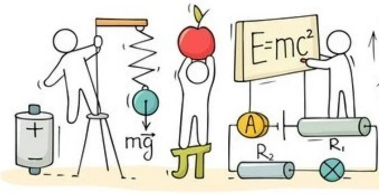
Generalized group structures for changes of quantum reference frames: ϑ-Poincaré case



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Based on:

G.F., F. Lizzi [arXiv:2507.05758] + forthcoming papers

Plan

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Preliminaries, paradoxes for classical RFs. Need generalized groups

Noncommutative (NC) spacetimes

NC ϑ-spaces (or "Moyal"); QFT attempts on them

Commutative Hopf algebras H_0, NC deformations $\hat{H}$, and QRFs

The Hopf algebra $H_0 \equiv Fun(\mathrm{P})$

The Hopf algebra $\hat{H} \equiv Fun_\vartheta(\mathrm{P})$, and QRFs

E^3_ϑ-covariant 1-particle QM on $\mathbb{R}^3_\vartheta$. Coherent states

Regular representation of $\hat{H} \equiv Fun_\vartheta(\mathrm{P})$; coherent states for QRFs

Summary and conclusions

Introduction

A RF is usually built using macroscopic bodies B with $N \gg 1$ atoms. Given two RFs $\mathfrak{R}, \mathfrak{R}'$, we can distinguish:

i) a set $X \equiv \{X_A\}$ of "collective" observables that pinpoint their relative position and orientation for all t;

ii) sets $Y \equiv \{Y_a\}$, $Y' \equiv \{Y'_a\}$ of "internal" observables used to record (e.g. on notebooks) the results of observations of other systems S (including other RFs) resp. made by $\mathfrak{R}, \mathfrak{R}'$.

E.g., for inertial RFs in flat spacetime, choose the origins O, O' as the CMs of B_1, B'_1 ('the labs'), $X = \{$relative displacement y, velocity $\vec{v}$,...$\}$.

$N \gg 1$ ensures that: observations made by $\mathfrak{R}, \mathfrak{R}'$ do not significantly affect X (while of course affect Y, Y' via the recording processes); Y, Y' are "large enough". **Classical (i.e. ideal) RFs:** $N = \infty$.

But the ultimate quantum nature of these bodies will spoil their classical (i.e. idealized) properties, via UR, etc; particularly manifest if $N \sim O(1)$.

Can we formulate a consistent theory of QRFs?

The idea of QRFs was first proposed by Aharonov & Susskind 1967, Aharonov & Kaufherr 1984. Ever since many hundreds papers.

They mostly use "Relational Quantum Mechanics" (Rovelli 1996, Loveridge et al 2018, Höhn et al. 2021,...): $\nexists$ unique "absolute" state of a system S; rather, $\exists$ one state relative to each observer.

Consequently, a composite system can be in an entangled state wrt QRF $\mathfrak{R}$, a factorized state wrt QRF $\mathfrak{R}'$, see e.g. "Quantum mechanics and the covariance of physical laws in quantum reference frames", by F. Giacomini, E. Castro-Ruiz, Č. Brukner, Nat. Commun. **10**, 494 (2019).

Use of spacetime observables relative to QRFs can heal QFT divergences:

- Chandrasekaran, Longo, Penington, Witten, JHEP02(2023)082;
- E. Witten, JHEP03(2024)077;
- Fewster, Janssen, Loveridge, Rejzner, Waldron, Comm. Math. Phys. 402 (2024), 1-41;
- De Vuyst, Eccles, Höhn, Kirklin, arXiv:2405.00114; arXiv:2412.15502;

propose operational frameworks for local measurements of QFs on a symmetric background wrt a QRF: under suitable assumptions the algebra of (relative) observables is a type II factor (instead of type III_1), i.e. has a semifinite, or even finite, (instead of an infinite) trace, what allows e.g. computing entropy.

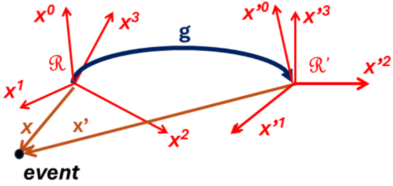
The approach to investigate properties of a QRF can be:

1. bottom-up: start from quantum properties of its microscopic constituents, operationally measuring spacetime coords wrt it.
2. top-down: study which classical properties of RFs are compatible with their quantum nature, or must be generalized, and how

Here we adopt 2., focusing on the group structure of changes of RFs.

Preliminaries, paradoxes for CRFs. Need generalized groups

Changes of classical reference frames (RF) $g : \mathfrak{R} \mapsto \mathfrak{R}'$ in space(time) make up a Lie group G:
the product gg' is the composition of g, g';
the unit is $\mathbf{1} : \mathscr{R} \mapsto \mathscr{R}$;
the inverse of g is $g^{-1} : \mathfrak{R}' \mapsto \mathfrak{R}$.



g sharply specifies where $\mathfrak{R}$ is located and how it moves wrt $\mathfrak{R}'$.

Let x, x' resp. be the (sets of) spacetime coordinates of a generic event wrt $\mathfrak{R}, \mathfrak{R}'$; g determines the 1-to-1 map $x \mapsto x'$. By pull-back the latter induces a map (*passive transf.*) between the dynamical variables used by $\mathfrak{R}, \mathfrak{R}'$ to describe a physical system S; e.g. for scalar fields the map

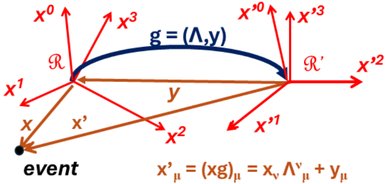
$$U(g) : \varphi \mapsto \varphi'$$

is determined by the eq. $\varphi(x) \stackrel{!}{=} \varphi'[x'(x)]$.

Enforcing this map assumes $\mathfrak{R}'$ has: i) got information about the description of S by $\mathfrak{R}$; ii) sharply determined g, i.e. how $\mathfrak{R}$ moves wrt $\mathfrak{R}'$.

Cartesian coords wrt inertial RFs on $M \equiv$ Minkowski spacetime: $x'_\mu=(xg)_\mu \equiv x_\nu \lambda_\mu^\nu + a_\mu$, $g \equiv (\lambda, a) \in G \equiv$ Poincaré group P,

$$U(g): \varphi \mapsto \varphi' \equiv \varphi \triangleleft g, \qquad \varphi'(x') \equiv \varphi\left(x' g^{-1}\right). \tag{1}$$

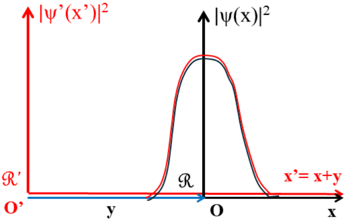


These maps apply also if S is quantum, e.g. a 0-spin elementary particle:

$$\hat{x} \mapsto \hat{x}' = \hat{x}\lambda + a, \qquad \hat{p} \mapsto \hat{p}' = \hat{p}\lambda, \qquad \rho \mapsto \rho', \qquad \psi \mapsto \psi'. \tag{2}$$

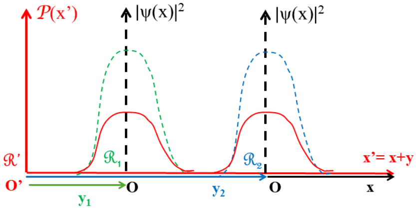
All pure (resp. mixed) states ρ_S ($\equiv$density operator) wrt $\mathfrak{R}$ are mapped into pure (resp. mixed) states ρ'_S wrt $\mathfrak{R}'$.

The wavefunctions $\psi(x) = {}_{\mathfrak{R}}\langle x|\Psi\rangle_{\mathfrak{R}}$, $\psi'(x') = {}_{\mathfrak{R}'}\langle x'|\Psi'\rangle_{\mathfrak{R}'}$ of S wrt resp. $\mathfrak{R}, \mathfrak{R}'$ fulfill $|\psi'(x')|^2 = |\psi(x)|^2$, and by Wigner Thm can be chosen so that $\psi'(x') = \psi(x)$.



However, if $\mathfrak{R}'$ has a coarse (i.e., probabilistic) knowledge about $\mathscr{R}$, then a pure state $\rho_S = |\Psi\rangle_{\mathfrak{R}\mathfrak{R}}\langle\Psi|$ is mapped into a mixed state ρ'_S.

If e.g. $\mathfrak{R}'$ knows exactly λ, i.e. the relative orientation and velocity of $\mathfrak{R}$, and that the origins' displacement is a_1, a_2 with probabilities $1/2$, then



$$\rho'_S = \tfrac{1}{2}\,|\Psi'_{a_1}\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle\Psi'_{a_1}| + \tfrac{1}{2}\,|\Psi'_{a_2}\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle\Psi'_{a_2}$$

$$\mathscr{P}(x') = \tfrac{1}{2}\left|\psi'_{a_1}(x')\right|^2 + \tfrac{1}{2}\left|\psi'_{a_2}(x')\right|^2;$$

here $\psi'_a(x') = \psi(x'-a)$, $\mathscr{P}(x') = \mathrm{Tr}\big(|x'\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle x'|\rho'\big)$ is the probability density to find the particle at position x' wrt $\mathfrak{R}'$.

More generally, if $\mathfrak{R}'$ knows that the origins' displacement is y with probability density $\widetilde{\rho_{\mathfrak{R}}}(y)$, then the state of S wrt $\mathfrak{R}'$ will be

$$\rho'_S = \int d^4a\, \widetilde{\rho_{\mathfrak{R}}}(a)\, |\Psi'_a\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle\Psi'_a| = \int d^4a\, \widetilde{\rho_{\mathfrak{R}}}(a)\, U(a)\,|\Psi\rangle\langle\Psi|\,U^{-1}(a); \quad (4)$$

this is pure iff $\widetilde{\rho_{\mathfrak{R}}}(a) = \delta_{\bar{a}}(a) \equiv \delta(a-\bar{a})$, for some $\bar{a} \in \mathbb{R}^4$.

Thus, **purity of states is a frame-dependent notion!**
To explain the paradox: $\widetilde{\rho_{\mathfrak{R}}}$ is a classical state (probability distribution) of $\mathfrak{R}$ wrt $\mathfrak{R}'$; if it is mixed, so is the state of $S \cup \mathfrak{R}$ wrt $\mathfrak{R}'$.

For generic states ρ_S of S wrt $\mathfrak{R}$ and $\rho_{\mathfrak{R}}$ of $\mathfrak{R}$ wrt $\mathfrak{R}'$ (4) generalizes to

$$\rho_S' = \int dg\, \widetilde{\rho_{\mathfrak{R}}}(g)\, U(g)\, \rho_S\, U^{-1}(g); \tag{5}$$

dg is the (left and right G-invariant) Haar measure on G.
As known, for $G = \mathrm{P}$ we can realize $U(g) = (\lambda, a)$ as the operator

$$U(\lambda, a) = \exp\left[\left(\ln\lambda\right)^{\sigma}_{\rho}\, \tfrac{i}{2}\, m^{\rho}_{\sigma}\right] \exp\left(a_\rho\, ip^\rho\right), \tag{6}$$

where the diff. operators $p^\rho = -i\partial^\rho$, $m^\rho_\sigma = i(x_\rho\partial^\sigma - x_\sigma\partial^\rho)$ are the images under the Schrödinger representation r of the generators P^ρ, M^ρ_σ of translations, Lorentz transformations in the Poincaré Lie algebra $\mathbf{g}$.

Let $y_\rho, \Lambda^\sigma_\rho$ be the coordinate functions on G defined on $g = (\lambda, a)$ by $y_\rho(g) = a_\rho$, $\Lambda^\sigma_\rho(g) = \lambda^\sigma_\rho$. Denoting by H_0 the abelian algebra of functions on G, which is generated by $y_\rho, \Lambda^\sigma_\rho$, and by $H_0' \equiv U\mathbf{g}$ the UEA of $\mathbf{g}$, we may rewrite (6) in the form $U(\lambda, a) = \mathscr{U}[\Lambda(\lambda), y(a)]$, where

$$\mathscr{U} = \mathscr{U}(\Lambda, y) := \exp\left[\tfrac{i}{2}\, m^\rho_\sigma \otimes \left(\ln\Lambda\right)^\sigma_\rho\right] \exp\left(ip^\rho \otimes y_\rho\right) \in \mathrm{r}\left(H_0'\right) \otimes H_0. \tag{7}$$

H_0, H_0' are actually *dual Hopf* algebras, and $\mathscr{U} = (\mathrm{r} \otimes \mathrm{id})(\mathscr{T})$, where $\mathscr{T}$ is the *canonical element* of the pair (H_0, H_0'). Eq. (5) can be rephrased as

$$\rho_S' = (\mathrm{id} \otimes \rho^c_{\mathfrak{R}'})\left[\mathscr{U}\,(\rho_S \otimes \mathbf{1})\, \mathscr{U}^{-1}\right]. \tag{8}$$

The set of states of classical $\mathfrak{R}$ wrt $\mathfrak{R}'$ gets a semigroup with product defined by convolution. δ_e plays the role of the unit element. Only pure states δ_g have inverse, $\delta_{g^{-1}}$. $G \leftrightarrow$ the set of pure states of CRFs.

Alternatively, it is more convenient to encode the group structure of G in the *Hopf algebra structure* of $Fun(G)$, as this allows to replace $Fun(G)$ by a noncommutative algebra, as we may need for dealing with Quantum Reference Frames (QRFs; i.e. RF whose ultimate quantum nature cannot be ignored) and for describing symmetries of a NC spacetime.

Generalizing (8), we postulate that the state ρ_S' of S w.r.t. $\mathfrak{R}'$ is obtained as

$$\rho_S' = \mathrm{Tr}_{\mathscr{H}_{\mathfrak{R}}}\left[(\mathbf{1}\otimes\rho_{\mathfrak{R}})\,\mathscr{U}\,(\rho_S\otimes\mathbf{1})\,\mathscr{U}^{-1}\right]. \qquad (9)$$

Below NC θ-deformations of: Euclidean space $\mathbb{R}^3$ and group $G = E^3$, ideally relating relatively immobile QRFs; Minkowski spacetime M and Poincaré group $G = \mathrm{P}$, ideally relating inertial QRFs. One can show that $\mathscr{U}(\hat{\Lambda}, \hat{y})$ has exactly the same form as (7):

$$\mathscr{U}(\hat{\Lambda},\hat{y}) = \exp\left[\tfrac{i}{2}\, m^{\rho}_{\sigma}\otimes\left(\ln\hat{\Lambda}\right)^{\sigma}_{\rho}\right]\exp\left(ip^{\rho}\otimes\hat{y}_{\rho}\right) \in \mathrm{r}\left(H'\right)\otimes H. \qquad (10)$$

Why NC spacetime?

Idea of noncommutative (NC) spacetime is rather old [Heisenberg].
Possible motivations:

1. framework where to reconcile the principles of QM and GR;
2. inthrinsic regularization mechanism of UV divergences in QFT (Heisenberg's motivation);
3. due to the quantum nature of RFs **(new!)**;
4. effective description in some low energy regime of string theory (e.g. D3-brane with a large B-field) or LQG (in flat spacetime limit).

1. In usual QFT no universal minimum for the localization Δx of events: $\Delta x \sim \hbar/\Delta p$ can be reduced at will by increasing the energy of the probe. On the other hand (argument due to [Bronstein,Mead,Wheeler]), by GR the energy concentration should not cause the formation of a black hole

$$\Rightarrow \qquad \Delta x \gtrsim l_p \quad \text{(Planck length)}. \tag{11}$$

Doplicher, Fredenhagen & Roberts [DFR95] propose more sophisticated bounds, and noncommuting x_i that could naturally imply such bounds.

Simplest NC spacetime: **constant commutators**

$$[\hat{x}_\mu, \hat{x}_\nu] = i\mathbf{1}\vartheta_{\mu\nu}, \tag{12}$$

$\vartheta_{\mu\nu} = -\vartheta_{\nu\mu}$. Theoretical laboratory to investigate QM, QFT on NC spaces. Note that (12) are translation invariant, not Lorentz-invariant.

Algebra $\widehat{\mathscr{X}}$ of functions on Moyal space: generated by $\mathbf{1}, \hat{x}_\mu$ fulfilling (12), with $\vartheta_{\mu\nu} \in \mathbb{R}$ (suitably extended). In [DFR95] $\vartheta_{\mu\nu} \in \mathscr{Z}(\widehat{\mathscr{X}})$ is dynamical.

Various inequivalent approaches to QFT on Moyal spaces. I would divide them according to: quantization approach, spacetime symmetries.

1. Path-integral quantization on Moyal-Euclidean spacetime: T. Filk , M. Douglas, A.S. Schwarz, N. Nekrasov, N. Seiberg, E. Witten, S. Minwalla, M. Van Raamsdonk, J. Gomis, L. Alvarez-Gaume, T. Mehen, M. Vazquez-Mozo, ...,R. Oeckl, J. Wess, P. Aschieri, P. Schupp, R.J. Szabo, M. Dimitrijevic,..., H. Grosse, R. Wulkenhaar,...
2. Field=operator-valued, Moyal-Minkowski spacetime. Quantization: canonical; or á la Wightman; ... DFR, Bahns, Piacitelli, Chaichian, Balachandran et al, Aschieri, Lizzi,Vitale, Abe, Zahn, GF & Wess,...

Various problems, some interesting features.
E.g. in 1: causality violation, non-unitarity (for $\vartheta_{0i} \neq 0$), UV-IR mixing of divergences, non-renormalizability, claimed changes of statistics, etc.
Some problems may arise because naively deformed Euclidean Feynman rules are not justified by a Wick rotation.

Standard or deformed Poincaré covariance? ...?

Doplicher-Fredenhagen-Roberts, *et* Bahns, Piacitelli,...: since 1995:
First canonical quantization of the free fields. $\vartheta_{\mu\nu} \rightsquigarrow Q_{\mu\nu}$ central Lorentz tensor (obeying some conditions), becoming on each irrep a set of fixed constants $\vartheta_{\mu\nu}$ (joint spectrum of $Q_{\mu\nu}$). $\Rightarrow$ Poincaré-covariant.
But with interacting fields Lorentz covariance is sooner or later lost.
Doplicher's speculations: $Q_{\mu\nu}$ finally related to v.e.v. of $R_{\mu\nu}$, in turn influenced by matter quantum fields through quantum eq.s of motion.

Oeckl 2000, Chaichian *et al* 2004, Wess 2004, Koch *et al* 2004:
(12) are not Poincaré -invariant; *but "twisted Poincaré"* invariant.

Then attempts to construct twisted Poincaré covariant quantum fields started: Chaichian *et al*, Balachandran *et al*, Lizzi-Vitale, Abe, Zahn, F.-Wess, F.,... Our framework.

The Hopf algebra $\left(H_0 \equiv Fun\ (\mathrm{P}), \varepsilon, \Delta, S\right)$

$$x_\mu \mapsto x'_\mu = (xg)_\mu = x_\nu \Lambda^\nu_\mu + y_\mu \equiv x_\nu \otimes \Lambda^\nu_\mu + \mathbf{1} \otimes y_\mu =: \Delta^r(x_\mu). \tag{13}$$

Regard: $\mathbf{1}, x_\mu$ as generators of $\mathscr{X}_0 := Fun(M)$; $\mathbf{1}_{H_0}, \Lambda^\nu_\mu, y_\mu$ as generators of the algebra $H_0 \equiv Fun\ (\mathrm{P})$ of functions on P. The transf. rule (13) is extended to all of $\mathscr{X}$ an algebra map (i.e. $\Delta^r(fg) = \Delta^r(f)\Delta^r(g)$, etc.), the *coaction* $\Delta^r : \mathscr{X} \to \mathscr{X} \otimes H_0, \quad f(x) \mapsto f(x') =: \left[\Delta^r(f)\right](x)$.
The group structure of P is encoded in the *counit* $\varepsilon : H_0 \to \mathbb{C}$, *coproduct* $\Delta : H_0 \to H_0 \otimes H_0$, *antipode* $S : H_0 \to H_0$, defined on the generators by

$$\begin{aligned} &\varepsilon(\Lambda^\nu_\mu) = \delta^\nu_\mu, \quad \Delta(\Lambda^\nu_\mu) = \Lambda^\nu_\rho \otimes \Lambda^\rho_\mu, \qquad S(\Lambda^\nu_\mu) = (\eta \Lambda^T \eta)^\nu_\mu \equiv \Lambda^{-1}{}^\nu_\mu, \\ &\varepsilon(y_\mu) = 0, \qquad \Delta(y_\mu) = y_\nu \otimes \Lambda^\nu_\mu + \mathbf{1}_{H_0} \otimes y_\mu, \quad S(y_\mu) = -y_\nu \Lambda^{-1}{}^\nu_\mu, \end{aligned} \tag{14}$$

which resp. give the identical, (twice) repeated, inverse change of frame. ε, Δ, S are extended as (anti-)algebra maps; fulfill many properties, e.g.

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta^r = \mathrm{id}\,, \quad (\Delta \otimes \mathrm{id}\,) \circ \Delta^r = (\mathrm{id} \otimes \Delta^r) \circ \Delta^r. \tag{15}$$

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Transf. (13) preserves $[x_\mu, x_\nu] = 0$. Does it preserve $[\hat{x}_\mu, \hat{x}_\nu] = i\mathbf{1}\vartheta_{\mu\nu}$?
Yes, if we "quantize" H_0, i.e. make it a NC Hopf algebra $\hat{H}$, such that $[\hat{x}'_\mu, \hat{x}'_\nu] = i\mathbf{1}\vartheta_{\mu\nu}$ holds as well $\Rightarrow$ **all inertial QRFs are equivalent!**

The Hopf algebra $\left(\hat{H} \equiv Fun_\vartheta(\mathrm{P}), \varepsilon, \Delta, S\right)$

$$\hat{x}_\mu \mapsto \hat{x}'_\mu = \hat{x}_\nu \Lambda^\nu_\mu + \hat{y}_\mu \equiv \hat{x}_\nu \otimes \Lambda^\nu_\mu + \mathbf{1} \otimes \hat{y}_\mu =: \Delta^r(\hat{x}_\mu). \tag{13}$$

Regard: $\mathbf{1}, \hat{x}_\mu$ as generators of $\widehat{\mathscr{X}}$; $\mathbf{1}_H, \Lambda^\nu_\mu, \hat{y}_\mu$ as generators of the algebra $\widehat{H} = Fun_\vartheta(\mathrm{P})$. The transf. rule (13) is extended to all of $\widehat{\mathscr{X}}$ an algebra map (i.e. $\Delta^r(fg) = \Delta^r(f)\Delta^r(g)$, etc.), the *coaction* $\Delta^r : \widehat{\mathscr{X}} \to \widehat{\mathscr{X}} \otimes \widehat{H}$, $f(\hat{x}) \mapsto f(\hat{x}')$. The *counit* $\varepsilon : \widehat{H} \to \mathbb{C}$, *coproduct* $\Delta : \widehat{H} \to \widehat{H} \otimes \widehat{H}$, *antipode* $S : \widehat{H} \to \widehat{H}$, defined on the generators by

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Transf. (13) preserves $[\hat{x}_\mu, \hat{x}_\nu] = i\mathbf{1}\vartheta_{\mu\nu}$ if [Oeckl 2000]:

$$[\Lambda^\rho_\mu, \cdot] = 0, \quad \Lambda^\mu_\rho \Lambda^\nu_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}\mathbf{1}_H, \quad [\hat{y}_\mu, \hat{y}_\nu] = i(\vartheta_{\mu\nu}\mathbf{1}_H - \vartheta_{\rho\sigma}\Lambda^\rho_\mu \Lambda^\sigma_\nu). \tag{16}$$

E^3_ϑ-covariant 1-particle QM on $\mathbb{R}^3_\vartheta$. Coherent states

As a warm-up, first E^3_ϑ-covariant 1-p QM on $\mathbb{R}^3_\vartheta$ with commutative time:

$$\mathscr{A}: \quad [\hat{x}_i,\hat{x}_j]=i\,\vartheta_{ij}, \qquad [\hat{p}^i,\hat{p}^j]=0, \qquad [\hat{x}_i,\hat{p}^j]=i\,\delta_i^j \tag{17}$$

$$H: \quad [Q_i^j,\cdot]=0, \quad Q^{-1}=Q^T, \quad [\hat{y}_i,\hat{y}_j]=i\left(\vartheta_{ij}\mathbf{1}-\vartheta_{jk}Q_i^jQ_j^k\right)=:i\,\chi_{ij}. \tag{18}$$

$$\Delta^r: \qquad \hat{x}_i\mapsto\hat{x}_i'=\hat{x}_jQ_i^j+\hat{y}_i, \qquad \hat{p}^i\mapsto\hat{p}'^i=Q^{-1}{}^i_j\,\hat{p}^j \tag{19}$$

$\mathscr{A}\simeq$ 3D Heisenberg algebra has a faithful irrep on $C^\infty(\mathbb{R}^3)$ via

$$\hat{x}_i\,\psi(x)=\left(x_i+\tfrac{i}{2}\vartheta_{ij}\partial^j\right)\psi(x), \qquad \hat{p}^i\,\psi(x)=-i\,\partial^i\,\psi(x), \tag{20}$$

where $\partial^j\equiv\frac{\partial}{\partial x_j}$. As $\vartheta\to 0$ we recover Schrödinger irrep of $\mathscr{A}_0$.
As for $\vartheta=0$: $\mathscr{S}(\mathbb{R}^3)$ carries a highly reducible (resp. irreducible) representation of $\mathscr{X}\subset\mathscr{A}$ (resp. $\mathscr{A}$); endowed with the scalar product

$$\langle\phi,\phi'\rangle:=\int_{\mathbb{R}^3}\overline{\phi(x)}\phi'(x)\,d^3x, \tag{21}$$

is a pre-Hilbert space whose completion is $\mathscr{L}^2(\mathbb{R}^3)\simeq$ Hilbert space of S .
Each of the $\hat{x}_i,\hat{p}^i$ has spectrum $\mathbb{R}$.

There is $C\in O(3)$ s.t. for $\hat{x}_i=C_i^h\hat{\mathrm{x}}_h$ the only nontrivial (17a) and UR are $\left[\hat{\mathrm{x}}_1,\hat{\mathrm{x}}_2\right]=i\zeta,\ \ \Delta\hat{\mathrm{x}}_1\Delta\hat{\mathrm{x}}_2\geq\frac{\zeta}{2}$. Setting $\partial^j\equiv\frac{\partial}{\partial\mathrm{x}_j}$, eq. (20) becomes

$$\hat{\mathrm{x}}_1\psi=\mathrm{x}_1\psi+\tfrac{i}{2}\zeta\,\partial_2\psi,\qquad \hat{\mathrm{x}}_2\psi=\mathrm{x}_2\psi-\tfrac{i}{2}\zeta\,\partial_1\psi,\qquad \hat{\mathrm{x}}_3\psi=\mathrm{x}_3\psi. \quad (22)$$

Proposition. *For all $\xi\equiv(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$ and $\nu\in\mathbb{R}^+$ the Gaussian*

$$\psi_{\xi,\nu}(x)=\left(\frac{8}{\pi^3\zeta^2\nu}\right)^{1/4}\exp\left[-\frac{(\mathrm{x}_1-\xi_1)^2+(\mathrm{x}_2-\xi_2)^2}{\zeta}-\frac{(\mathrm{x}_3-\xi_3)^2}{\nu}\right] \quad (23)$$

is an eigenvector of $b=\hat{\mathrm{x}}_1+i\hat{\mathrm{x}}_2$ with eigenvalue $z\equiv\xi_1+i\xi_2$ and has

$$\langle\hat{\mathrm{x}}_i\rangle=\xi_i,\qquad \Delta\hat{\mathrm{x}}_1=\Delta\hat{\mathrm{x}}_2=\sqrt{\tfrac{\zeta}{2}}.\quad \Delta\hat{\mathrm{x}}_3=\sqrt{\tfrac{\nu}{2}}. \quad (24)$$

$\{\psi_{\xi,\nu}\}_{\xi\in\mathbb{R}^3}$ is a complete [in $\mathscr{L}^2(\mathbb{R}^3)$] family of states centered around $\xi\in\mathbb{R}^3$, which labels all classical translations. The generalized functions

$$\psi_\xi(x):=\lim_{\nu\to0}\psi_{\xi,\nu}(x)=\sqrt{\frac{2}{\pi\zeta}}\exp\left[-\frac{(\mathrm{x}_1-\xi_1)^2+(\mathrm{x}_2-\xi_2)^2}{\zeta}\right]\delta(\mathrm{x}_3-\xi_3) \quad (25)$$

in addition saturate the URs

$$\Delta\hat{\mathrm{x}}_1\Delta\hat{\mathrm{x}}_2\geq\tfrac{\zeta}{2},\qquad \sum_{i=1}^3\Delta\hat{x}_i^2\geq\zeta=\sqrt{-\tfrac{1}{2}\mathrm{tr}(\vartheta^2)}. \quad (26)$$

Remarks:

- The $\psi_{\xi,\nu}$ are also Schrödinger coherent states for $\hat{\mathrm{x}}_i, \hat{\mathrm{p}}^i$, i.e. saturate also the HUR, with $\langle\hat{\mathrm{p}}^i\rangle = 0$, $\Delta\hat{\mathrm{p}}^1 = \Delta\hat{\mathrm{p}}^2 = 1/\sqrt{2\zeta}$, $\Delta\hat{\mathrm{p}}^3 = 1/\sqrt{2\nu}$.
- As $\vartheta \to 0$ we have $\zeta \to 0$, and each ψ_ξ goes to a Dirac delta,

$$\psi_\xi(x) \to \delta^{(3)}(\mathrm{x} - \xi). \tag{27}$$

The algebra H (18) is faithfully represented on the space V of smooth functs $f(y,q)$ of commuting variables y_i, q^i_j, with $q \in SO(3)$, via

$$Q^i_j f(y,q) = q^i_j f(y,q), \qquad \hat{y}_i f(y,q) = \left(y_i + \tfrac{i}{2}\chi_{ij}\partial^j\right) f(y,q); \tag{28}$$

is a reducible representation of $\hat{H}$ with correct commutative limit $\vartheta \to 0$. In fact, rhs(28b) is the star-product $y_i \star f(y,q)$ induced by the twist $\mathscr{F} = \exp[\frac{i}{2}\vartheta_{ij} p^i \otimes p^j]$, leading to $\hat{y}_i \star \hat{y}_j - \hat{y}_j \star \hat{y}_i = i\chi_{ij}$.

We require $f \in V$ a fast decay $f(y,q) \stackrel{|y|\to\infty}{\longrightarrow} 0$; the scalar product

$$\langle f, f' \rangle := \int_{E^3} \overline{f(g)} f'(g)\, dg, \tag{29}$$

where dg is the Haar measure on E^3, makes V a pre-Hilbert space, whose completion is $\mathscr{L}^2(E^3) \simeq \mathscr{H}_{\mathfrak{R}'} \equiv$ Hilbert space of states of $\mathfrak{R}'$ w.r.t. $\mathfrak{R}$.

$\exists D(q)\in O(3)$ s.t. the only nontrivial (18c) for $\hat{y}_i\equiv D_i^h\hat{\mathrm{y}}_h$ is $\left[\hat{\mathrm{y}}_1,\hat{\mathrm{y}}_2\right]=i\mu$.

Proposition. *If $\chi(q)\neq 0$, for all $\alpha\in\mathbb{R}^3$ and $\nu\in\mathbb{R}^+$ the Gaussian*

$$\psi^T_{\alpha,\nu,q}(y)=\left(\frac{8}{\pi^3\mu^2(q)\nu}\right)^{1/4}\exp\left[-\frac{(\mathrm{y}_1-\alpha_1)^2+(\mathrm{y}_2-\alpha_2)^2}{\mu(q)}-\frac{(\mathrm{y}_3-\alpha_3)^2}{\nu}\right] \quad (30)$$

is and eigenvector of $b=\hat{\mathrm{y}}_1+i\hat{\mathrm{y}}_2$ with eigenvalue $z\equiv\alpha_1+i\alpha_2$ and has

$$\langle\hat{\mathrm{y}}_i\rangle=\alpha_i,\qquad \Delta\hat{\mathrm{y}}_3=\sqrt{\frac{\nu}{2}},\quad \Delta\hat{\mathrm{y}}_1=\Delta\hat{\mathrm{y}}_2=\sqrt{\frac{\mu}{2}}. \quad (31)$$

$\{\psi^T_{\alpha,\nu,q}\}_{\alpha\in\mathbb{R}^3}$ is a complete family of states in $\mathscr{L}^2(G^T)$ ($G^T\simeq\mathbb{R}^3_y$ is the translation subgroup) centered around $\alpha\in\mathbb{R}^3$. The generalized functions

$$\psi^T_{\alpha,q}(y):=\sqrt{\frac{2}{\pi\mu(q)}}\exp\left[-\frac{(\mathrm{y}_1-\alpha_1)^2+(\mathrm{y}_2-\alpha_2)^2}{\mu(q)}\right]\delta(\mathrm{y}_3-\alpha_3), \quad (32)$$

obtained from (30) taking the limit $\nu\to 0$, saturate the URs

$$\Delta\hat{\mathrm{y}}_1\Delta\hat{\mathrm{y}}_2\geq\mu/2,\qquad \sum_{i=1}^3\Delta\hat{y}_i^2\geq\mu=\sqrt{-\tfrac{1}{2}\mathrm{tr}(\chi^2)}. \quad (33)$$

Remark As $q\to I_3$ (or $\vartheta\to 0$), $\mu\to 0$, and the RF change gets "classical"

$$\psi^T_{\alpha,q}(y)\to\delta^{(3)}(\mathrm{y}-\alpha). \quad (34)$$

The rhs(33), as χ_{ij}, depends on q; here is a q-independent upper bound:

Proposition. *The uncertainties of the coherent states (32) satisfy*

$$\sum_{i=1}^{3} \Delta\hat{y}_i^2 \leq 2\zeta = \sqrt{-2\,\mathrm{tr}(\vartheta^2)}. \tag{35}$$

Assume that S is in a pure state $\psi \in \mathscr{L}^2(\mathbb{R}^3)$ w.r.t. $\mathfrak{R}$, and $\mathfrak{R}$ is w.r.t. $\mathfrak{R}'$ in the (pure) product state $\psi^T_{\alpha,\nu,q_0} \otimes \Phi_{q_0,\nu'} \in \mathscr{L}^2(G^T) \otimes \mathscr{L}^2(G^H)$, where $q_0 \in SO(3)$ and $\left\{\Phi_{q_0,\nu'}\right\}_{\nu' \in \mathbb{R}}$ is a family s. t. $|\Phi_{q_0,\nu'}|^2 \xrightarrow{\nu' \to 0} \delta(q-q_0)$.

We find that for $q_0 \neq \mathbf{1}_3$ and generic ψ the state of S wrt $\mathfrak{R}'$ is mixed; in fact, the kernel $\rho'_S(x,x') \equiv \langle x|\rho'_S|x'\rangle$ of the state (density operator) ρ'_S of S w.r.t. $\mathfrak{R}'$ cannot be put in a factorized form $\rho'_S(x,x') = \phi(x)\,\overline{\phi(x')}$. However, if $\psi(x) = N\, e^{ix_j k^j}$ = eigenstate of momentum, then ρ'_S is pure, because $\rho'_S(x,x') = |N|^2\, e^{i(x_i q_0{}^i_j k^j)}\, \overline{\left[e^{i(x_i q_0{}^i_j k^j)}\right]}$.

Regular repr. of $\hat{H} \equiv Fun_\vartheta(\mathrm{P})$; coherent states for QRFs

Abbreviating $\chi := \mathbf{1}_H\vartheta - \Lambda^T\vartheta\Lambda$, $\hat{H}$ is generated by $\Lambda^\nu_\mu, \hat{y}_\mu$ fulfilling

$$[\Lambda^\nu_\mu, \cdot\,] = 0, \qquad \Lambda\eta\Lambda^T = \mathbf{1}_H\eta, \qquad [\hat{y}_\mu, \hat{y}_\nu] = i\,\chi_{\mu\nu}. \tag{36}$$

It can be faithfully represented on the space V of smooth functions $f(y,\lambda)$ of real commuting variables y_μ, λ^ν_μ with $\lambda \in SO(1,3)$, e.g. by

$$\Lambda^\nu_\mu\, f(y,\lambda) = \lambda^\nu_\mu\, f(y,\lambda), \qquad \hat{y}_\mu\, f(y,\lambda) = \left(y_\mu + \frac{i}{2}\chi_{\mu\rho}\frac{\partial}{\partial y_\rho}\right) f(y,\lambda); \tag{37}$$

reducible representation of $\hat{H}$ with the correct commutative limit $\vartheta \to 0$. In fact, rhs(37b) is the star-product $y_\mu \star f(y,\Lambda)$ induced by the twist $\mathscr{F} = \exp[\frac{i}{2}\vartheta_{\mu\rho}p^\mu \otimes p^\rho]$, leading to $\hat{y}_\mu \star \hat{y}_\nu - \hat{y}_\nu \star \hat{y}_\mu = i\chi_{\mu\nu}$.
There is λ-dependent $D \in O(4)$ with only nontrivial (36c) for $\hat{\mathrm{y}}_\mu = \hat{y}_\rho D^\rho_\mu$

$$\left[\hat{\mathrm{y}}_0, \hat{\mathrm{y}}_3\right] = -\left[\hat{\mathrm{y}}_3, \hat{\mathrm{y}}_0\right] = i\kappa, \qquad \left[\hat{\mathrm{y}}_1, \hat{\mathrm{y}}_2\right] = -\left[\hat{\mathrm{y}}_2, \hat{\mathrm{y}}_1\right] = i\mu, \tag{38}$$

where κ, μ are λ-dependent linear combinations of the $\vartheta_{\mu\nu}$; (37) gets

$$\hat{\mathrm{y}}_0 = y_0 + \frac{i\kappa}{2}\frac{\partial}{\partial y_3}, \quad \hat{\mathrm{y}}_3 = y_3 - \frac{i\kappa}{2}\frac{\partial}{\partial y_0}, \quad \hat{\mathrm{y}}_1 = y_1 + \frac{i\mu}{2}\frac{\partial}{\partial y_2}, \quad \hat{\mathrm{y}}_2 = y_2 - \frac{i\mu}{2}\frac{\partial}{\partial y_1}.$$

$$b_0=\frac{\hat{y}_0+i\hat{y}_3}{\sqrt{2\kappa}},\quad b_0^\dagger=\frac{\hat{y}_0-i\hat{y}_3}{\sqrt{2\kappa}},\quad b_1=\frac{\hat{y}_1+i\hat{y}_2}{\sqrt{2\mu}},\quad b_1^\dagger=\frac{\hat{y}_1-i\hat{y}_2}{\sqrt{2\mu}} \tag{39}$$

are ladder operators fulfilling the CCR

$$[b_a,b_b]=[b_a^\dagger,b_b^\dagger]=0,\qquad [b_a,b_b^\dagger]=\delta_{ab}. \tag{40}$$

Proposition. *If* $\det\chi\neq 0$, *then* $\forall\alpha\equiv(\alpha_0,\alpha_3,\alpha_1,\alpha_2)\in\mathbb{R}^4$ *the Gaussian*

$$\psi^T_{\alpha,\lambda}(y)=\frac{2}{\pi\sqrt{\kappa(\lambda)\mu(\lambda)}}\exp\left[-\frac{(y_0-\alpha_0)^2+(y_3-\alpha_3)^2}{\kappa(\lambda)}-\frac{(y_1-\alpha_1)^2+(y_2-\alpha_2)^2}{\mu(\lambda)}\right] \tag{4$$

is an eigenvector of $b_0\equiv(\hat{y}_0+i\hat{y}_3)/\sqrt{2\kappa}$, $b_1=(\hat{y}_1+i\hat{y}_2)/\sqrt{2\mu}$ *with eigenvalues* $z_0\equiv(\alpha_0+i\alpha_3)/\sqrt{2\kappa}$, $z_1\equiv(\alpha_1+i\alpha_2)/\sqrt{2\mu}$. *Correspondingly,*

$$\langle\hat{y}_\mu\rangle=\alpha_\mu,\qquad \Delta\hat{y}_0=\Delta\hat{y}_3=\sqrt{\frac{\kappa}{2}},\qquad \Delta\hat{y}_1=\Delta\hat{y}_2=\sqrt{\frac{\mu}{2}}. \tag{42}$$

It saturates the URs, following from (38),

$$\Delta\hat{y}_0\Delta\hat{y}_3\geq\tfrac{\kappa}{2},\quad \Delta\hat{y}_1\Delta\hat{y}_2\geq\tfrac{\mu}{2},\quad \sum_{\mu=0}^{3}\Delta\hat{y}_\mu^2\geq\mu+\kappa=\sqrt{2\mathrm{Pf}(\chi)-\tfrac{1}{2}\mathrm{tr}(\chi^2)}. \tag{43}$$

$\{\psi_{\alpha,\lambda}\}_{\alpha\in\mathbb{R}^4}$ *is a complete family of coherent states for the* $\hat{y}_i$ *centered around the 4 parameters* $\alpha\in\mathbb{R}^4$, *which label all classical translations.*

Rhs(43) depends on the specific Lorentz transformation λ. **Remarks:**

- To compute the rhs(43) *no need* to put (36c) in canonical form (38).
- As $\lambda \to I_4$ (or $\vartheta \to 0$) the $\kappa, \mu \to 0$, and the RF change gets "classical":

$$\psi^{T}_{\alpha,\lambda}(y) \to \delta^{(4)}(\mathrm{y}-\alpha). \tag{44}$$

Proposition. *The $\psi^{T}_{\alpha,\lambda}$ satisfy the rotation-independent upper bounds*

$$\sum_{\mu=0}^{3} \Delta\hat{y}_{\mu}^{2} \leq \left[\sqrt{\gamma^2-1}+\gamma+1\right](b+e), \tag{45}$$

$$\sum_{\mu=0}^{3} \Delta\hat{y}_{\mu}^{2} \leq 2(e+b) \qquad \text{if } \lambda = \text{pure rotation}, \tag{46}$$

$$\sum_{\mu=0}^{3} \Delta\hat{y}_{\mu}^{2} \leq \sqrt{2\gamma(\gamma-1)}\,(b+e) \quad \text{if } \lambda = \text{ pure boost}, \tag{47}$$

where $\gamma \equiv \lambda_0^0 = 1/\sqrt{1-v^2/c^2}$, $v \equiv$ speed of the origin of $\mathfrak{R}'$ wrt $\mathfrak{R}$, e, b are the norms of the 3-vectors $\boldsymbol{e}, \boldsymbol{b}$ of components $e^i \equiv \vartheta_{0i}$, $b^i \equiv \frac{1}{2}\varepsilon^{ijk}\vartheta_{jk}$.

Note: the upper bounds at the rhs depend on λ only via $\lambda_0^0 \equiv \gamma$.

Summary and conclusions

Physical theories are covariant under changes of reference frames (RFs). An *ordinary* change $\mathfrak{R} \mapsto \mathfrak{R}'$ of *classical* RFs can be seen as a point g in a Lie group manifold G.

If the state of $\mathfrak{R}$ w.r.t. $\mathfrak{R}'$ is mixed (a statistical distribution on G), or more generally if $\mathfrak{R}$ or $\mathfrak{R}'$ are quantum RFs (i.e., use "clocks", "rulers" that are themselves quantum systems), then one can describe the associated "unsharp" changes of RFs only via some *generalized group* structure.

The notion of a Hopf algebra, and of its (co)representation, is a possible one, naturally associated with NC spacetimes. We have shown the first steps in formulating quantum theories on the NC ("Moyal") ϑ-Minkowski space, which is covariant the under the ϑ-Poincaré Hopf algebra (here formulated as NC algebra $\hat{H}$ of "functions on the group"); in particular, coherent states for $\hat{H}$ best approximate sharp changes of classical RFs.

We have also shown that the state of a generic system S may be pure relative to a RF $\mathfrak{R}$ and mixed relative to another one $\mathfrak{R}'$: for CRFs this occurs if the state of $\mathfrak{R}$ w.r.t. $\mathfrak{R}'$ is mixed; for QRFs, this in general occurs even if the latter state is pure.

Thank you for your attention!