Codifferential Calculi and Quantum Homogeneous Spaces

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The Heckenberger–Kolb Differential Calculi

Theorem (Heckenberger–Kolb '06)

Let $B = \mathcal{O}_q(G/L_S)$ be a quantized irreducible flag manifold (e.g. $G/L_S = S^2$ or $G/L_S = \mathrm{Gr}(m, n)$). Then there exist q-deformations $\Omega_q^{(\bullet,0)}(G/L_S)$ and $\Omega_q^{(0,\bullet)}(G/L_S)$ of the holomorphic and antiholomorphic algebraic Dolbeault complex.

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Theorem (Heckenberger–Kolb '07)

There exists an isomorphism

$$\Omega_q^{(0,\bullet)}(G/L_S) \cong \left(\underline{C_\bullet^S}\right)^\circ$$

where $C_\bullet^S$ denotes the parabolic BGG resolution of the trivial $U_q(\mathfrak{g})$-module (a similiar isomorphism exists for the holomorphic complex).

First Order Differential Calculi

- From now on k is a field with $\mathrm{char}(k) \neq 2$.

Definition

Let B be a k-algebra. A *first order differential calculus (FODC)* over B consists of a B-bimodule Ω^1 and a k-linear derivation $d\colon B \to \Omega^1$ such that the map

$$B \otimes B \to \Omega^1, \quad a \otimes b \mapsto ad(b)$$

is surjective.

First Order Codifferential Calculi

Definition

Let C be a coalgebra over k, and let $\mathscr{W}_1$ be a C-bicomodule

- A k-linear map $\delta\colon \mathscr{W}_1 \to C$ is called *coderivation* if

$$\Delta(\delta(w)) = \delta(w_{(0)}) \otimes w_{(1)} + w_{(-1)} \otimes \delta(w_{(0)})$$

  for all $w \in \mathscr{W}_1$.

- Let $\delta\colon \mathscr{W}_1 \to C$ be a coderivation. The pair $(\mathscr{W}_1, \delta)$ is called *first order codifferential calculus (FOCC)* if the map

$$\mathscr{W}_1 \to C \otimes C, \quad w \mapsto w_{(-1)} \otimes \delta(w_{(0)})$$

  is injective.

The Universal FOCC

Definition

The *universal FOCC* over C consists of the C-bicomodule $\mathscr{W}_1^u := C \otimes C/\mathrm{im}(\Delta)$, and the coderivation

$$\delta^u \colon \mathscr{W}_1^u \to C, \quad [c \otimes d] \mapsto \varepsilon(c)d - \varepsilon(d)c.$$

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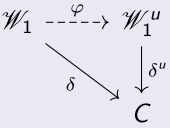
Proposition (Doi '81)

Let $\mathscr{W}_1$ be a C-bicomodule and let $\delta\colon \mathscr{W}_1 \to C$ be a coderivation, then there exists a unique C-bicolinear map φ such that the diagram



commutes. Moreover, if $(\mathscr{W}_1, \delta)$ is a FOCC, then φ is injective, that is every FOCC is a subspace of the universal FOCC.

Higher Order Codifferential Calculi

Definition

A *differential graded coalgebra (DGC)* is a graded coalgebra $C_\bullet = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} C_n$ together with a degree -1 differential $\delta \colon C_\bullet \to C_\bullet$ such that

$$\Delta(\delta(c)) = \delta(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes \delta(c_{(2)})$$

for all homogeneous elements $c \in C_\bullet$.

Remark

- The degree zero part C_0 together with the restriction of Δ to C_0 is a regular (non graded) coalgebra, and every component $C_n, n \geq 0$ is a bicomodule over C_0.
- The degree 1 differential $\delta_1 \colon C_1 \to C_0$ is a coderivation.
- If (C_1, δ_1) is a FOCC, and $C_\bullet$ is conilpotently cogenerated by C_1 relativ to C_0, we call $(C_\bullet, \delta_\bullet)$ a *codifferential calculus*.

The Maximal Prolongation

Problem

Given a FOCC $(\mathscr{W}_1, \delta)$, can we find a canonical dg coalgebra $(C_\bullet, \delta_\bullet)$ such that $(C_1, \delta_1) = (\mathscr{W}_1, \delta)$, mimicking maximal prolongations for FODC's?

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Definition

The *C-relative tensor coalgebra* of a C-bicomodule M is given by $T^c_C(M) := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M^{\Box_C n}$ together with the coproduct

$$\Delta(\underline{m}) = \underline{m}_{(-1)} \otimes \underline{m}_{(0)} + \underline{m}_{(0)} \otimes \underline{m}_{(1)} + \sum_{i=1}^{n-1} m_1 \Box \ldots \Box m_i \otimes m_{i+1} \Box \ldots \Box m_n$$

where $\underline{m} = m_1 \Box \ldots \Box m_n \in M^{\Box_C n}$ (recall: For C-bicomodules M, N we have $m \otimes n \in M \Box_C N$ if $\Delta_M(m) \otimes n = m \otimes {}_N\Delta(n)$).

- The maximal prolongation of an arbitrary FOCC $(\mathscr{W}_1, \delta)$ will be constructed as a graded sub coalgebra of $T^c_C(\mathscr{W}_1)$. By universality of ${\mathscr{W}_1}^u$, we can assume $\mathscr{W}_1 \subseteq {\mathscr{W}_1}^u$.

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- Let $\mathscr{W}_2$ be the largest C-subbicomodule such that its image under

$$\mathscr{W}_1 \square_C \mathscr{W}_1 \to {\mathscr{W}_1}^u, \quad w_1 \square w_2 \mapsto [\delta(w_1) \otimes \delta(w_2)]$$

is contained in $\mathscr{W}_1$.

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is contained in $\mathscr{W}_1$.

- We define δ_2 as the unique linear map that completes the diagram

$$\begin{array}{ccc} \mathscr{W}_1 \Box_C \mathscr{W}_1 & \longrightarrow & {\mathscr{W}_1}^u \\ \uparrow & & \uparrow \\ \mathscr{W}_2 & \overset{\delta_2}{\dashrightarrow} & \mathscr{W}_1 \end{array} .$$

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- The component of degree $n \geq 3$ is defined as the following subspace of ${\mathscr{W}_1}^{\Box_C n} \subseteq T^c_C(\mathscr{W}_1)$

$$\mathscr{W}_n := \bigcap_{i+j+2=n} {\mathscr{W}_1}^{\Box_C i} \Box_C \mathscr{W}_2 \Box_C {\mathscr{W}_1}^{\Box_C j}.$$

- The differential on $\mathscr{W}_\bullet$ is defined as follows:

$$\delta_1 := \delta\colon \mathscr{W}_1 \to C, \quad \delta_2\colon \mathscr{W}_2 \to \mathscr{W}_1 \text{ (as in the previous slide)}$$

$$\delta_n(w_1 \square \ldots \square\, w_n) := \sum_{i=1}^{n-1} (-1)^i w_1 \otimes \cdots \otimes \delta_2(w_i \otimes w_{i+1}) \otimes \cdots \otimes w_n \quad (n \geq 3).$$

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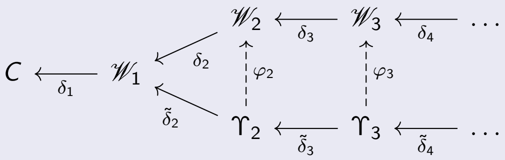
Theorem (B '25)

The pair $(\mathscr{W}_\bullet, \delta_\bullet)$ is a codifferential pre-calculus. Moreover, for any other prolongation $(\Upsilon_\bullet, \tilde{\delta}_\bullet)$ of $(\mathscr{W}_1, \delta)$, the identity $\mathrm{id}_{C\oplus\mathscr{W}_1}$ extends uniquely to a morphism of dg coalgebras $\varphi\colon \Upsilon_\bullet \to \mathscr{W}_\bullet$.



We call $(\mathscr{W}_\bullet, \delta_\bullet)$ the *maximal prolongation* of $(\mathscr{W}_1, \delta_1)$.

Quantum Homogeneous Spaces

Definition

Let $\pi\colon A \twoheadrightarrow \overline{A}$ be a Hopf algebra surjection, and consider the right $\overline{A}$-coaction

$$A \to A \otimes \overline{A}, \quad a \mapsto a_{(1)} \otimes \pi(a_{(2)}).$$

The coinvariant subalgebra $B := A^{\mathrm{co}\,\overline{A}}$ is called *principal quantum homogeneous spaces* if $B \subseteq A$ is a faithfully flat Hopf–Galois extension.

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- For our purpuses we dualize this setup:
- From now on let U be a Hopf algebra and let $H \subseteq U$ be a Hopf subalgebra. Further let
$$C := U \otimes_H k \cong U/UH^+$$
where $H^+ = H \cap \ker(\varepsilon)$.
- We also assume that U is faithfully flat over H.
- Denote by ${}^C_U\mathcal{M}^C$ respectively ${}_H\mathcal{M}^C$ the categories of C-bicovariant left U-modules, respectively right C-covariant left H-modules.

Schneider's Categorical Equivalence

Theorem (Schneider '90, B '25)

- The induction and coinvariant functors

$$ {}^C_U\mathcal{M}^C \underset{U\otimes_H -}{\overset{{}^{\mathrm{co}\,C}(-)}{\rightleftarrows}} {}_H\mathcal{M}^C $$

  form an equivalence of categories.
- Let ${}_H\mathcal{M}^0$ be the full subcategory of ${}_H\mathcal{M}^C$ of objects with trivial right C-action. Then the above equivalence restricts to a monoidal equivalence

$$ ({}^C_U\mathcal{M}^0, \square_C, C) \underset{U\otimes_H -}{\overset{{}^{\mathrm{co}\,C}(-)}{\rightleftarrows}} ({}_H\mathcal{M}^0, \otimes, k) \ . $$

Equivariant Codifferential Calculi

Definition

- A FOCC $(\mathscr{W}_1, \delta)$ is *U-equivariant* if $\mathscr{W}_1 \in {}^C_U\mathcal{M}^C$ and δ is U-linear.
- A codifferential calculus $(C_\bullet, \delta_\bullet)$ is *U-equivariant* if every $C_n \in {}^C_U\mathcal{M}^C$ and the differential $\delta_\bullet$ is U-linear.

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Proposition

Let $(C_\bullet, \delta_\bullet)$ be an U-equivariant codifferential calculus, then the FOCC (C_1, δ_1) is U-equivariant. Conversely, if $(\mathscr{W}_1, \delta)$ is a U-equivariant FOCC, then its maximal prolongation is U-equivariant.

Hermisson Classification of FOCC's

Definition

A *quantum tangent space* is a subspace $T \subseteq C^+ = \ker(\varepsilon)$ such that $HT \subseteq T$ and $\Delta(T) \subseteq (T \oplus k[1]) \otimes C$.

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Remark

Any quantum tangent space $T \subseteq C^+$ is a right C-comodule via $t \mapsto t_{(1)}^+ \otimes t_{(2)}$ (here $(-)^+$ denotes the projection onto C^+). In addition $T \in {}_H\mathcal{M}^C$.

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Theorem (B '25)

There is a 1:1-correspondence

$$\{U\text{-equivariant FOCC's}\} \overset{1:1}{\leftrightarrow} \{\text{quantum tangent spaces}\}$$

given by $(\mathscr{W}_1, \delta) \mapsto \delta({}^{\mathrm{co}\ C}\mathscr{W}_1)$ and $T \mapsto (U \otimes_H T, \delta)$ with $\delta(x \otimes_H t) = xt$. Moreover, the universal FOCC corresponds to $U \otimes_H C^+$.

Remark

If $\mathscr{W}_1 = U \otimes_H T$, with $T \in {}_H\mathcal{M}^0$, then by the monoidality of the categorical equivalence, we have $\mathscr{W}_1 \square_C \mathscr{W}_1 \cong U \otimes_H (T \otimes T)$.

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Theorem (B '25)

Let $\mathscr{W}_1 = U \otimes_H T$, with $T \in {}_H\mathcal{M}^0$, then the following hold.

- Consider the map

$$\hat{\delta} \colon T \otimes T \to U \otimes_H C^+, \quad [x] \otimes [y] \mapsto x_{(1)} \otimes_H [S(x_{(2)})y]$$

  and let $R := \hat{\delta}^{-1}(U \otimes_H T) \subseteq T \otimes T$, then $\mathscr{W}_2 \cong U \otimes_H R$.

- The maximal prolongation can be written as

$$\mathscr{W}_\bullet \cong U \otimes_H C(T, R)$$

  where

$$C(T, R) = k \oplus T \oplus R \oplus \bigoplus_{n \geq 3} \bigcap_{i+j+2=n} T^{\otimes i} \otimes R \otimes T^{\otimes j} \subseteq T^c(T).$$

The Podleś Cocalculus

- Recall that $U := U_q(\mathfrak{sl}_2)$ is the $\mathbb{C}$-algebra given by the generators E, F, K, K^{-1} together with the relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2 EK, \quad KF = q^{-2} FK$$
$$[E, F] = \frac{1}{q - q^{-1}}(K - K^{-1}).$$

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- The algebra $U_q(\mathfrak{sl}_2)$ is a Hopf algebra, with coproduct given

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(E) = E \otimes K + 1 \otimes E$$
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- Let $H := U_q(\mathfrak{h})$ be the subalgebra of U generated by K and K^{-1}. Fact: $U_q(\mathfrak{sl}_2)$ is faithfully flat over $U_q(\mathfrak{h})$.

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- The space

$$T := \mathrm{span}_{\mathbb{C}}\{[E], [F]\}.$$

Is a quantum tangent space in the quotient $C := U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{h})} \mathbb{C}_{\mathrm{triv}}$, and since $[E]$ and $[F]$ are primitive, we have $T \in {}_H\mathcal{M}^0$.

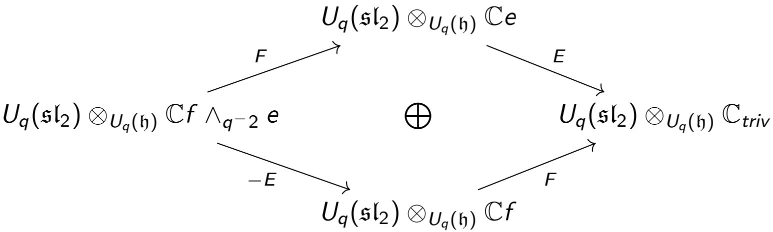
- Recall the map $\hat{\delta}\colon T \otimes T \to U \otimes_H C^+, \quad [x] \otimes [y] \mapsto x_{(1)} \otimes_{U_q(\mathfrak{h})} [S(x_{(2)})y]$.
- Computing the values of $\hat{\delta}$ on the basis elements of $T \otimes T$, we get

$$\begin{aligned}
\hat{\delta}([E] \otimes [E]) &= q^{-2} E \otimes_{U_q(\mathfrak{h})} [E] - q^{-2} \otimes_{U_q(\mathfrak{h})} [E^2] \\
\hat{\delta}([F] \otimes [F]) &= F \otimes_{U_q(\mathfrak{h})} [F] - 1 \otimes_{U_q(\mathfrak{h})} [F^2] \\
\hat{\delta}([E] \otimes [F]) &= q^2 E \otimes_{U_q(\mathfrak{h})} [F] - q^2 \otimes_{U_q(\mathfrak{h})} [EF] \\
\hat{\delta}([F] \otimes [E]) &= F \otimes_{U_q(\mathfrak{h})} [E] - 1 \otimes_{U_q(\mathfrak{h})} [EF]
\end{aligned}$$

- It follows that $R = \hat{\delta}^{-1}(U \otimes_H T)$ is given by

$$R = \mathrm{span}_{\mathbb{C}}\{[F] \otimes [E] - q^{-2}[E] \otimes [F]\}.$$

- Letting $e := [E], f := [F]$ and $f \wedge_{q^{-2}} e := f \otimes e - q^{-2} e \otimes f$, the resulting codifferential calculus can be written in the following form:



(here F denotes the map $1 \otimes_{U_q(\mathfrak{h})} f \wedge_{q^{-2}} e \mapsto F \otimes_{U_q(\mathfrak{h})} e$, and similiarly for E)

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$$
\begin{array}{ccccc}
 & & U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{h})} \mathbb{C}e & & \\
 & \overset{F}{\nearrow} & & \overset{E}{\searrow} & \\
U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{h})} \mathbb{C}f \wedge_{q^{-2}} e & & \bigoplus & & U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{h})} \mathbb{C}_{triv} \\
 & \underset{-E}{\searrow} & & \underset{F}{\nearrow} & \\
 & & U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{h})} \mathbb{C}f & &
\end{array}
$$

(here F denotes the map $1 \otimes_{U_q(\mathfrak{h})} f \wedge_{q^{-2}} e \mapsto F \otimes_{U_q(\mathfrak{h})} e$, and similiarly for E)

Thanks for the attention!