = \frac{1}{2}\max(Q^2, \tilde{Q}^2) - 1$$

2

Compute the long-range forces (contributions to the RFC) relating EFT amplitudes to worldsheet three-point functions;

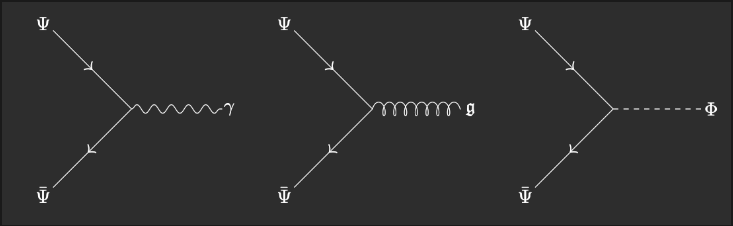
Long-range forces

Long-range forces

$$\mathcal{F} = \mathcal{F}_{\text{GRAV}+\Phi^0} + \mathcal{F}_{\text{GAUGE}} + \mathcal{F}_{\Phi^i \text{ MODULI}}$$

are mediated by massless states.

F depends on EFT vertices



$$\langle \overline{\Psi}\Psi\gamma^A \rangle \sim \delta^{AB} \mathfrak{e}^a_B Q_a \,, \qquad \langle \overline{\Psi}\Psi\mathfrak{g} \rangle \sim \kappa_d m \,, \qquad \langle \overline{\Psi}\Psi\Phi^I \rangle \sim \delta^{IJ} g^i_J \frac{\partial m}{\partial \phi^i}$$

These are mapped to worldsheet three-point functions.

Correlators from the worldshheet

Factorizing the worldsheet theory as $\mathcal{T} = \mathcal{T}_{\text{EXT}} \otimes \mathcal{T}_{\text{INT}}$

- $\mathcal{T}_{\text{EXT}}$: "external" universal free theory $\sim$ Free Boson
- $\mathcal{T}_{\text{INT}}$: "internal" interacting modular-invariant unitary CFT with $c = 26 - D$

The correlators factorize as

$$\langle V_1 V_2 V_3 \rangle = \langle V_1^{\text{EXT}} V_2^{\text{EXT}} V_3^{\text{EXT}} \rangle_{\text{EXT}} \cdot \langle V_1^{\text{INT}} V_2^{\text{INT}} V_3^{\text{INT}} \rangle_{\text{INT}}$$

Massless states contributing to long-range interactions are mapped to "internal" CFT operators

EFT state	worldsheet operator	$(h, \tilde{h})$
graviton $g_{\mu\nu}$	1	(0,0)
photons A_μ^a	$J^a(z)$	(1,0)
	$\tilde{J}^a(\bar{z})$	(0,1)
scalars (moduli) Φ^i	$\phi^i(z, \bar{z})$	(1,1)

Universal terms

$\mathcal{F}_{\text{GRAV}+\Phi^0}$: since graviton and dilaton operators in the internal CFT are trivial $g_{\text{INT}} = \Phi^0_{\text{INT}} = \widehat{1} \implies$ reduces to two-point functions + external normalization factors

$$\mathcal{F}_{\text{GRAV}+\Phi^0} \sim \langle \Psi\Psi g \rangle \langle gg \rangle^{-1} \langle g \Psi\Psi \rangle = -k_D m^2$$

$\mathcal{F}_{\text{GAUGE}}$: Modes of the worldsheet currents $J^a_0, \tilde{J}^{\tilde{b}}_0$ extract charges of the Ψ states

$$\oint \frac{dz}{2\pi} \langle \Psi J^a(z) \Psi \rangle = \langle \Psi J^a_0(z) \Psi \rangle = Q^a \langle \Psi\Psi \rangle \implies \langle \Psi J^a(z) \Psi \rangle = \frac{i}{z} Q^a \Psi\Psi \rangle$$

$$\mathcal{F}_{\text{GAUGE}} = N_J \langle \Psi\Psi J^a \rangle \langle J^a J^b \rangle^{-1} \langle J^b \Psi\Psi \rangle = \frac{2k_D^2}{\alpha'} \left(\delta_{ab} Q^a Q^b + \delta_{\tilde{a}\tilde{b}} \tilde{Q}^{\tilde{a}} \tilde{Q}^{\tilde{b}} \right)$$

Moduli contribution

$\mathcal{F}_{\Phi^i}$ MODULI: This is the interesting to compute.
(1,1) primaries that contribute are

- $\lambda^{a\tilde{b}}(z,\bar{z}) = J^a(z)\tilde{J}^{\tilde{b}}(\bar{z})$

$$\langle \Psi \lambda^{a\tilde{b}}(z,\bar{z}) \Psi \rangle = \frac{1}{|z|^2} Q^a \tilde{Q}^{\tilde{b}}$$

- Neutral current primaries $\chi(z,\bar{z})$,
  i.e. $J_1^a|\chi\rangle = \tilde{J}_1^{\tilde{a}}|\chi\rangle = 0 \leftrightarrow \langle \chi \lambda^{a\tilde{b}} \rangle = 0$

$$\langle \Psi \chi(z,\bar{z}) \Psi \rangle = ?$$

Sugawara decomposition

We decompose the CFT factoring its current part.

$$T(z) = \widehat{T}(z) + T^J(z)\,, \quad T^J(z) = \frac{1}{2}\delta_{ab} : J^a(z)J^b(z) :$$

We know the weights of the charged states, $L_0^J|\Psi\rangle = \frac{1}{2}Q^2|\Psi\rangle$. Furthermore, we had factorized the moduli so that $\chi(z,\bar{z})$ are neutral current primaries.

Operator	$(h^J, \tilde{h}^J)$	$(\widehat{h}, \widetilde{\widehat{h}})$
$\Psi(z,,\bar{z})$	$\left(\frac{1}{2}Q^2, \frac{1}{2}\tilde{Q}^2\right)$	(0,0)
$\chi(z,,\bar{z})$	$(0,0)$	(1,1)

$$\Longrightarrow \underbrace{\langle\Psi\chi\Psi\rangle = \langle\Psi\Psi\rangle\langle\chi\rangle = 0}_{\text{too quick, but we show it carefully}}$$

Self-force

Putting all the pieces together

$$\mathcal{F} = -\frac{4k_D^2}{\alpha' m m'}\left(\frac{\alpha'}{2}mm' - \delta_{ab}Q^a Q^b\right)\left(\frac{\alpha'}{2}mm' - \delta_{\tilde{a}\tilde{b}}\tilde{Q}^{\tilde{a}}\tilde{Q}^{\tilde{b}}\right)$$

Recall massive spectrum contains $\frac{\alpha'}{4}m^2 = \frac{1}{2}\max(Q^2, \tilde{Q}^2) + N - 1$

- $N = 0$ $\mathcal{F}_{\text{SELF}} > 0$: if $|Q^2 - \tilde{Q}^2| > 2$
- $N = 0$ $\mathcal{F}_{\text{SELF}} = 0$: if $|Q^2 - \tilde{Q}^2| = 2$
- $N = 0$ $\mathcal{F}_{\text{SELF}} < 0$ if $|Q^2 - \tilde{Q}^2| = 0$:
- $N = 1$ $\mathcal{F}_{\text{SELF}} = 0$: always
- $N = 2$ $\mathcal{F}_{\text{SELF}} < 0$ always

For every site in the period lattice, we have states with $\mathcal{F}_{\text{SELF}} = 0 \implies$

Sublattice RFC

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Show that if states are self-repulsive everywhere in moduli space, then they must be superextremal. Œ

Black Hole solutions

Consider a refined statement of the WGC.
A state of mass m and charge q is superextremal if

$$M = \Lambda m \leq M_{\text{EXT}}(Q = \Lambda q)$$

RHS is the mass of the lightest BH of charge Q.

A spherically symmetric black hole solution

$$\dot{\delta} s^2 = -e^{2\psi(r)} f(r) \dot{\delta} t^2 + e^{-\frac{2}{d-3}\psi(r)} \left[\frac{\dot{\delta} r^2}{f(r)} + r^2 \dot{\delta} \Omega_{d-2}^2 \right]$$

$$F_2^a = \frac{\mathfrak{f}^{ab}(\phi(r)) Q_b}{V_{d-2}} \frac{e^{2\psi(r)}}{r^{d-2}} \dot{\delta} t \wedge \dot{\delta} r \,, \quad f(r) = 1 - \frac{r_h^{d-3}}{r^{d-3}}$$

Black Hole Extremality Bound

The black hole mass can be written as a functional

$$M_{\text{BH}\cdot} = \frac{1}{2}\int_0^{z_h} e^{2\psi}\underbrace{\left[\mathfrak{f}^{ab}Q_aQ_b - k_N M(\phi)^2 - G^{ij}M_{,i}M_{,j}\right]}_{\star}\dot{\delta} z - fe^{\psi}M(\phi)\Big|_0^{z_h}$$

If $\star \geq 0$, then $M_{\text{BH}} \geq -fe^{\psi}M(\phi)\Big|_0^{z_h}, = M(\phi_\infty) = M_{\text{EXT}}(Q)$

Now suppose there is a self-repulsive particle, i.e.

$$\mathfrak{f}^{ab}q_aq_b - k_N m^2 - G^{ij}m_{,i}m_{,j} \geq 0$$

Then taking $M(\phi) = \Lambda m(\phi) \implies M_{\text{BH}} \geq \Lambda m(\phi)$.

Superextremal states

We had in fact found the first excited states

$$N = 1 \quad \mathcal{F}_{\text{SELF}} = 0\text{:always (for any } Q)$$

to have vanishing self-force. So they satisfy (saturate)

$$\frac{Q_{\text{BH}}}{M_{\text{BH}}} \leq \frac{Q}{M(\phi)} = \frac{\Lambda q}{\Lambda m(\phi)} = \frac{q}{m}$$

which implies

$$\frac{\alpha'}{4} M_{\text{BH}}^2 \geq \underbrace{\frac{1}{2} \max(Q^2, \tilde{Q}^2)}_{N=1 \text{ states, } \mathcal{F}=0} > \underbrace{\frac{1}{2} \max(Q^2, \tilde{Q}^2) - 1}_{N=0 \text{ states, } \mathcal{F}=?}$$

Lightest states for each Q are <u>strictly</u> superextremal.

Summary

Results

The Weak Gravity Conjecture and its refinements was postulated and checked in many examples..

Various ingredients had appeared before but there was no comprehensive "proof".

- Modular invariance on the worldsheet / spectral flow
- Self-force computation for charged states $\rightarrow$ RFC
- Black hole solution relating RFC and WGC

Proof of Sublattice WGC
(flat space, perturbative, bosonic string)

Caveats and next steps

- Are there any issues with our perturbative results?
  - higher genus spectrum / modular invariance (claim: ✓)
  - generic background, cosmological constant (?)
  - existence of perturbative black hole solutions (stay tuned: [25xx.yyyyy] w/ Ben Heidenreich)
- How to repeat the derivation for superstrings?
  - more complicated structure, along the way find general features of superstring theory (stay tuned some more: [25/6zz.wwwww] w/ Ben Heidenreich)
- Can we relate our results to other conjectures in the swampland program?
  - leverage worldsheet approach

Thanks!

Backup

Universal contributions

The first factor

$$\mathcal{F}_{\text{GRAV}+\Phi^0} \sim N_G \langle \Psi\Psi g \rangle \langle gg \rangle^{-1} \langle g\Psi\Psi \rangle$$

is automatic since graviton and dilaton operators in the internal CFT are trivial $g_{\text{INT}} = \Phi^0_{\text{INT}} = \widehat{1}$. Therefore,

$$\langle g_{\text{INT}} \Phi^0_{\text{INT}} \Phi^i_{\text{INT}} \rangle = 0$$

there is no mixing between graviton, dilaton and other moduli. $\mathcal{F}_{\text{GRAV}+\Phi^0}$ comes purely from the external (normalization) factor.

$$\mathcal{F}_{\text{GRAV}+\Phi^0} = -k_D m^2$$

Gauge contribution

The worldsheet currents extract the charges of the Ψ states as the eigenvalues of the current modes $J_0^a, \tilde{J}_0^{\tilde{b}}$.

$$\mathcal{F}_{\text{GAUGE}} \sim N_J \langle \Psi \Psi J^a \rangle \langle J^a J^b \rangle^{-1} \langle J^b \Psi \Psi \rangle$$

The two-point function of the current is simply $\langle J^a J^b \rangle = \delta^{ab} \langle 1 \rangle$.
The three-point function $\langle \Psi J \Psi \rangle$ comes from

$$\oint \frac{dz}{2\pi} \langle \Psi J^a(z) \Psi \rangle = \langle \Psi J_0^a(z) \Psi \rangle = Q^a \langle \Psi \Psi \rangle \implies \langle \Psi J^a(z) \Psi \rangle = \frac{i}{z} Q^a \Psi \Psi \rangle$$

$$\begin{aligned} \mathcal{F}_{\text{GAUGE}} &= N_J \delta_{ab} Q^a Q^b + \tilde{N}_J \delta_{\tilde{a}\tilde{b}} \tilde{Q}^{\tilde{a}} \tilde{Q}^{\tilde{b}} \\ &= \frac{2k_D^2}{\alpha'} \left(\delta_{ab} Q^a Q^b + \delta_{\tilde{a}\tilde{b}} \tilde{Q}^{\tilde{a}} \tilde{Q}^{\tilde{b}} \right) \end{aligned}$$

Moduli contribution

Finally, we get to the interesting term to compute.

$$\mathcal{F}_{\Phi^i \text{ MODULI}} \sim N_{\Phi^i} \langle \Psi\Psi\phi^i \rangle \langle \phi^i \phi^j \rangle^{-1} \langle \phi^j \Psi\Psi \rangle$$

$(1,1)$ primaries in the internal CFT are

- $\lambda^{a\tilde{b}}(z,\bar{z}) = J^a(z)\tilde{J}^{\tilde{b}}(\bar{z})$ and
- Neutral current primaries $\chi(z,\bar{z})$, i.e. $J_1^a|\chi\rangle = \tilde{J}_1^{\tilde{a}}|\chi\rangle = 0 \leftrightarrow \langle \chi \lambda^{a\tilde{b}} \rangle = 0$

From the currents

$$\langle \Psi \lambda^{a\tilde{b}}(z,\bar{z})\Psi \rangle = \frac{1}{|z|^2} Q^a \tilde{Q}^{\tilde{b}}$$

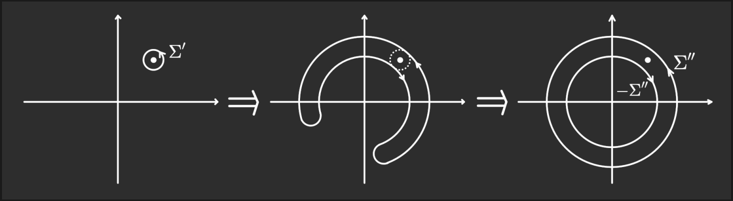
We still have the remaining contribution

$$\langle \Psi\chi(z,\bar{z})\Psi \rangle$$

Contour argument

Since $\chi(z,\bar{z})$ has weight (1,1) and is a neutral current primary

$$|\chi\rangle = L_0|\chi\rangle = \widehat{L}_0|\chi\rangle \implies \chi(z,\bar{z}) = \oint \frac{dz'}{2\pi i}(z'-z)\widehat{T}(z')\chi(z,\bar{z})$$



$$\chi(z,\bar{z}) = \left[\oint_{\Sigma''} \frac{dz'}{2\pi i}(z'-z)\widehat{T}(z'), \chi(z,\bar{z})\right] = [\widehat{L}_0 - z\widehat{L}_{-1}, \chi(z,\bar{z})]$$

$$\implies \langle \Psi \chi \Psi \rangle = \langle \Psi | [\widehat{L}_0 - z\widehat{L}_{-1}, \chi(z,\bar{z})] | \Psi \rangle = 0$$