Matrix ensembles from fuzzy physics

the good, the bad, the ugly

Juraj Tekel

Department of theoretical physics
Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS
Comenius University
Bratislava

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work with: M. Šubjaková, S. Kováčik, D. Prekrat, B. Bukor, K. Magdolenová, N. Pagliaroli

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Take home message

Take home message

Take home message

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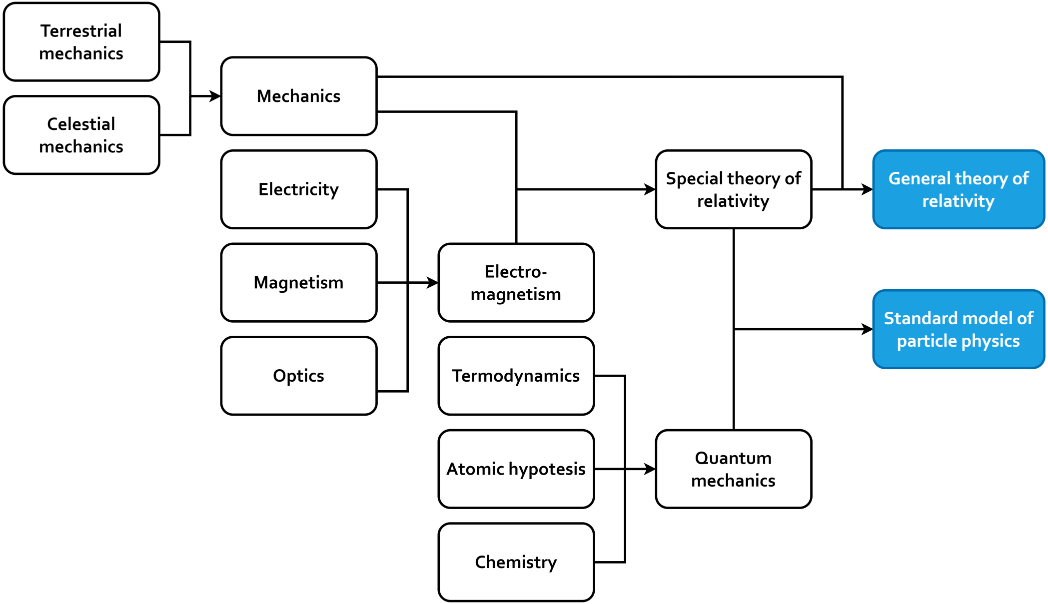
- Fuzzy spaces are examples of spacetimes with quantum structure.
- Plenty of interesting things happen on such spaces.
- Physics is described by random matrix ensembles.
  Analyzing these is technically challenging, but doable.

Quick motivation


Terrestrial mechanics
Celestial mechanics
Mechanics
Electricity
Magnetism
Optics
Electro-magnetism
Termodynamics
Atomic hypotesis
Chemistry
Special theory of relativity
Quantum mechanics
General theory of relativity
Standard model of particle physics

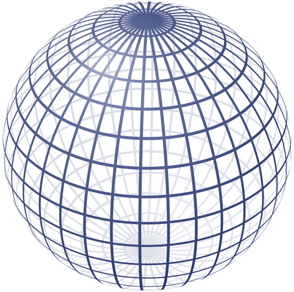
Quantum structure of spacetime

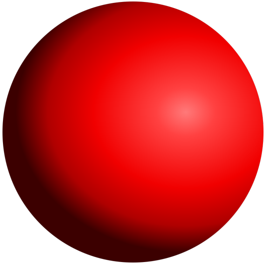
- We need a quantum theory of gravity.
- Quantization of general relativity leads to a nonrenormalizable theory.
- We have reasons to believe that future theory of quantum gravity will have a different notion of spacetime.
  No distinction between points under certain length scales. [Hossenfelder 1203.6191]
- Reasons:
  - gravitational Heisenberg microscope,
  - instability of quantum gravitational vacuum, [Doplicher, Fredenhagen, Roberts '95]
  - emergent spacetimes.
- Fuzzy spaces are very important examples of such spacetimes.

Fuzzy spaces

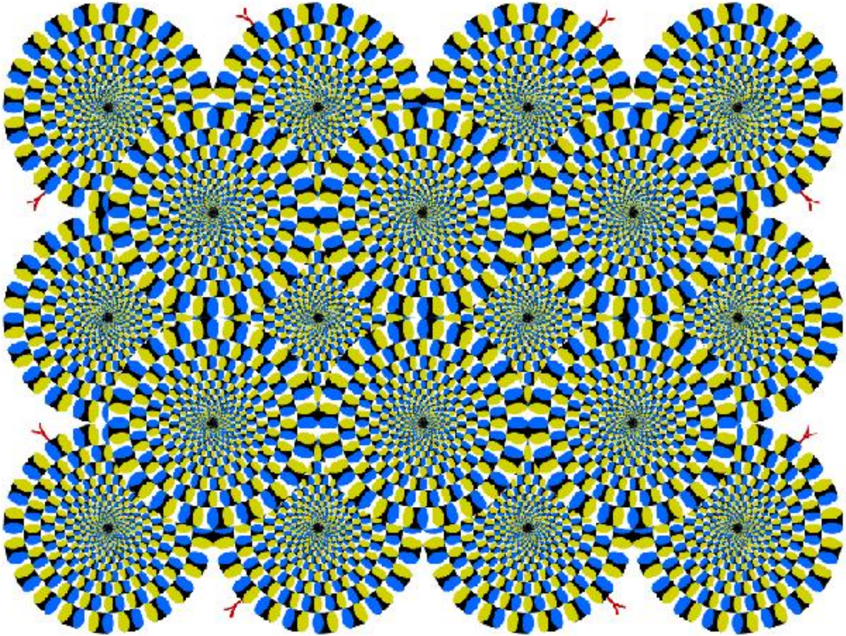
Fuzzy spaces

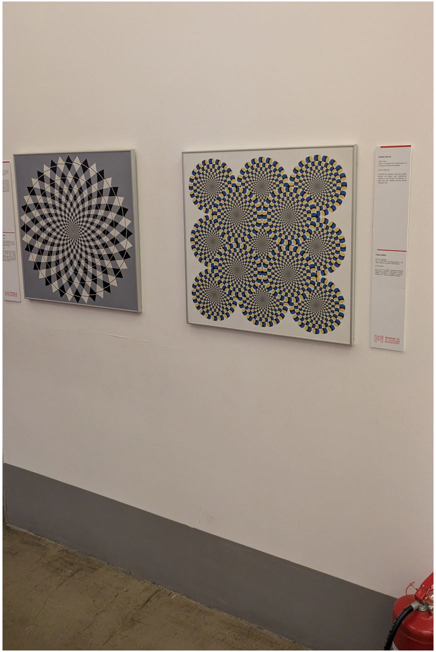
- We divide the space into N cells. Function on the fuzzy space is given by a matrix M and the eigenvalues of M represent the values of the function on these cells.





- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.

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Fuzzy spaces

- Regularization of infinities in the standard QFT.
  [Heisenberg ∼1930; Snyder 1947, Yang 1947]
- Regularization of field theories for numerical simulations.
  [Panero 2016]
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  [Seiberg Witten 1999; Douglas, Nekrasov 2001]
- Solutions of various matrix formulations of the string theory (IKKT, BFSS, BMN).
  [Steinacker 2013, 2024]
- Geometric unification of the particle physics and theory of gravity.
  [van Suijlekom 2015]
- An effective description of various systems in a certain limit (eg. QHE).
  [Karabali, Nair 2006]
- Toy models of spaces with discrete quantum structure.

Fuzzy sphere [Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder 1990s]

- Functions on the usual sphere are given by

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ ,$$

where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)\ .$$

- If we truncate the possible values of l in the expansion

$$f=\sum_{l=0}^{L}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ ,$$

we will not be able to see any features of functions under certain length scales.
- Expressions defined in this way are not closed under multiplication.

- Number of independent functions with $l \leq L$ is $(L+1)^2$, the same as the number of $N \times N$ hermitian matrices.
- We have a map $\varphi : Y_{lm} \to M$ and we define the product

$$Y_{lm} \star Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$

- Opposing to some lattice discretization this space still possess a full rotational symmetry.
- In the limit N or $L \to \infty$ we recover the original sphere.

Fuzzy spaces

- For the fuzzy sphere S_N^2 we define

$$\hat{x}_i \hat{x}_i = r^2 \quad , \quad [\hat{x}_i, \hat{x}_j] = i\theta \varepsilon_{ijk} \hat{x}_k \ , \ i,j = 1,2,3 \ .$$

- The conditions can be realized as an $N = 2s+1$ dimensional representation of $SU(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2-1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \sim \frac{2}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2-1} s(s+1) = r^2 \ .$$

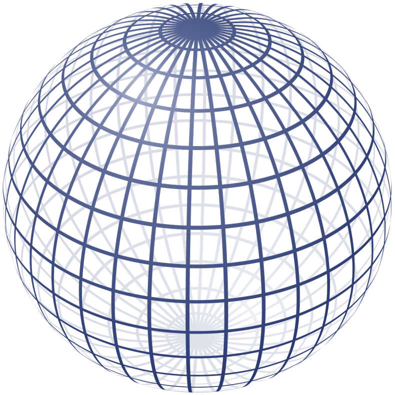
- The group $SU(2)$ still acts on $\hat{x}_i$'s and this space enjoys a full rotational symmetry. Most importantly nonzero commutators imply uncertainty relations for positions $\Delta x_i \Delta x_j \neq 0$.
- In a similar fashion it is possible to construct an analogous deformation of the plane

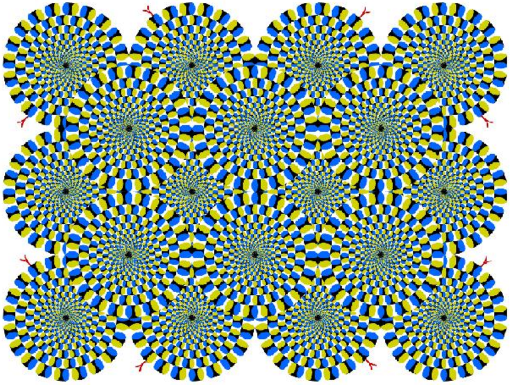
$$[\hat{x}_i, \hat{x}_j] = i\theta \varepsilon_{ij} = i\theta_{ij} \ , \ i = 1,2 \ .$$

Construction uses the $\star$-product

$$f \star g = f\, e^{\frac{i}{2} \overleftarrow{\partial}\, \theta\, \overrightarrow{\partial}} g = fg + \frac{i\theta^{\mu\nu}}{2} \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} + \cdots$$

Fuzzy spaces





Fuzzy field theories

Fuzzy scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi)\right]$$

and path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi\, F(\Phi) e^{-S(\Phi)}}{\int d\Phi\, e^{-S(\Phi)}} \ .$$

- We construct the noncommutative theory as an analogue with
  - field $\rightarrow$ matrix,
  - functional integral $\rightarrow$ matrix integral,
  - spacetime integral $\rightarrow$ trace,
  - derivative $\rightarrow$ L_i commutator.

Fuzzy scalar field theory

- **Commutative**

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi) \right] ,$$

$$\langle F \rangle = \frac{\int d\Phi\, F(\Phi) e^{-S(\Phi)}}{\int d\Phi\, e^{-S(\Phi)}} .$$

- **Noncommutative** (for S_F^2)

$$S(M) = \frac{4\pi R^2}{N} \mathrm{Tr} \left[\frac{1}{2} M \frac{1}{R^2} [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + V(M) \right] ,$$

$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} .$$

[Balachandran, Kürkçüoğlu, Vaidya 2005; Szabo 2003; Ydri 2016]

Random matrices ensembles

[M.L. Mehta 2004; B. Eynard, T. Kimura, S. Ribault 2015; G. Livan, M. Novaes, P. Vivo 2017]

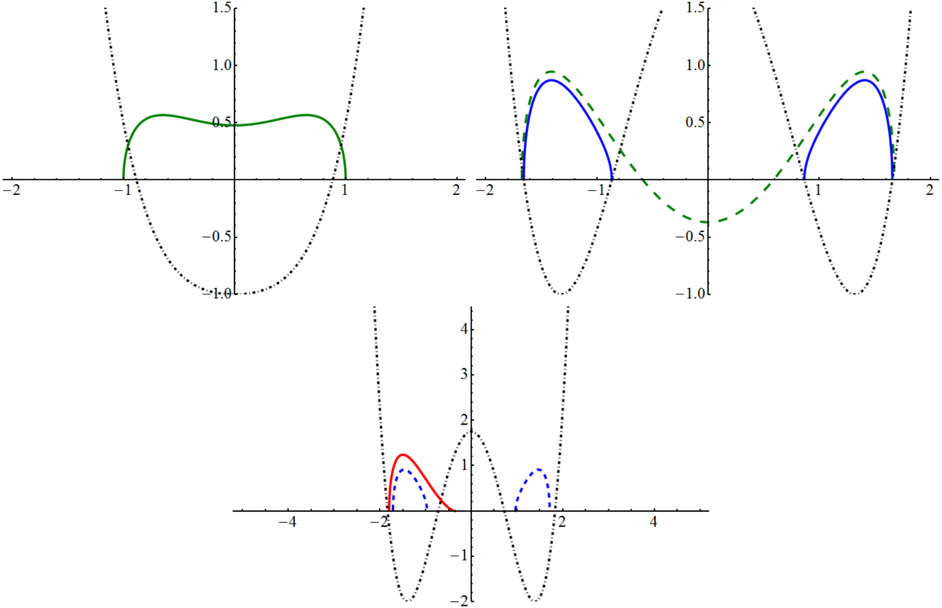
- Matrix model = ensemble of random matrices, e.g. ensemble of $N \times N$ hermitian matrices with

$$P(M) \sim e^{-N\mathrm{Tr}(V(M))}\ ,\ \text{usually}\ V(x) = \frac{1}{2} r\, x^2 + g\, x^4$$

- Expectation values

$$\langle f \rangle = \frac{1}{Z} \int dM\, P(M) f(M)$$

  can be analyzed
    - numerically using Hamiltonian Monte Carlo,
    - analytically in the large N limit using saddle point equation.
- One usually looks for eigenvalue distribution $\rho(x)$.



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Fuzzy field theories ensembles I
Full matrix model

Fuzzy field theory model

- Recall the action of the fuzzy scalar field theory

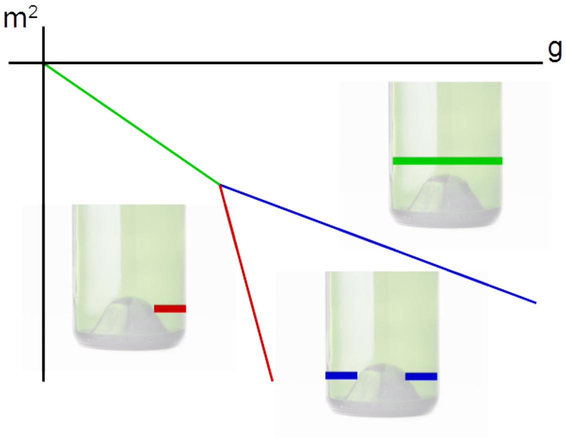
$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)\ . \qquad (1)$$

This is a particular case of a matrix model since we need

$$\int dM\, F(M) e^{-S(M)}\ .$$

- "Matrix model begs to be put on a computer".

$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + gM^4\right)$$



Fuzzy field theory model - UV/IR mixing

- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  [Gubser, Sondhi 2001; Chen, Wu 2002]
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is spontaneously broken.
- This has been established in numerous numerical works for variety different spaces.
  [Martin 2004; García Flores, Martin, O'Connor 2006, 2009; Panero 2006, 2007; Ydri 2014; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero 2014; Mejía-Díaz, Bietenholz, Panero 2014; Medina, Bietenholz, D. O'Connor 2008; Bietenholz, Hofheinz, Nishimura 2004; Lizzi, Spisso 2012; Ydri, Ramda, Rouag 2016; Kováčik, O'Connor 2018]
  [Panero 2015]
- The key property of the noncommutative field theories is the UV/IR mixing phenomenon, which arises as a result of the nonlocality of the theory.
  [Minwalla, Van Raamsdonk, Seiberg 2000; Vaidya 2001; Chu, Madore, Steinacker 2001]

Fuzzy field theories ensembles II
Perturbative model

Perturbative model

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M. [O'Connor, Sämann 2007; Sämann 2010]
- The most recent result is [Sämann 2015]

$$\begin{aligned} S_{eff}(\Lambda) =& \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] - \\ & - \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 - \\ & - \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2 \quad , \quad \text{where } c_n = \frac{1}{N}\sum_i \lambda_i^n \end{aligned} \qquad (2)$$

- Yields a very unpleasant behaviour close to the origin of the parameter space. [JT '15]

Fuzzy field theories ensembles III
Nonperturbative model

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. [Steinacker 2005]
- There is a unique parameter independent effective action that reconstructs this rescaling. [Polychronakos 2013]

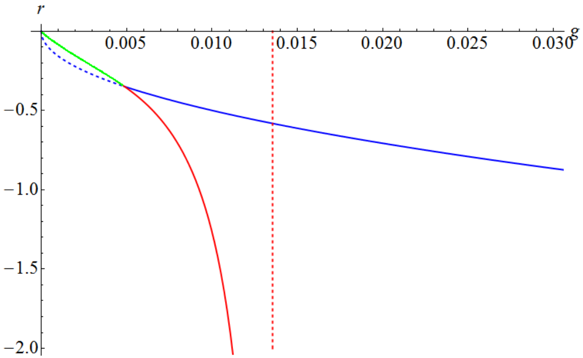
$$S_{eff}(\Lambda) = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}\ .$$

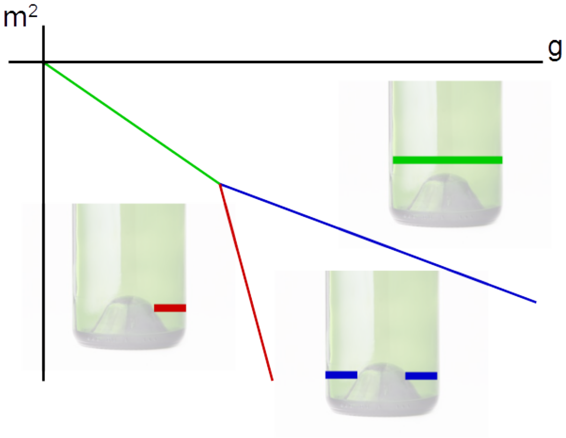
Can be generalized to more a more complicated kinetic term $\mathcal{K}$.

- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ we obtain a matrix model

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r\,\mathrm{Tr}\,(M^2) + g\,\mathrm{Tr}\,(M^4)\quad,\quad F(t) = \log\left(\frac{t}{1-e^{-t}}\right)\ . \qquad (3)$$

[Šubjaková, JT PoS CORFU2019; JT '14 '15 '18; Šubjaková, JT '20]





[JT '18; Šubjaková, JT 2020]

- Taking a lesson from

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}r\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right) \quad , \quad F(t) = \log\left(\frac{t}{1-e^{-t}}\right)$$

we could try to complete the perturbative action

$$S_{eff} = F\left[c_1, t_2, t_3, t_4 - 2t_2^2\right] = \frac{1}{2}\log\left(\frac{t_2}{1-e^{-t_2}}\right) + F_3(t_3) + F_4(t_4 - 2t_2^2) \tag{4}$$

and

$$F_4(y_4) = \alpha_0 \log(y_4) + \alpha_1 + \frac{\alpha_2}{y_4} + \frac{\alpha_3}{y_4^2} + \dots \; .$$

- Any attempt to complete the perturbative expansion in the spirit of the non-perturbative model is not capable of solving the above problems and does not lead to a phase diagram that is in complete agreement with the numerical simulations. Most importantly the location of the triple point can not be brought closer to the numerical value. [Šubjaková, JT '22]

Fuzzy field theories ensembles IV
Removal of stripes

Removal of stripes

- There are more complicated field theory models on fuzzy sphere and NC plane, where the UV/IR mixing is not present. [Dolan, O'Connor, Prešnajder '01; Grosse Wulkenhaar '00's].
- These can be recast as matrix models and at least some their aspect studied.

$$S(M)=\frac{1}{2}\mathrm{Tr}\left(M\mathcal{K}M\right)+\frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right)+g\,\mathrm{Tr}\left(M^4\right)\ ,\ \mathcal{K}=(1+ag)C_2+bg\,C_2^2\ ,\ C_2=[L_i,[L_i,\cdot]]\,. \quad (5)$$

[Šubjaková, JT '20]

$$S=\mathrm{Tr}\left(M[X,[X,M]]+M[Y,[Y,M]]\right)-g_r\mathrm{Tr}\left(RM^2\right)-g_2\mathrm{Tr}\left(M^2\right)+g_4\mathrm{Tr}\left(M^4\right)\ . \quad (6)$$

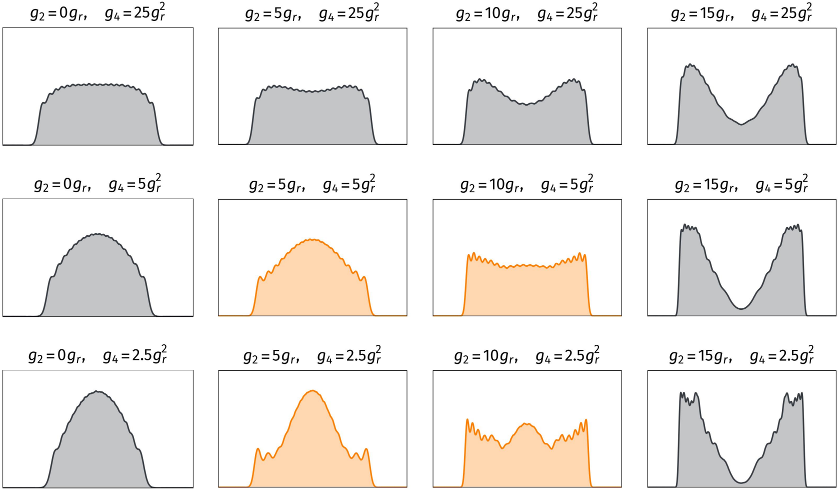
[Bukor, JT '23]

[Prekrat, Todorović-Vasović, Ranković, Kováčik, JT '22; Bukor, Prekrat, JT '25]

$$S(\Lambda)=N\left(-g_2c_2+8g_rc_2+g_4c_4-\frac{32}{3}g_r^2c_4+\frac{1024}{45}g_r^4c_8-\frac{(8g_r)^6}{2835}c_{12}\right)+$$
$$+\frac{32}{3}g_r^2c_2^2+\frac{1024}{15}g_r^4c_4^2-\frac{4096}{45}g_r^4c_6c_2+\frac{2(8g_r)^6}{945}c_2c_{10}-\frac{(8g_r)^6}{189}c_4c_8+\frac{2(8g_r)^6}{567}c_6^2\,.$$

New phase in GW model?

[Bukor, Prekrat, JT '25]



Fuzzy field theories ensembles V
Beyond phase structure

Correlation functions

- Behaviour of

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{Z} \int d\phi \, \langle \vec{x} | \, \phi \, | \vec{x} \rangle \, \langle \vec{y} | \, \phi \, | \vec{y} \rangle \, e^{-S(\phi)}$$

in the matrix model can be studied numerically.
[Hatakeyama, Tsuchiya '17; Hatakeyama, Tsuchiya, Yamashiro '18 '18]

- At the "standard" phase transition, the behaviour of the correlation functions at short distances differs from the commutative theory and seems to agree with the tricritical Ising model.
A different behaviour at long distances.

Entanglement entropy

- In local theories $S(A) \sim A$.
  [Ryu, Takayanagi '06]
  In non-local theories this can change.
  [Barbon, Fuertes '08; Karczmarek, Rabideau '13; Shiba, Takayanagi '14]
- Problem on the fuzzy sphere has been studied numerically.
  [Karczmarek, Sabella-Garnier '13; Sabella-Garnier '14; Okuno, Suzuki, Tsuchiya '15; Suzuki, Tsuchiya '16; Sabella-Garnier '17; Chen, Karczmarek '17]
- For free fields, the EE follows volume law rather than area law.
  In the interacting case much smaller EE than in the free case.

Fuzzy field theories ensembles VI
Other spaces

Other spaces

- To study entanglement entropy, we need to extended the model to $\mathbb{R} \times S_F^2$, i.e. $M(t)$

$$S(M) = \int dt \mathrm{Tr} \left(-\frac{1}{2} M \partial_t^2 M + \frac{1}{2} M [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + g M^4 \right) \quad (8)$$

[Medina, Bietenholz, O'Connor '07; Ihl, Sachse, Sämann '10]

- This is matrix quantum mechanics, different but similar methods apply. [Jevicki, Sakita '80]
- We are trying to apply the second moment approximation here. For EE free theory where $\mathcal{R} = 0$, is enough. [Bukor, JT work in progress]

Other spaces

- The field theory on other spaces differs in the definition of the kinetic term.

$$S(M) = \text{Tr}\left(\frac{1}{2}M\mathcal{K}M + \frac{1}{2}m^2M^2 + gM^4\right) \ . \qquad (9)$$

  Second moment approximation applicable.
- Numerical results available for fuzzy disc [Lizzi, Spisso '12] and torus [Mejía-Díaz, Bietenholz, Panero '14].
- Perturbative models have been derived for $\mathbb{C}P^2, \mathbb{C}P^3$ [Sämann '10], disc [Rea, Sämann '15].

Dirac ensembles and random fuzzy geometries

Quantum dynamics of finite spectral triples

- Noncommutative geometry can be described by a spectral triple [Connes '94]

$$(\mathcal{A}, \mathcal{D}, \mathcal{H}) \ .$$

- For certain finite geometries the Dirac operator can be constructed using (anti)commutators with p hermitian and q antihermitian matrices (and some Clifford module baggage) to form a (p, q) geometry [Barrett '15].
- Path integral over geometries given by weight

$$\int d\mathcal{D} e^{-S(\mathcal{D})}$$

and becomes (multi)matrix integral. The simplest nontrivial choice is $S(\mathcal{D}) = \mathrm{Tr}\left(g\,\mathcal{D}^2 + \mathcal{D}^4\right)$.
[Barrett, Glaser '16; Khalkhali '20s; D'Arcangelo '22; Glaser '23]
- Toy model of fluctuating dynamical geometry.

(1, 0) Dirac ensemble

[Khalkhali, Pagliaroli '21; Bukor, Kováčik, Magdolenová, Pagliaroli, JT work in progress]

- In the simplest $(1,0)$ case the Dirac operator is given by

$$\mathcal{D}\cdot = \{M,\cdot\}\ .$$

- Then simply $d\mathcal{D} = dM$. The action for M is given by

$$S(M) = N\left(2g\,c_2 + 2c_4\right) + 2g\,c_1^2 + 8c_1c_3 + 6c_2^2\ . \tag{10}$$

  Simpler model analyzed before [Bukor, JT '25].

- Can be analyzed numerically, analytically and using bootstrap.

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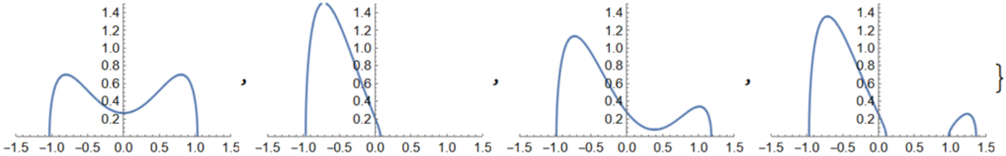
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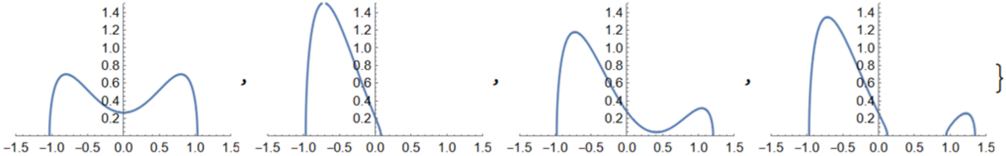
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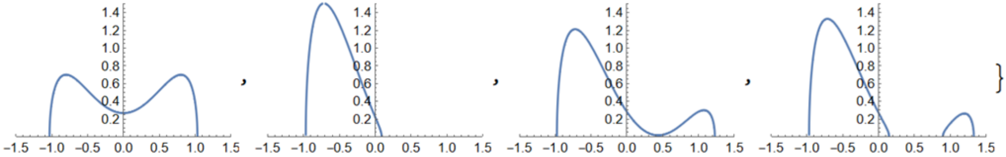
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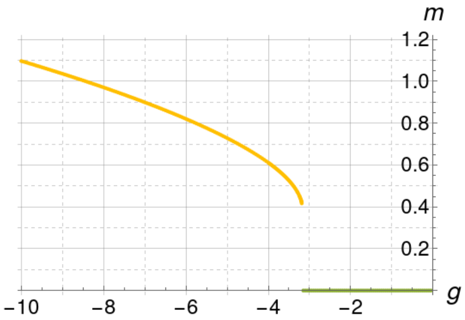
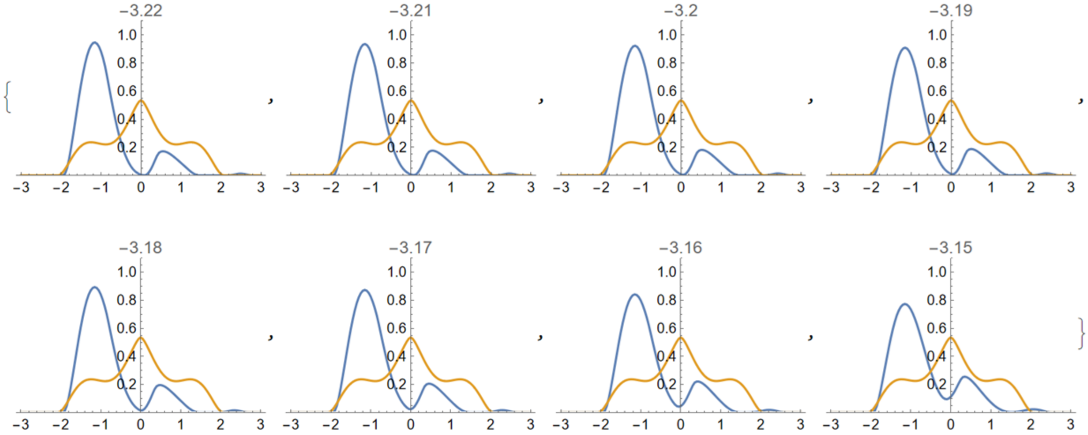
}

(1, 0) Dirac ensemble

- The ensamble
$$S(M) = N\left(2gc_2 + 2c_4\right) + 2gc_1^2 + 8c_1c_3 + 6c_2^2$$
has a stable asymmetric 2-cut regime for $g < -3.18702$.
- Similar results obtained numerically before [D'Arcangelo '22].





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$(1, 3)$ Dirac ensemble is fuzzy sphere

- More complicated spaces are described by multi matrix models. Symetric regime has been analyzed before, but no results for asymmetric regime.
-

$$(1, 3) \tag{11}$$

geometry is the fuzzy sphere!
- Not much hope for analytical results, but bootstrap might be useful.

One final ensemble - gauge theory on NC plane

[Buric, Grosse, Madore '10]

- start with the 3-dim action on ϵ-tHA:

$$S_{YM} = \frac{1}{16g^2}\mathrm{tr}(F(*F) + (*F)F)$$

$$[X, Y] = i\epsilon(\mathbb{1} - Z)$$

$$\epsilon P_1 = Y \qquad \epsilon P_2 = -X$$

$$\downarrow$$

"compactification" to $z = 0$

$$\downarrow$$

$$S_{YM} = \frac{1}{2}\mathrm{tr}\Big(\frac{1-\epsilon^2}{g^2}F_{12}^2 + (D\phi)^2 + (5-\epsilon^2)\mu^2\phi^2 - $$
$$- 2(1-\epsilon^2)\frac{\mu}{g}F_{12}\phi - 4\epsilon F_{12}\phi^2 + \epsilon^2\{P + gA, \phi\}^2\Big) \quad \mathbf{(12)}$$

where

$$D_\alpha\phi = i[P_\alpha + gA_\alpha, \phi] \qquad F_{12} = ig[P_1, A_2] - ig[P_2, A_1] + ig^2[A_1, A_2]$$

Gauge theory on NC plane

- Standard analysis of this model suggests that it is not renormalizable even with the GW trick [Buric, Nenadovic, Prekrat '16].
- A rather complicated three matrix model.
- Can we see that in the phase structure of the corresponding matrix model – is there a striped phase? [work in progress]

Take home message

Take home message and 2do list

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- Plenty of interesting things happen on such spaces.
- Physics is described by random matrix ensembles.
  Analyzing these is technically challenging, but doable.

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Thank you for your attention!