# Complex Structures for the Full Flag Manifold of Quantum SU(3)

(Joint work with R. Ó Buachalla and A. Carotenuto)

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# Hopf Algebra

**Definition**
A Hopf algebra is a quadruple $(\mathcal{A}, \Delta, \epsilon, S)$ where:

- $\mathcal{A}$ is a unital algebra.
- $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ and $\epsilon : \mathcal{A} \longrightarrow \mathbb{C}$ are algebra homomorphisms called coproduct and counit respectively.
- $S : \mathcal{A} \longrightarrow \mathcal{A}$ an algebra anti-morphism called antipode.

such that:

- $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ (coassociativity)
- $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$ (counit condition)
- $m \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$ (antipode condition)

**Example**

Let $G$ be a finite group. By $\mathcal{O}(G)$ we denote the set of all complex-valued functions on $G$.

Together with the pointwise operations $\mathcal{O}(G)$ turns into an algebra:

$$\begin{aligned}(f_1 + f_2)(g) &:= f_1(g) + f_2(g)\\ (f_1 f_2)(g) &:= f_1(g) f_2(g)\end{aligned}$$

Moreover, together with the coproduct, counit and antipode defined as follows:

$$\Delta(f)(g, g') := f(gg'), \qquad \varepsilon(f) := f(e), \qquad S(f)(g) := f(g^{-1}),$$

for all $g, g' \in G$ and $f \in \mathcal{O}(G)$, and $e$ denotes the identity element of $G$, $\mathcal{O}(G)$ turns into a commutative Hopf algebra.

**Example**

Let $q \in \mathbb{C}^{\times}$ be a non-zero complex number and set $\nu := q - q^{-1}$. The algebra of quantum matrices $\mathbb{C}_q[\mathrm{M}_n]$ is defined as the quotient algebra:

$$\mathbb{C}_q[\mathrm{M}_n] := \mathbb{C}\langle u_{ij} |\ i, j = 1, \cdots, n \rangle / \mathrm{I},$$

where $\mathrm{I}$ is the ideal generated by following relations (often called Manin relations):

$$u_{ik} u_{jk} - q u_{jk} u_{ik}, \qquad u_{ki} u_{kj} - q u_{kj} u_{ki}, \qquad 1 \leq i < j \leq n,\ 1 \leq k \leq n,$$
$$u_{il} u_{jk} - u_{jk} u_{il}, \qquad u_{ik} u_{jl} - u_{jl} u_{ik} - \nu u_{il} u_{jk}, \qquad 1 \leq i < j \leq n,\ 1 \leq k < l \leq n.$$

$\mathbb{C}_q[\mathrm{M}_n]$ turns into a bialgebra together with the coproduct and counit defined as follows:

$$\Delta(u_{ij}) := \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \qquad \varepsilon(u_{ij}) := \delta_{ij}.$$

# Quantum Special Linear Group

Define the quantum determinant to be:

$$\det_q := \sum_{\sigma \in \mathrm{S}_n} (-q)^{l(\sigma)} u_{1\sigma(1)} \cdots u_{n\sigma(n)},$$

where $l(\sigma)$ denotes the length of $\sigma$.

The quantum special linear group $\mathbb{C}_q[\mathrm{SL}_n]$ is defined as the quotient algebra:

$$\mathbb{C}_q[\mathrm{SL}_n] := \mathbb{C}_q[\mathrm{M}_n]/\langle \det_q - 1 \rangle$$

$\mathbb{C}_q[\mathrm{SL}_n]$ turns into a Hopf algebra together with the antipode $S$ defined as follows:

$$S(u_{ij}) = (-q)^{i-j} \sum_{\sigma \in \mathrm{S}_{n-1}} (-q)^{l(\sigma)} u_{k_1\sigma(l_1)} \cdots u_{k_{n-1}\sigma(l_{n-1})},$$

where $\{k_1, \cdots, k_{n-1}\} := \{1, \cdots, n\} - \{j\}$ and $\{l_1, \cdots, l_{n-1}\} := \{1, \cdots, n\} - \{i\}$ as ordered sets.

Furthermore, $\mathbb{C}_q[\mathrm{SL}_n]$ also admits a $*$-structure given by:

$$(u_{ij})^* := S(u_{ji}).$$

$\mathbb{C}_q[\mathrm{SL}_n]$ togther with this $*$-structure is called quantum special unitary group $\mathbb{C}_q[\mathrm{SU}_n]$.

**Example**

Let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then, together with the following data:

$$\Delta(X) := X \otimes 1 + 1 \otimes X, \qquad \varepsilon(X) = 0, \qquad S(X) = -X, \qquad \forall\, X \in \mathrm{U}(\mathfrak{g})$$

$\mathrm{U}(\mathfrak{g})$ turns into a Hopf algebra.

# Drinfeld–Jimbo Quantized Enveloping Algebras

Let $\mathfrak{g}$ be a finite dimensional, complex, semi-simple Lie algebra of rank $l$, and $(a_{ij})$ denotes its Cartan matrix, and we fix $q_i := q^{(\alpha_i,\alpha_i)/2}$.

The Drinfeld–Jimbo quantized enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ is the algebra generated by the elements $E_i, F_i, K_i$ and $K_i^{-1}$ subject to the following relations:

$$K_iE_j = q_i^{a_{ij}}E_jK_i, \qquad K_iF_j = q_i^{-a_{ij}}F_jK_i, \qquad K_iK_j = K_jK_i,$$
$$K_iK_i^{-1} = 1 = K_i^{-1}K_i, \qquad E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

where $i, j \in \{1, \cdots, l\}$, and the quantum Serre relations:

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# Drinfeld–Jimbo Quantized Enveloping Algebras

**Theorem**
$\mathrm{U}_q(\mathfrak{g})$ *admits a unique Hopf algebra structure with comultiplication* $\Delta$*, counit* $\varepsilon$ *and antipode* $S$ *defined as:*

$$\Delta(E_i) := E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta(F_i) := F_i \otimes 1 + K_i^{-1} \otimes F_i,$$
$$\Delta(K_i) := K_i \otimes K_i, \qquad \Delta(K_i^{-1}) := K_i^{-1} \otimes K_i^{-1},$$

$$\varepsilon(K_i) := 1, \qquad \varepsilon(E_i) := \varepsilon(F_i) = 0,$$

$$S(E_i) := -E_i K_i^{-1}, \qquad S(F_i) := -K_i F_i, \qquad S(K_i) := K_i^{-1}.$$

**Theorem**
*Any finite-dimensional irreducible representation of a Drinfeld–Jimbo algebra is a weight representation and a representation with highest weight. Such a representation is uniquely determined by its highest weight.*

# Dual Pairing Between $\mathcal{O}_q(\mathrm{SU}_n)$ And $\mathrm{U}_q(\mathfrak{sl}_n)$

Let $G$ and $H$ be two Hopf algebras. A dual pairing is a bilinear map

$$\langle -, - \rangle : G \otimes H \longrightarrow \mathbb{C}$$

such that, for all $g, g' \in G$ and $h, h' \in H$:

$$\langle g, hh' \rangle = \langle g_{(1)}, h \rangle \langle g_{(2)}, h' \rangle, \qquad \langle gg', h \rangle = \langle g, h_{(1)} \rangle \langle g', h_{(2)} \rangle,$$
$$\langle g, 1_H \rangle = \varepsilon(g), \qquad \langle 1_G, h \rangle = \varepsilon(h), \qquad \langle S(g), h \rangle = \langle g, S(h) \rangle.$$

**Example**

A dual pairing of Hopf algebras between $\mathcal{O}_q(\mathrm{SL}_n)$ and $\mathrm{U}_q(\mathfrak{sl}_\mathfrak{n})$ is given by:

$$\langle u_{i+1,i}, E_i \rangle = 1, \qquad \langle u_{i,i+1}, F_i \rangle = 1,$$
$$\langle u_{ii}, K_j \rangle = q^{\delta_{j+\mathbf{1},i} - \delta_{ij}}, \qquad \langle u_{ii}, K_j^{-1} \rangle = q^{\delta_{ij} - \delta_{j+\mathbf{1},i}},$$

and requiring all other pairings to be zero, where $u_{ij}$ denotes the generators of $\mathcal{O}_q(\mathrm{SL}_n)$ and $E_i$, $F_i$, $K_i$ are the generators of $\mathrm{U}_q(\mathfrak{sl}_\mathfrak{n})$. Furthermore, this pairing respects the $*$-structure.

# Full Flag Manifold $\mathcal{O}_q(\mathrm{F}_3)$

The dual pairing described above gives a natural $\mathrm{U}_q(\mathfrak{sl}_{n+1})$-module structure on $\mathcal{O}_q(\mathrm{SU}_{n+1})$:

$$f \triangleright b := \sum b_{(1)} \langle f, b_{(2)} \rangle \qquad \text{for } f \in \mathrm{U}_q(\mathfrak{sl}_{n+1}),\ b \in \mathcal{O}_q(\mathrm{SU}_{n+1}).$$

The full quantum flag manifold $\mathcal{O}_q(\mathrm{F}_3)$ is defined as the subalgebra of $\mathrm{U}_q(\mathfrak{h})$-invariants:

$$\begin{aligned} \mathcal{O}_q(\mathrm{F}_3) &:= {}^{\mathrm{U}_q(\mathfrak{h})}\mathcal{O}_q(\mathrm{SU}_3), \\ &= \Big\{ b \in \mathcal{O}_q(\mathrm{SU}_3) \,|\, f \triangleright b = \varepsilon(f) b,\ \forall\ f \in \mathrm{U}_q(\mathfrak{h}) \Big\}. \end{aligned}$$

where $\mathrm{U}_q(\mathfrak{h}) := \mathrm{span}\{K_i^{\pm}\} \subset \mathrm{U}_q(\mathfrak{sl}_3)$.

**Theorem**
$\mathcal{O}_q(\mathrm{F}_3)$ *as a subalgebra of* $\mathcal{O}_q(\mathrm{SU}_3)$ *is generated by the following elements:*

$$z_{ij}^{\alpha_1} := u_{i1} u_{j1}^* = u_{i1} S(u_{1j}), \qquad z_{ij}^{\alpha_2} := u_{i3} u_{j3}^* = u_{i3} S(u_{3j}), \qquad \textit{for } i, j = 1, 2, 3.$$

# First-Order Differential Calculus

A first-order differential calculus over an algebra $\mathcal{B}$ is a pair $(\Omega^1, \mathrm{d})$, where, $\Omega^1$ is an $\mathcal{B}$-bimodule, and,

$$\mathrm{d} : \mathcal{B} \longrightarrow \Omega^1,$$

is a linear map obeying:

(i) $\mathrm{d}(ab) = \mathrm{d}a.b + a.\mathrm{d}b \qquad \forall\ a, b \in \mathcal{B}$,

(ii) $\Omega^1 = \mathrm{span}\{a\mathrm{d}b : a, b \in \mathcal{B}\}$,

(iii) $\ker \mathrm{d} = \mathbb{K}.1$.

**Example (Universal First-Order Differential Calculus)**

Given any algebra $\mathcal{B}$, define:

$$\Omega^1_u := \ker(\mathrm{m}) = \{\sum a \otimes b \in \mathcal{B} \otimes \mathcal{B} : \sum ab = 0\},$$

$$\mathrm{d}_u(a) := 1 \otimes a - a \otimes 1, \qquad \forall\ a \in \mathcal{B}.$$

It is easy to verify that the pair $(\Omega^1_u, \mathrm{d}_u)$ is a first-order differential calculus.

**Theorem**
*Any FODC* $(\Omega^1, \mathrm{d})$ *on* $\mathcal{B}$ *is isomorphic to some quotient calulus* $(\Omega^1_u/N, \mathrm{d}_N)$ *where* $N$ *is a sub-bimodule and* $\mathrm{d}_N := \pi \circ \mathrm{d}_u$*.*

**Definition**
A differential calculus on an algebra $\mathcal{A}$ is a triplet $(\Omega, \wedge, \mathrm{d})$, where $\Omega = \oplus_n \Omega^n$ is a graded-algebra,

$$\wedge : \Omega \otimes \Omega \longrightarrow \Omega \qquad \text{and} \qquad \mathrm{d} : \Omega \longrightarrow \Omega$$

are linear maps, such that:

- (i) $\Omega^k \wedge \Omega^l \subset \Omega^{k+l}, \qquad \mathrm{d}(\Omega^k) \subset \Omega^{k+1}, \qquad \forall\, k, l \in \mathbb{N}_0,$
- (ii) The wedge product $\wedge$ is associative,
- (iii) $\mathrm{d}^2 = 0$, and $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge \mathrm{d}\eta$, for all $\eta, \omega \in \Omega$, $\omega$ being homogeneous,
- (iv) $\Omega^0 = \mathcal{A}, \qquad \Omega^n = \mathrm{span}\{a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n : a_0, \cdots a_n \in \mathcal{A}\}$.

# Left-Covariant FODC

Let $\mathcal{A}$ be a Hopf algebra. A first order differential calculus $(\Omega^1, \mathrm{d})$ on a left $\mathcal{A}$-comodule algebra $\mathcal{B}$ is called left $\mathcal{A}$-covariant if $\Omega^1$ also admit a left coaction $\Phi_L$ such that:

$$\Phi_L(a\sigma b) = \Delta_L(a)\Phi_L(\sigma)\Delta_L(b) \qquad \text{for all } a, b \in \mathcal{B}, \ \sigma \in \Omega^1,$$

and the following diagram commute:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Delta_L} & \mathcal{A} \otimes \mathcal{B} \\ \downarrow{\scriptstyle \mathrm{d}} & & \downarrow{\scriptstyle \mathrm{id} \otimes \mathrm{d}} \\ \Omega^1 & \xrightarrow{\Phi_L} & \mathcal{A} \otimes \Omega^1. \end{array}$$

**Theorem (Woronowicz, 1989):** Let $\mathcal{A}$ be a Hopf algebra, $\mathcal{R}$ be a right ideal contained in $\ker \varepsilon$ and $N = r^{-1}(\mathcal{A} \otimes \mathcal{R})$ where $r : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ defined as: $r(a \otimes b) = (a \otimes 1)\Delta(b)$. Then, $N$ is a sub-bimodule of $\Omega^1_u$ and $(\Omega^1_u/N, \mathrm{d}_N)$ is a left-covariant FODC on $\mathcal{A}$. Moreover, every left-covariant FODC can be obtained in this way.

# Quantum Homogeneous Space

Let $(\mathcal{A}, \mathrm{m}, \Delta, \eta, \varepsilon)$ and $(\mathcal{H}, \mathrm{m}_{\mathcal{H}}, \Delta_{\mathcal{H}}, \eta_{\mathcal{H}}, \varepsilon_{\mathcal{H}})$ be Hopf algebras, and $\pi : \mathcal{A} \longrightarrow \mathcal{H}$ be a surjective Hopf algebra morphism. We view $\mathcal{A}$ as a right $\mathcal{H}$-comodule algebra via coaction

$$\Delta_{\mathcal{A}} := (\mathrm{id} \otimes \pi) \circ \Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{H}.$$

With all this datum, the space of coinvariants:

$$\mathcal{B} := \mathcal{A}^{co(\mathcal{H})} = \{a \in \mathcal{A} : \Delta_{\mathcal{A}}(a) = a \otimes 1\}$$

is a right coideal subalgebra of $\mathcal{A}$. We call $\mathcal{B}$ a quantum homogeneous space, if $\mathcal{A}$ is faithfully flat as a right $\mathcal{B}$-module.

# Left-Covariant FODC on quantum homogeneous spaces

**Theorem (Hermisson, 2002)**
*Let $\mathcal{B} = \mathcal{A}^{co\mathcal{H}}$ be a quantum homogeneous space. For any $I^{(1)} \subset \mathcal{B}^+ := \mathcal{B} \cap \ker(\epsilon)$ in $\mathcal{M}^{\mathcal{H}}_{\mathcal{B}}$, define:*

$$\Omega^1 := \mathcal{A} \square_{\mathcal{H}} \mathcal{B}^+ / I^{(1)}$$

*with $\mathcal{B}$-bimodule structure and left $\mathcal{A}$-coaction as:*

$$b(a^i \otimes [c^i])b' := ba^i b'_{(1)} \otimes [c^i b'_{(2)}], \qquad {}_{\Omega^1}\Delta := \Delta \otimes \mathrm{id},$$

*and, $\mathrm{d} : \mathcal{B} \longrightarrow \Omega^1$ defined as:*

$$\mathrm{d}(b) := b_{(1)} \otimes \pi_I((b_{(2)})^+).$$

*Then, $(\Omega^1, \mathrm{d})$ is a left $\mathcal{A}$-covariant FODC on $\mathcal{B}$. Moreover, every left $\mathcal{A}$-covariant FODC on $\mathcal{B}$ is of this form.*

# Quantum Tangent Space

A quantum tangent space for $\mathcal{B} = {}^W\mathcal{A}$ is a subspace $T \subseteq \mathcal{B}^\circ$ such that $T \oplus \mathbb{C}1$ is a right coideal of $\mathcal{B}^\circ$ and $WT \subseteq T$.

For any quantum tangent space $T$, a right $\mathcal{B}$-ideal of $\mathcal{B}^+$ is given by

$$I^{(1)} := \{x \in \mathcal{B}^+ \,|\, X(x) = 0, \text{ for all } X \in T\}.$$

We call $V^1 = \mathcal{B}^+/I^{(1)}$, the cotangent space of $T$.

**Theorem (Heckenberger, Kolb, 2003)**
*There is a bijective correspondence between isomorphism classes of finite-dimensional tangent spaces and finitely-generated left $\mathcal{A}$-covariant FODCi on $\mathcal{B}$.*

**Theorem (Heckenberger, Kolb, 2006)**
*For quantum grassmannians, there exists a unique covariant differential calculus of classical dimension.*

# Quantum Tangent Space Generated by Lusztig's Root Vectors

**Theorem (R. Ó Buachalla, P. Somberg, 2025)**
*For a particular choice of reduced decomposition of the longest element of the Weyl group, the space spanned by the Lusztig's root vectors is a quantum tangent space for $\mathcal{O}_q(\mathrm{SU}_n)$, whose restriction to the case of quantum grassmannians gives the anti-holomorphic HK quantum tangent space.*

**Example**

For the case of $\mathfrak{sl}_3\mathbb{C}$, and the choice $w = w_2 w_1 w_2$ (this is the choice set by R. Ó B. and P. S.) of reduced decomposition of the longest element $w$ of the Weyl group $W \cong \mathrm{S}_3$, the list of root vectors is given by:

$$E_{\alpha_1} := E_1, \qquad E_{\alpha_2} := E_2, \qquad \text{and} \qquad E_{\alpha_1+\alpha_2} := [E_2, E_1]_{q^{-1}}.$$

and we denote by:

$$T^{(0,1)} := \mathrm{span}\{E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_1+\alpha_2}\}$$

# A Tangent Space for $\mathcal{O}_q(\mathrm{F}_3)$

We define:

$$T^{(1,0)} := (T^{(0,1)})^*,$$

where $*$ is the $*$-structure on $\mathrm{U}_q(\mathfrak{sl}_3)$. We see it is spanned by the elements:

$$F_{\alpha_1} := E^*_{\alpha_1} = K_1 F_1, \qquad F_{\alpha_2} := E^*_{\alpha_2} = K_2 F_2,$$
$$F_{\alpha_1+\alpha_2} := E^*_{\alpha_1+\alpha_2} = q^{-1} K_1 K_2 [F_1, F_2]_{q^{-1}}$$

Now, we take our quantum tangent space $T$ to be:

$$T := T^{(1,0)} \oplus T^{(0,1)}$$

# Differential Calculus on $\mathcal{O}_q(\mathrm{F}_3)$

**Theorem (A. Carotenuto, R. Ó Buachalla, J. Razzaq, 2025)**

*Let $V^\bullet$ denote the quantum exterior algebra for the maximal prolongation of $\Omega^1(\mathrm{F}_3)$. Then, a full set of relations for $V^\bullet$ is given by following three sets of identities:*

$$e_\gamma \wedge e_\beta = -q^{(\beta,\gamma)} e_\beta \wedge e_\gamma, \qquad f_\gamma \wedge f_\beta = -q^{-(\beta,\gamma)} f_\beta \wedge f_\gamma, \qquad \textit{for all } \beta \leq \gamma \in \Delta^+,$$

$$e_\gamma \wedge f_\beta = -q^{(\beta,\gamma)} f_\beta \wedge e_\gamma, \qquad \textit{for all } \beta \neq \gamma \in \Delta^+, \ \textit{or for } \beta = \gamma = \alpha_1 + \alpha_2,$$

$$e_{\alpha_1} \wedge f_{\alpha_1} = -q^2 f_{\alpha_1} \wedge e_{\alpha_1} - \nu f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2},$$

$$e_{\alpha_2} \wedge f_{\alpha_2} = -q^2 f_{\alpha_2} \wedge e_{\alpha_2} + \nu f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2},$$

*where an order $\leq$ on the set of positive roots $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ is fixed as follows:*

$$\alpha_2 \leq \alpha_1 + \alpha_2 \leq \alpha_1.$$

A first-order almost complex structure for a $*$-FODC $\Omega^1(\mathcal{B})$ over an algebra $\mathcal{B}$ is a direct sum decomposition of $\mathcal{B}$-bimodules,

$$\Omega^1(\mathcal{B}) \simeq \Omega^{(1,0)} \oplus \Omega^{(0,1)}$$

such that $(\Omega^{(1,0)})^* = \Omega^{(0,1)}$ or equivalently $(\Omega^{(0,1)})^* = \Omega^{(1,0)}$.

An almost complex structure for a differential $*$-calculus $\Omega^\bullet(A)$ is an $\mathbb{N}_0^2$-algebra grading $\Omega^\bullet(A) = \oplus_{(p,q)}\Omega^{(p,q)}$ such that:
$(i)$ $\Omega^k(A) = \oplus_{p+q=k}\Omega^{(p,q)}$, $(ii)$ $(\Omega^{(p,q)})^* = \Omega^{(q,p)}$.

Define the projections of differential operator $\mathrm{d}$ as follows:

$$\partial := \mathrm{proj}_{\Omega^{(p+1,q)}} \circ \mathrm{d}, \qquad \overline{\partial} := \mathrm{proj}_{\Omega^{(p,q+1)}} \circ \mathrm{d},$$

An almost complex structure is said to be integrable if $\mathrm{d} = \partial + \overline{\partial}$. Moreover, an integrable almost complex structure is called a complex structure.

**Theorem (A. Carotenuto, R. Ó Buachalla, J. Razzaq, 2025)**

*The first-order differential calculus $\Omega^1_q(\mathrm{F}_3)$ admits, up to identification of opposite structures, two covariant first-order almost complex structures. Explicitly, one decomposition of $V^1$ is given by:*

$$V^{(1,0)} = \mathrm{span}_{\mathbb{C}}\Big\{e_{\alpha_1},\, e_{\alpha_2},\, e_{\alpha_1+\alpha_2}\Big\}, \quad V^{(0,1)} := \mathrm{span}_{\mathbb{C}}\Big\{f_{\alpha_1},\, f_{\alpha_2},\, f_{\alpha_1+\alpha_2}\Big\},$$

*and the other is given by:*

$$V^{(1,0)} = \mathrm{span}_{\mathbb{C}}\Big\{e_{\alpha_1},\, f_{\alpha_2},\, e_{\alpha_1+\alpha_2}\Big\}, \quad V^{(0,1)} := \mathrm{span}_{\mathbb{C}}\Big\{f_{\alpha_1},\, e_{\alpha_2},\, f_{\alpha_1+\alpha_2}\Big\},$$

*Moreover, both of these FOACSs extends to an integrable almost complex structure on $\Omega^\bullet_q(\mathrm{F}_3)$.*

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