Quantum Geometry of Data (& Spacetime)

Harold Steinacker

Department of Physics, University of Vienna

EISA Corfu, september 2025

Motivation:

quantum gravity & quantum structure of spacetime:

(most reasonable :) answer is known: IKKT model

$\rightarrow$ covariant quantum spacetime, based on quantum geometry

quantum geometry of data:

QCML = Quantum Cognitive Machine Learning

same mathematical framework
data science meets quantum geometry

Abanov, Candelori, HS, Wells, Musaelian etal, arXiv:2507.21135

outline:

- framework:
  quantum (matrix) geometry
- data science application:
  Quantum Cognitive Machine Learning (QCML)
  project in collaboration with Qognitive Inc.
- physics application:
  quantum spacetime & quantum gravity through IKKT model

Quantum spaces

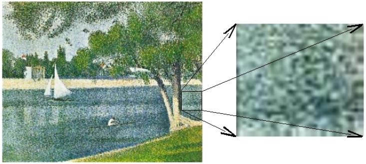
core of Quantum Mechanics, appropriate for matrix models

- QM: quantized phase space $[Q, P] = i\hbar \mathbb{1}$
- Moyal-Weyl quantum plane $\mathbb{R}^{2n}_\theta$

$$[X^\mu, X^\nu] = i\theta^{\mu\nu}\mathbb{1}, \qquad X^\mu = \begin{pmatrix} Q^i \\ P_j \end{pmatrix}$$

NC algebra of observables = quantized functions

quantum cells, uncertainty, finite dof per volume



metric structure: $\begin{cases} \text{Dirac or Laplace op} \to \text{spectral geometry} \\ \text{or} \\ \text{matrix configuration} \quad (\approx \text{embedding, local}) \end{cases}$

quantum (matrix) geometry

... defined in terms of a matrix configuration $\{X^1, ..., X^D\}$

commuting matrices	$\to$	classical lattice
(mildly) noncommuting matrices	$\to$	quantum geometry

efficient, "smooth", suitable for computer (& data science!)

Quantum (matrix) geometry

definitions:

- a matrix configuration is a set of D selfadjoint matrices
  $\{X^a \in \mathrm{End}(\mathcal{H}),\ a = 1, \ldots, D\}$ (often: $\mathcal{H} \cong \mathbb{C}^N$)
- for $x \in \mathbb{R}^D$ define

$$H_x = \sum_a (X^a - x^a \mathbf{1})^2 \qquad \geq 0$$

displacement Hamiltonian

(cf. shifted harmonic osc!)

ground states:

$$H_x|x\rangle = \lambda(x)|x\rangle$$

...quasi-coherent states

- $\Box := [X^a, [X^b, .]]\delta_{ab}$ ...Matrix Laplacian
  (similarly d'Alembertian $\delta_{ab} \to \eta_{ab}$)

cf. Berenstein-Dzienkowski 1204.2788 Ishiki 1503.01230, HS 2009.03400, HS book

$|x\rangle$ smooth on $\tilde{\mathbb{R}}^n$ ... nondeg. ground states

abstract quantum space:

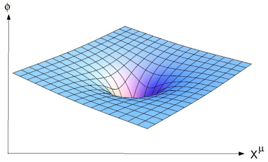
$$\mathcal{B} := \bigcup_{x\in\tilde{\mathbb{R}}^n} \{|x\rangle\} \qquad \ldots U(1) \quad \text{bundle}$$

$$\boxed{\mathcal{M} := \mathcal{B}/_{U(1)} \hookrightarrow \mathbb{C}P^{N-1}}$$

quantum manifold if $\mathcal{M} \subset \mathbb{C}P^{N-1}$ submanifold

embedding in target space:

$$\mathbf{x}^a = \langle x|X^a|x\rangle : \ \mathcal{M} \to \mathbb{R}^D$$



embedded quantum space: $\tilde{\mathcal{M}} := \mathbf{x}^a(\mathcal{M})$

matrix configurations $\{X^a\}$ describe quantized embedding map

$$X^a \sim \mathbf{x}^a : \ \mathcal{M} \to \mathbb{R}^D \qquad .$$

more general: symbol map

$$\begin{aligned} \mathrm{End}(\mathcal{H}) &\to \mathcal{C}(\mathcal{M}) \\ \Phi &\mapsto \langle x|\Phi|x\rangle =: \phi(x) \end{aligned}$$

extra structure: $U(1)$ bundle $\mathcal{B} \to \mathcal{M}$

connection 1-form $iA = \langle x|d|x\rangle, \qquad iA_\mu = \langle x|\partial_\mu|x\rangle$

(cf. Berry connection)

$$h_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} + i\omega_{\mu\nu}) = (\partial_\mu + iA_\mu)\langle x|\,(\partial_\nu - iA_\nu)|x\rangle$$

... hermitian tensor

$\mathcal{M}$ inherits $\left\{ \begin{array}{l} \text{closed 2-form } \omega = dA \\ \text{“quantum” metric } g \end{array} \right\}$ via pull-back from $\mathbb{C}P^{N-1}$

---

note: everything exact, no approx, no limit
expect: “almost-commuting” matrix configurations approximate
embedded symplectic manifolds

examples:

- Moyal-Weyl quantum plane $\mathbb{R}^D_\theta$: $[X^a, X^b] = i\theta^{ab}$
  $|x\rangle$ ... standard coherent states
- fuzzy S^2_N

$$X^a = \Pi_N(J^a), \quad a = 1, 2, 3 \qquad \ldots N-\text{dim. irrep of } SU(2)$$

  can show: $\mathcal{M} = \{|x\rangle\} \cong S^2$
  (no approx! minimal S^2_N = Bloch sphere)
- quantized coadjoint orbits $\mathcal{O}$

$$X^a = \Pi_{\mathcal{H}}(T^a) \qquad \ldots\text{large irrep of semi-simple } G$$

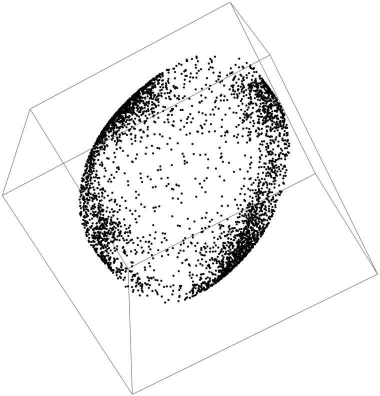
  $\mathcal{M} = \{|x\rangle\} \cong \mathcal{O}$ ... coherent states (Perelomov)
- generic deformations thereof

visualization:

choose random point cloud $x_{(i)}$ in some cube $Q \subset \mathbb{R}^D$
plot expectation values $\langle x_{(i)}|X^a|x_{(i)}\rangle$ of corresponding $|x_{(i)}\rangle$

e.g.: deformed fuzzy sphere for $N = 11$

$$X^1 = J_1$$
$$X^2 = 1.1 J_2 + 0.02 J_1^3$$
$$X^3 = 0.9 J_3 + 0.05 J_2^2$$



semi-classical correspondence

$$\mathrm{End}(\mathcal{H}) \quad \Longleftrightarrow \quad L^2(\mathcal{M})$$

such that $[\Phi, \Psi] \sim i\{\phi, \psi\}$ $\qquad (\mathcal{M}, \omega)$ symplectic

$\mathrm{End}(\mathcal{H})$ is Hilbert space via $\langle \Phi, \Psi \rangle = \mathrm{Tr}(\Phi^\dagger \Psi)$

$L^2(\mathcal{M})$ is Hilbert space via $\langle \phi, \psi \rangle = \int\limits_{\mathcal{M}} \Omega\, \phi^* \psi$

intuition: $\mathcal{M}$ comprises N “quantum cells”

$$\mathrm{Tr}(\Phi) \approx \int\limits_{\mathcal{M}} \Omega\, \phi(x) \approx \sum_i \phi(x_i)$$

justified to some extent for “almost-commuting matrix configurations” in subspace of $End(\mathcal{H})$

HS, 2009.03400; HS, book

almost-local quantum spaces

$Loc(\mathcal{H}) \subset End(\mathcal{H})$	$\sim$	$\mathcal{C}_{\mathrm{IR}}(\mathcal{M}) \subset L^2(\mathcal{M})$
Φ	$\sim$	$\phi(x) = \langle x\vert\Phi\vert x\rangle$
X^a	$\sim$	$\mathbf{x}^a(x)$
$[.,.]$	$\sim$	$i\{.,.\}$
Tr	$\sim$	$\int_{\mathcal{M}}$
$\Box = [X^a,[X_a,.]]$	$\sim$	$e^{-\sigma}\Box_G$

"almost-local" = approx. diagonal w.r.t. $|x\rangle$

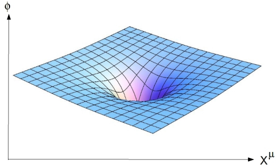
misleading in UV regime! string modes $|x\rangle\langle y|$ dominate

summary: quantum (matrix) geometry

= matrix configuration $\{X^a\}$

defines quantized embedding map

$$X^a \sim x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D$$



natural framework for Matrix Models → dynamical quantum spaces

---

efficient way to encode (high-dimensional) geometric data !

Data analysis

- data involving a large number of features
  represented as points in feature space $\mathbb{R}^D$ ("target space")
- often: concentration of measure
  data points concentrate near manifold with relatively low intrinsic dimension
- goal: capture underlying data manifold & its properties

new approach ($\geq$ 2024):

QCML = Quantum Cognition Machine Learning

combining concepts & tools of quantum geometry
with ideas in quantum cognition

encodes data as quantum geometry,
observables as Hermitian matrices

Abanov, Candelori, HS, Wells, Musaelian etal, arXiv:2507.21135

what is QCML?

- dataset $\mathcal{X} = \{x_{(1)}, \ldots, x_{(t)}, \ldots, x_{(T)}\}$ consisting of T points
  $x_{(t)} \in \mathbb{R}^D$ (target space, "feature space")
- matrix configuration $X = \{X^1, \ldots, X^D\}$ learned from dataset
- associate to each data point $x_{(t)} \in \mathcal{X}$ a quantum state:

$$\boxed{|x_{(t)}\rangle \in \mathcal{H} = \mathbb{C}^N \qquad = \text{ground state of } H_{x_{(t)}}}$$

(= quasicoherent state)

(recall $H_x = \sum (X^a - x^a)^2$)

- expectation values

$$X(x) = \left(\langle x|X^1|x\rangle, \ldots, \langle x|X^D|x\rangle\right)$$

(highly non-linear map $\tilde{\mathbb{R}}^D \to \mathbb{R}^D$!)

dataset $\mathcal{X}$ → QCML point cloud

$$\mathcal{X}_X = \{X(x_{(t)}) \mid x_{(t)} \in \mathcal{X}\} \qquad \subset \mathbb{R}^D$$

- deviation of $X(x_{(t)})$ from data point $x_{(t)}$ measured by displacement

$$d^2(x) = \|X(x) - x\|^2 = \sum_a \left(X^a(x) - x^a\right)^2.$$

- quantum fluctuations: variance

$$\sigma^2(x) = \sum_a \sigma_a^2(x), \quad \sigma_a^2(x) = \langle x|X_a^2|x\rangle - \langle x|X_a|x\rangle^2.$$

- loss function

$$L[X] = \sum_{x\in\mathcal{X}} \left(d^2(x) + w\cdot\sigma^2(x)\right) \quad \stackrel{w=1}{=} \lambda(x)$$

- training: matrix configuration X optimized to minimize $L[X]$:

$$\boxed{X_a = \underset{X_a\in\mathrm{Mat}(N)}{\mathrm{argmin}}\ L[X]}$$

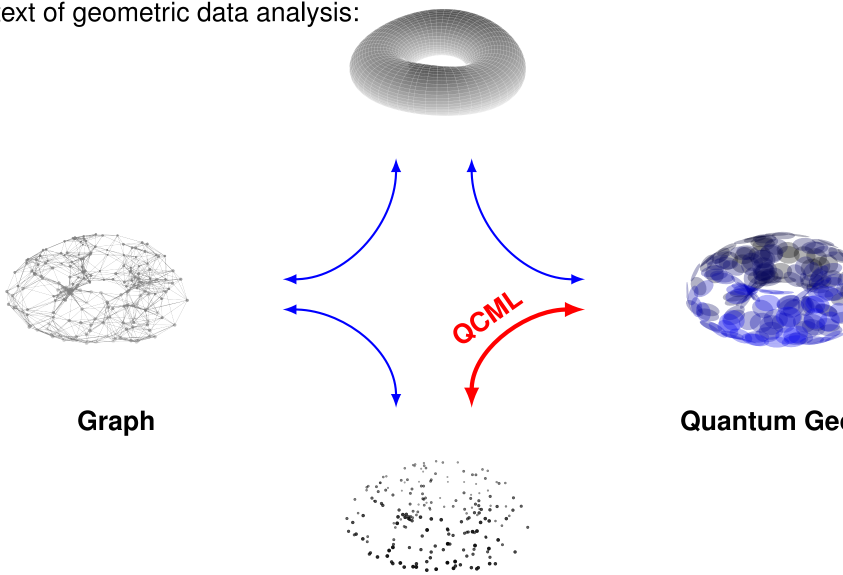
trained matrices X^a define quantum geometry,

data points optimally reproduced by QCML point cloud (=expectation values)

key advantages: extra structure from Hilbert space!

- optimal approximation of dataset by quantum space
- provides smooth non-linear map $x \in \tilde{\mathbb{R}}^D \to \mathbb{R}^D$,
  $\approx$ projection of x to closest point on data manifold $\mathcal{M}$, inference
- can extract geometric structure:
  intrinsic dimension, topology, reduction/abstraction, ...
- allows to model incompatible observables
  (cf. quantum cognition)
- very efficient, intrinsically smooth, no lattice artifacts
  may overcome curse of dimensionality

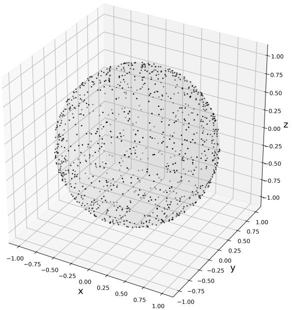
(recovers K-means for commuting matrix configurations) context of geometric data analysis:



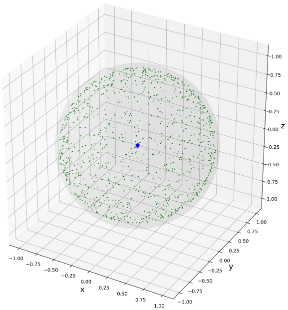
Example 1: Fuzzy sphere S^2_N from random points on a sphere.

dataset: 1000 points distributed uniformly over unit sphere $S^2 \subset \mathbb{R}^3$

train three 4×4 matrices X_1, X_2, X_3 using QCML $\quad (N = 4)$



(a) 1000 points on the surface of a unit sphere



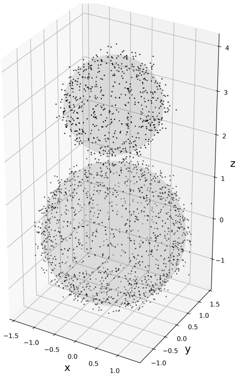
(b) QCML point cloud

trained matrices X_a are approximately spin $\frac{3}{2}$ generators of $SU(2)$,
$\approx$ recover fuzzy sphere:

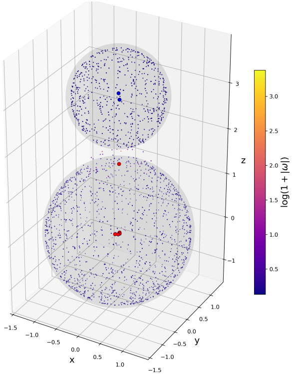
$$\mathrm{spec}(X_3) \approx \{-1.50, -0.49, +0.51, +1.52\},$$
$$\|[X_a, X_b] - i\epsilon_{abc}X_c\| \approx 0.16, \quad \|\sum_a X_a^2 - j(j+1)\| \approx 0.11.$$

Example 2: two disconnected spheres with noise

dataset: 2000 points sampled uniformly near surfaces of two spheres, with random noise.

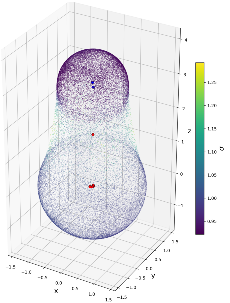


(a) 2000 points near surfaces of two spheres with noise.

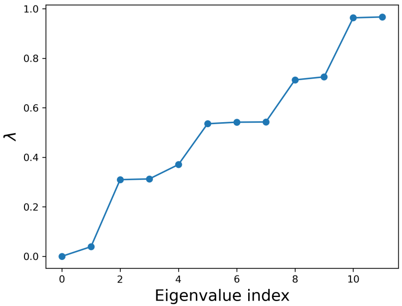


(b) QCML point cloud via trained $\mathbf{X}^a$.

quantum geometry:



(a) Quantum geometry point cloud, colored by uncertainty $\sigma(x)$.



(b) first eigenvalues of matrix Laplacian. A spectral gap separates the first two eigenvalues, indicating two almost disconnected components.

two nearly perfect spheres connected by a bridge,
uncertainty is higher in the bridge region

The matrix Laplacian

given matrix config. $\{X_a\}$, define

$$\nabla_a := [X_a, \cdot\,] \sim i\{x_a, \cdot\,\}$$

...quantized Hamiltonian vector fields on $\mathcal{M}$

$$\Delta = \sum_a [X_a, [X_a, \cdot]] = \nabla_a \nabla_a$$

...Hermitian, positive-def. operator acting on $\mathrm{Mat}(N)$

analogous to Laplace-Beltrami operator on classical manifold $\mathcal{M}$

spectrum and eigenmatrices (= eigenmaps):

$$\Delta Y_i = \lambda_i Y_i, \qquad \mathrm{spec}(\Delta) = \{\lambda_0, \lambda_1, \ldots, \lambda_{\max}\}.$$

can show

$$\Delta \sim \rho^2 \Delta_G$$

$G_{\mu\nu}$ ... effective metric (cf. HS 1003.4134), ρ ... dilaton relevance of matrix Laplacian in data analysis:

- allows to separate disconnected (topolog.) components
- encodes spectral geometry

  eff. dimension from Weyls law
- lowest eigenmaps $\Delta Y_i = \lambda_i Y_i$ provide

  reduced (abstract) matrix configuration Y_i

  reduced quantum space $\mathcal{M}_Y \subset \mathbb{C}P^{N-1}$

  ... abstract model for intrinsic quantum geometry, zero modes and separating components of $\mathcal{M}$

consider reducible matrix configuration

$$X^a = \begin{pmatrix} X^a_{(1)} & 0 & 0 \\ 0 & X^a_{(2)} & 0 \\ 0 & 0 & X^a_{(3)} \end{pmatrix}.$$

projectors P_i on irreducible blocks are in 1-to-1 correspondence to **zero modes** of Δ:

$$\Delta P_i = 0$$

cf. example 2!

topological properties of $\mathcal{M}$

$$\mathcal{B} = \{|x\rangle,\ x \in \tilde{\mathbb{R}}^D\} \qquad \ldots U(1) \text{ bundle over } \mathcal{M}$$

$A \quad = \langle x|d|x\rangle$ ... $U(1)$ (Berry) connection on hermitian line bundle

$\omega \quad = dA$ ... (Berry) curvature on line bundle

well-defined **topological invariants**

e.g. Chern numbers:

$$c_1 := \int_{S^2} \frac{\omega}{2\pi} \quad \in \mathbb{Z},$$

can be computed numerically: sphere around singularities (=degen.)

e.g. $\sum c_i = n$ for fuzzy sphere S^2_n cf. examples 1, 2

well-defined integers (topology) from finite matrix configs

(cf. HS, book)

metric properties of $\mathcal{M}$

$$h_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} + i\omega_{\mu\nu}) = (\partial_\mu + iA_\mu)\langle x| \, (\partial_\nu - iA_\nu)|x\rangle$$

... pull-back of symplectic form ω and (quantum) metric g from $\mathbb{C}P^N$

can be extracted numerically

allows e.g. to measure intrinsic dimension of $\mathcal{M}$, etc.

Candelori etal, arXiv:2409.12805

overcoming the curse of dimensionality

high-dimensional features (lattices) require exponential growth of ressources

avoided in quantum spaces:

e.g. minimal fuzzy $\mathbb{C}P^{N-1}$: smooth $2(N-1)$ -dim. quantum manifold encoded using N^2-1 matrices of size N.

(in fact just $2N+1$ matrices)

required resources grow more slowly - often linearly - with dim. & with non-trivial features

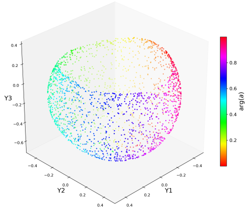
Example: high-dimensional data sets

choose 100 reference points $\{z_i = (x_i, y_i)\} \subset \mathbb{C}$ in unit disk $\to$ $\mathbb{R}^{200}$
mapped using 2000 random conformal maps from unit disk to itself

$$z \mapsto e^{i\theta} \frac{a - z}{1 - \bar{a}z},$$

$\to$ 2000 points on 2-dim manifold $\mathcal{M} \subset \mathbb{R}^{200}$ as input to QCML

reduced matrix config. Y^1, Y^2, Y^3 recovers intrinsic disk structure



generalized to conformal maps on $\mathbb{C}^n$ for $n = 2, 3, 4, 5$
successfully extract intrinsic dimensions

real-life example: Wisconsin breast cancer data

569 data points, 30 features (characterize cell nucleus)

choose Hilbert space dim. $N = 8$

QCML gives an intrinsic dimension estimate of 2, using both local quantum metric g as well as spectral dimension from Δ

low eigenmaps Y_i comprise most of correlations $Tr(X_a Y_i)$,

capture dominant features

further aspects:

- distance on $\mathcal{M}$ from g captures *intrinsic proximity* between states, encoded in $\langle y|x\rangle$

  $\neq$ distance in feature space $\mathbb{R}^D$!

  distinct points on $\mathcal{M}$ may be mapped to *same point in* $\mathbb{R}^D$

  $\rightarrow$ coherent modeling of different objects w/ same features
- non-commuting observable can model incompatible features (cognition!)
- naturally smooth, no lattice artifacts

  naturally extrapolates smooth manifold structure
- efficient, implemented in practical applications

$|Q\rangle$OGNITIVE

The IKKT matrix model

Ishibashi, Kawai, Kitazawa, Tsuchiya 1996

$$S[Y,\Psi] = Tr\Big([Y^a,Y^b][Y^{a'},Y^{b'}]\eta_{aa'}\eta_{bb'} + \bar\Psi\Gamma_a[Y^a,\Psi]\Big)$$

$$Y^a = Y^{a\dagger} \in Mat(N,\mathbb{C})\,, \qquad a = 0,...,9$$

$\Psi \in \mathrm{Mat}(N,\mathbb{C}) \otimes \mathbb{C}^{32}$ ... Majorana-Weyl spinor

gauge symmetry $Y^a \to U^{-1}Y^aU$, $ISO(9,1)$, SUSY

- related to IIB string theory
- class. solutions Y^a typically noncommutative
  $\to$ quantum spacetime $\mathcal{M}^{3,1}$, dynamical
- $\to$ gauge theory, UV finite for dimension $\leq 3+1$

Gravity is a quantum effect on quantum spacetime

for spacetimes Y^a with structure $\mathcal{M}^{3,1} \times \mathcal{K} \subset \mathbb{R}^{9,1}$

SUSY $\to$ mild quantum effects:

Einstein-Hilbert action (+ extra) in the 1-loop effective action on $\mathcal{M}^{3,1}$ (cf. Sakharov '67)

$$\Gamma_{1-\mathrm{loop}} \ni \int\limits_{\mathcal{M}} T_{\nu\lambda}{}^{\mu} T_{\nu\lambda}{}^{\mu} + \ldots \sim \int\limits_{\mathcal{M}} d^4x\sqrt{G}\,\frac{1}{G_N}\mathcal{R}[G] + \ldots$$

Planck scale

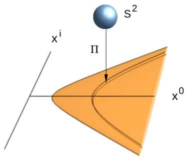
$$G_N \sim \frac{\rho^2}{c_{\mathcal{K}}^2 m_{\mathcal{K}}^2}$$

set by Kaluza-Klein mass scale on $\mathcal{K}$

finite, no UV divergence !

- combination of $S_{EH} + S_{YM}$ leads to modification of gravity in IR — K. Kumar, HS 2312.01317

  in progress Kawai, Ho, HS
- most reasonable $\mathcal{M}^{3,1}$ = (minimal) covariant quantum spacetime

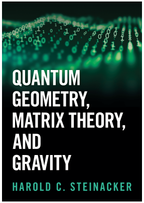


- stabilization of $\mathcal{K}$: either
  - 1-loop effects — A. Manta, T. Tran, HS 2411.02598
  - large R charge (internal rotation) — A. Manta, HS 2512.xxxxx.

lots to be done, near-realistic, rich & approachable framework

literature for quantum geometry in physics:

- short introductory review: HS arXiv:1911.03162
- systematic exposition: Book
  "Quantum geometry, Matrix Theory, and Gravity"



- Lorentzian FLRW quantum spacetime $\mathcal{M}_n^{1,3}$:
  M. Sperling, HS 1901.03522
  A. Manta, HS 2502.02498; Ch. Gass, HS 2503.1956
- one-loop effective action & emergent gravvity:
  HS 2303.08012, 2110.03936
- cosmological aspects
  Battista, HS : 2207.01295 ff, Karczmarek, HS 2207.00399
- no-ghost-theorem: HS 1901.03522
  HS, T. Tran 2203.05436, 2305.19351 , 2311.14163, 2312.16110
- 1-loop quantization of $\mathfrak{hs}$ Yang-Mills: HS, T. Tran 2405.09804

Quantum geometry = NC operators X^a and (quasi)coherent states $|x\rangle$

powerful & broad framework, huge potential

Thank you