Mixed in Translations

Fedele Lizzi

Università di Napoli Federico II

INFN Napoli

Work in collaboration with Gaetano Fiore

Corfu 2025

In both the classical and quantum cases, continuous symmetries are described by Lie groups, or in the case of infinitesimal transformations, Lie algebras. This generalises to quantum symmetries of quantum spaces.

For most of this talk I will nevertheless be dealing with usual group symmetries of usual spaces. Gaetano in the next talk will discuss the quantum angle. For most of the talk I will be in the simplest possible case: translations on the line.

Groups are topological Hausdorff spaces, manifolds, with an extra structure which enables the multiplications of two elements, the presence of the identity and of the inverse.

Any topological space can be described by the relations among its points, which sets are open, closed, the concept of convergence and the like.

Many of the people in this audience like the alternative, and equivalent, description given by the commutative algebras of continuous function on the space. Given a commutative algebra over the complexes, with an involution (conjugation) and a Banach norm, this is always the algebra of continuous functions over a topological space

For the sake of completeness and to fix notations I recap some key concepts

Given an algebra $\boxed{\mathcal{A}}$ the set of points of the topological space can be reconstructed as states: maps $\boxed{\rho : f \in \mathcal{A} \to \mathbb{C}}$ with the following properties:

$\boxed{\rho(f^* f) \geq 0}$, unit norm, i.e. $\boxed{\|\rho\| = \sup_{\|f\| \leq 1} \rho(f) = 1}$, if the algebra contains the identity $\boxed{\rho(1) = 1}$.

States can be combined, given a number $\boxed{0 < \lambda < 1}$ the sum $\boxed{\lambda\rho_1 + (1 - \lambda)\rho_2}$ is still a state.

Some states cannot be written as sum of two other states. Those are called pure states.

Pure states correspond to the points. Namely, at each pure state $\boxed{\delta}$ corresponds a point.

The topology of the set of points is defined by the concept of convergence. A set of points/states $\boxed{\delta_n}$ converges to $\boxed{\delta}$ if

$$\boxed{\lim_n \delta_n = \delta \text{ if } \forall f \in \mathcal{A} \text{ then } \lim_n \delta_n(f) = \delta(f)}$$

The second limit is well defined since it is the convergence of complex numbers, and this is used for the definition.

Let us look at the example of the real line $\boxed{\mathbb{R}}$.

We show that pure states are the points for the algebra of continuous functions vanishing at infinity.

The norm of the function $\boxed{f(a)}$ is the *sup norm*: $\boxed{\|f\| = \sup_a |f(a)|}$

A positive normalised density distribution $\boxed{\tilde{\rho}(a)}$ defines a state $\boxed{\rho}$ as $\boxed{\rho(f) = \int \mathrm{d}a\, \tilde{\rho}(a) f(a)}$

These states can be the sum of two other states. In several different ways in fact

Also $\boxed{\delta}$ distributions define states.

$$\boxed{\delta_{a_0}(f) = \int \mathrm{d}a\, \delta(a - a_0) f(a) = f(a_0)}$$

They **cannot** be written as sum of two other states. They are pure states.

Let us now apply this point of view to Lie Groups

A Lie group is not just a topological space and a manifold, it has further structures: product, identity, inverse.

Given two points $\boxed{a_1, a_2 \in G}$ the product $\boxed{\mu : G \otimes G \to G}$ defines a third element $\boxed{\mu(a_1, a_2)}$.

In the case of translations we use the notation $\boxed{a_1 + a_2}$.

Given a point there exists the inverse map: $\boxed{G \to G}$ indicated $\boxed{a_0^{-1}}$, or for translations $\boxed{-a_1}$.

There is a special point, the identity, which for translations we call $\boxed{0}$.

At the level of the algebra this structure is encoded in the properties of a Hopf Agebra

Groups are important because they act as symmetries on physical systems. This is done via a representations on vector spaces.

A representation associates to any element of the group a unitary operators acting on a vector space. For our translations we will use as vector space $\boxed{L^2}$ functions on the line $\boxed{\Psi(x)}$

To each element of the group we associate a unitary operator as

$$\boxed{\pi(a_0)\Psi(x) = U_{a_0}\Psi(x) = e^{a_0\partial_x}\Psi(x) = \Psi(x + a_0)}$$

Interpreting $\boxed{\Psi(x) = \langle x|\Psi\rangle}$ as a “wave function”, we have the representation of pure states of the group over pure states of the vector space. We will call the vector space of the $\boxed{\Psi}$’s the *carrier space*.

We can also act with the element of the group on density matrices of the carrier space.

To distinguish the density matrices we use the notation $\boxed{\rho}$ for these other density matrices.

Do not confuse the states $\boxed{\rho}$ on the group introduced earlier, with the states $\boxed{\rho}$ on the carrier space.

Then

$$\boxed{\pi(a_0)(\rho) = e^{a_0^\mu \frac{\partial}{\partial x^\mu}} \rho e^{-a_0^\mu \frac{\partial}{\partial x^\mu}} = U(a_0)\rho U(a_0)^\dagger}$$

For $\boxed{\rho = |\Psi\rangle\langle\Psi|}$ it reduces to the previous case, and of course $\boxed{\langle x\,|\Psi\rangle\langle\Psi|\,x\rangle = |\Psi(x)|^2}$. Being unitary the representation transforms pure states in pure states. All this is well known.

We have therefore associated a transformation to the pure states i.e. to the points of the manifold. If we want to understand the action of quantum groups, and somehow connect them to some classical limit, we need to generalise this action to states which are not the evaluation map at a point.

Consider therefore nonpure states for the algebra of functions on the group. We are not considering a single point on the group manifold, which corresponds to an element of the group, but a density probability on the group.

In other words we are not considering translations by a definite amount, but rather a certain probability to have a particular translation.

We want to find an action of these mixed states on the carrier space, some sort of "$\boxed{\pi(\rho)}$". We will see that there is no group structure for the space of states, and therefore we cannot find a representation as unitary operators. We will nevertheless define an action of the space of state on the carrier space and use the same symbol $\boxed{\pi}$.

Start with the simplest nonpure state

$$\boxed{\widetilde{\rho}_{a_1,a_2}(a) = \frac{1}{2}\delta(a-a_1) + \frac{1}{2}\delta(a-a_2)}$$

This state is just the average of the function $\boxed{f}$ in two different points. We are averaging two different translations.

$$\boxed{\rho_{a_1,a_2}(f) = \frac{1}{2}f(a_1) + \frac{1}{2}f(a_2)}$$

Consider now the action on the carrier space. We need reproduce the weighted sum of two translation. The action is:

$$\boxed{\pi(\rho_{a_1,a_2})(\rho) = \frac{1}{2}U(a_1)\rho U(a_1)^\dagger + \frac{1}{2}U(a_2)\rho U(a_2)^\dagger}$$

For the case $\boxed{\rho = |\Psi\rangle\langle\Psi|}$, in the position representation.

$$\boxed{\langle x|\, \pi(\rho_{a_1,a_2})\rho\, |x\rangle = \ \mathrm{Tr}\ |x\rangle\langle x|\, \pi(\rho_{a_1,a_2})\rho = \frac{1}{2}\left|\Psi(x + a_1)\right|^2 + \frac{1}{2}\left|\Psi(x + a_2)\right|^2}$$

This is still a density matrix, but even if we started with a pure state, we end up with a mixed state!

There is no contradiction in this, because the whole state we started with (group plus representation space) was mixed.

We can easily generalise this to a generic $\boxed{\rho(a)}$:

$$\boxed{\pi(\rho)\mathrm{\rho} = \int \mathrm{d}a\tilde{\rho}(a)U(a)\mathrm{\rho}U(a)^\dagger}$$

and for pure states

$$\boxed{\mathrm{Tr}\ |\mathrm{x}\rangle\langle\mathrm{x}|\,\pi(\rho)\,|\Psi\rangle\langle\Psi| = \int \mathrm{d}a\,\tilde{\rho}(a)\,|\Psi(\mathrm{x}+a)|^2}$$

Smeared translations smear states.

Are these transformations another group?

We should look for the product, the identity and the inverse.

The group I am considering is abelian, therefore certain aspects are simplified, but the substance will remain in the nonabelian case.

At the level of the algebra the product among points is dually reflected in the coproduct in the algebra:

$$\boxed{\Delta f = \sum_i f_i \otimes f_i \Rightarrow \Delta f(a_1, a_2) = f(a_1 + a_2)}$$

In the generic case the + must be substituted by the group product.

Given two generic states $\boxed{\rho_1, \rho_2}$, we define the product state as

$$\boxed{(\rho_1\rho_2)(f) = \int \mathrm{d}a\mathrm{d}a' \widetilde{\rho}_1(a)\widetilde{\rho}_2(a')\Delta f(a, a') = \int \mathrm{d}a\mathrm{d}a' \widetilde{\rho}_1(a)\widetilde{\rho}_2(a') f(a + a')}$$

For pure states $\boxed{\widetilde{\rho}_i(a) = \delta(a - a_i)}$

$$\boxed{(\rho_1\rho_2)(f) = f(a_1 + a_2)}$$

It is straightforward to check that $\boxed{(\rho_1\rho_2)(f)}$ can be written as a single $\boxed{\rho}$ with a density function $\boxed{\widetilde{\rho}(a)}$ given by the convolution:

$$\boxed{\widetilde{(\rho_1\rho_2)}(a) = \int \mathrm{d}b \; \widetilde{\rho}_1(b) \, \widetilde{\rho}_2(a - b)}$$

this expression is generic. Purity of the states plays no role.

The identity functional has the density $\boxed{\widetilde{\rho}(a) = \delta(a)}$ as before.

It reflects the *counit*, and is denoted by $\boxed{\varepsilon}$, i.e. $\boxed{\varepsilon(f) = f(0)}$.

It remains to verify the existence of the inverse to give a full group structure.

This is given by the antipode which in this simple case is $\boxed{S(f)(a) = f(-a)}$.

To the pure state with density $\boxed{\delta(a - a_1)}$ there corresponds the inverse pure state with density $\boxed{\delta(-a - a_1)}$. This corresponds to a transaltions of $\boxed{-a_1}$.

This does not work for mixed states!

Take the simplest example $\boxed{\rho_{a_1,a_2}}$. If we act first with the state, and then with the state $\boxed{\rho_{-a_1,-a_2}}$ the result is

$$\boxed{\rho_{-a_1,-a_2}\left(\rho_{a_1,a_2}(\rho)\right) = \rho_{-a_1,-a_2}\left(\frac{1}{2}U(a_1)\rho U(a_1)^\dagger + \frac{1}{2}U(a_2)\rho U(a_2)^\dagger\right)}$$

$$\boxed{= \frac{1}{2}\rho + \frac{1}{4}U(a_1 - a_2)\rho U(a_1 - a_2)^\dagger + \frac{1}{4}U(a_2 - a_1)\rho U(a_2 - a_1)^\dagger}$$

Therefore instead than a group we have a semigroup.

In the dual algebraic setting, a semigroup amounts to a *bialgebra*, i.e. an algebra endowed with compatible coproduct and counit. The antipode is missing with respect to the Hopf algebra case.

As an example start from a pure gaussian state in the carrier space:

$$\boxed{\Psi(x) = \frac{1}{\sqrt{\alpha\sqrt{2\pi}}} e^{-\frac{x^2}{4\alpha^2}}}$$

Consider $\boxed{\tilde{\rho}_{a_1,a_2}}$ we considered earlier, with $\boxed{a_1 = 0}$. The translated mixed state has the following probability density of finding the particle in $\boxed{x}$:

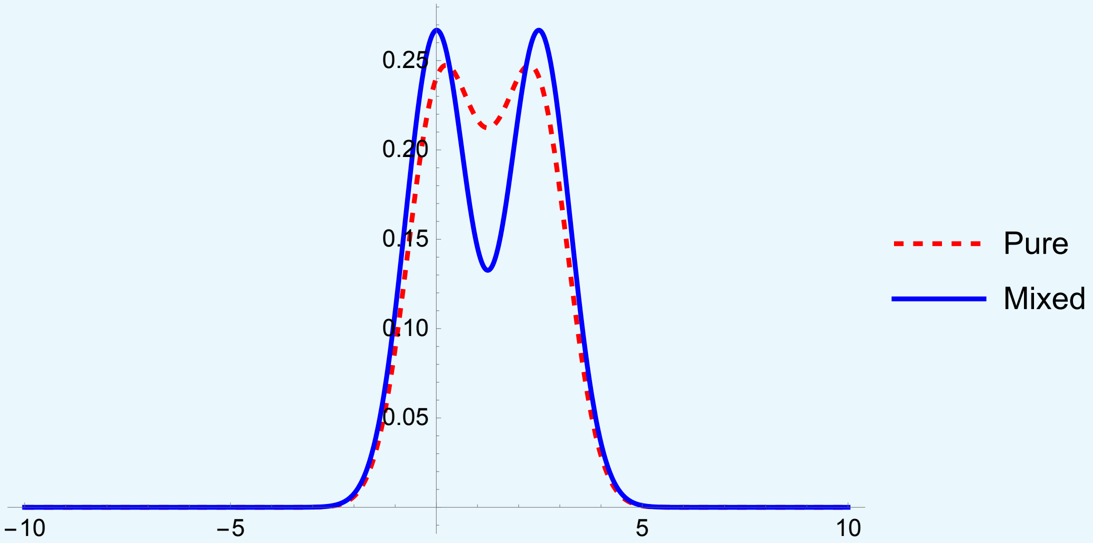
$$\boxed{\frac{1}{2}\left|\Psi(x)\right|^2 + \frac{1}{2}\left|\Psi(x + a_2)\right|^2 = \frac{e^{-\frac{(x-a_2)^2}{2\alpha^2}} + e^{-\frac{x^2}{2\alpha^2}}}{2\sqrt{2\pi}\alpha}}$$

Compare with the pure state case of a single wave functions sum of two Gaussians

$$\boxed{\left|\Psi_{a_2}(x)\right|^2 = \frac{\left(e^{-\frac{(x-a_2)^2}{4\alpha^2}} + e^{-\frac{x^2}{4\alpha^2}}\right)^2}{2\sqrt{2\pi}\alpha\left(e^{-\frac{a_2^2}{8\alpha^2}} + 1\right)}}$$

The absence of the mixed terms in the mixed state has the effect to divide the two maxima in a sharper way. It is more “classical”.

Out[•]=



The probability density for the pure and mixed states. The chosen parameters are $\boxed{\alpha = 0.75, a_2 = 2.5}$.

The case in which also the non pure translation is a Gaussian: $\boxed{\tilde{\rho}(a) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(a-a_0)^2}{2\sigma^2}}}$

The probability is position space is: $\boxed{\int da\, \tilde{\rho}(a)\, |\Psi(x+a)|^2 = \frac{1}{\sqrt{\sigma^2+\alpha^2}\sqrt{2\pi}} e^{-\frac{(x-a_0)^2}{2(\sigma^2+\alpha^2)}}}$

This appears as a simple spreading of the wave packet, but this would be misleading, the state resulting from the nonpure translation is non pure and cannot be described by a single function of $\boxed{x}$.

Comparing with the pure state obtained translating in a similar Gaussian way a Gaussian wave function $\boxed{\Psi_{\mathrm{transl}}(x) = N \int da\, \tilde{\rho}(a) \Psi(x+a)}$

In this case we have $\boxed{\left|\Psi_{\mathrm{transl}}(x)\right|^2 = \frac{e^{-\frac{(a_0-x)^2}{\sigma^2+2\alpha^2}}}{\sqrt{\pi}\sqrt{\sigma^2+2\alpha^2}}}$

Since $\boxed{2(\sigma^2+\alpha^2) > \sigma^2 + 2\alpha^2}$ we see that the pure state is more localised than the mixed one.

Consider now time translations for a free particle.

Let us go in the momentum representation, for which the time evolution is particularly simple, since momentum is conserved.

The improper state of momentum $\boxed{p}$ is described by the density matrix $\boxed{|p\rangle\langle p|}$

The vectors $\boxed{|p\rangle}$ are also eigenvectors of the Hamiltonian $\boxed{\widehat{H}}$ since

$$\boxed{\widehat{H}\,|p\rangle = E\,|p\rangle = \frac{p^2}{2m}\,|p\rangle}$$

In this basis translations by a fixed time $\boxed{t_0}$ and energy $\boxed{E}$ are given by the multiplicative operator

$$\boxed{U(E,t_0)\,|p\rangle = \mathrm{e}^{\mathrm{i}\frac{Et_0}{\hbar}}\,|p\rangle}$$

Consider now the one parameter translation group parametrised by the energy $\boxed{E}$, not by time, which we fix once for all at the value $\boxed{t_0}$.

Physically we are considering an ensemble of particles at different energies and considering their time translated states.

The value $\boxed{E = 0}$ is the particle at rest. Conventionally we can consider also negative values of $\boxed{E}$ as evolving back in time. In this way we have a group.

Since the evolution is a multiplication by a phase, the density matrix $\boxed{|p\rangle\langle p|}$ is invariant. This is conservation of momentum and energy for a pure eigenstate.

Adapt the previous reasoning to this case, namely assume that we do not know precisely the energy for the translation, but we have a spread of energies which depends on a parameter $\boxed{\beta}$ with the dimensions of time.

In particular consider

$$\boxed{\tilde{\rho}(\beta) = \sqrt{\frac{\beta}{4\pi E\hbar}}\mathrm{e}^{-\frac{E\beta}{\hbar}}}$$

Such that

$$\boxed{\int_0^\infty \mathrm{d}E\,\tilde{\rho}(\beta) = 1}$$

The seemingly strange form of this density is explained if we express the two relations in terms of $\boxed{p}$ (taking into account the difference in the measure):

$$\boxed{\tilde{\rho}(\beta) = \sqrt{\frac{\beta}{2m\hbar\pi}}\,\mathrm{e}^{-\frac{p^2}{2m\hbar}\beta}}$$

which is properly normalised with the transformed measure:

$$\boxed{\int_{-\infty}^{\infty} \mathrm{d}p\, \tilde{\rho}(\beta) = 1}$$

This latest expression shows that we are considering a smeared translation for which the overall momentum vanishes, since the Gaussian is centered in 0.

Since we are in the basis in which $\boxed{\hat{p}}$ and $\boxed{\hat{H}}$ are diagonal, the steps which lead to the new density matrix are trivial.

The unitary operators $\boxed{U(E, t_0)}$ are just multiplicative, and cancel each other

$$\boxed{\rho = \int \mathrm{d}p \sqrt{\frac{\beta}{2m\hbar\pi}} \mathrm{e}^{-\frac{p^2}{2m\hbar}\beta} |p\rangle\langle p|}$$

In the $\boxed{p}$ basis the time $\boxed{t_0}$ does not appear anymore. All we needed was that the translation was smeared.

As before we started with a pure state and ended up with a mixed one

Make the standard identification ($\boxed{K_B}$ Boltzmann constant)

$$\boxed{\beta = \frac{\hbar}{K_B T}}$$

We have found a state with a thermal distribution of momenta: a thermal state!

$$\boxed{\sqrt{\frac{1}{2\pi m K_B T}} e^{-\frac{p^2}{2mK_B T}}}$$

This is the distribution of momenta in a gas at temperature $\boxed{T}$.

The appearance of a thermal state in this context has a very suggestive interpretation.

For a free particle precise knowledge of momentum implies precise knowledge of energy.

This is in conflict with the generalised energy/time uncertainty for which

$$\boxed{\Delta H \Delta t \geq \frac{\hbar}{2}}$$

As in the case of coherent states the best we can do is to consider a Gaussian smearing.

The analogy is only heuristic, there is not time operator conjugated to the Hamiltonian by a commutation relation.

Since $\boxed{t_0}$ has disappeared, no matter the time we consider for the evolution, if there is a time translation it has to be smeared, and this leads to thermal state.

The free parameter we still have is the amount of smearing: $\boxed{\beta}$. A large $\boxed{\beta}$ means a precise knowledge of the energy (nearly zero),and consequently a low temperature. $\boxed{\beta}$ small gives a high temperature, and a poor knowledge of the energy.

We have therefore found the relation between time, energy and temperature in a novel way, without the usual techniques of going to a strip in the complex plane, or other similar techniques. In this case the thermal state is a necessary consequence of the time energy uncertainty.

Final Considerations

The original motivation of this work was the description of the action of quanum groups, and in particular θ-Poincaré which will be discussed in the next talk.

We needed to comprehend the action of a group (or its generalisation) in the cases in which we could not (or wanted not) identify the transformation with a single point of a topological space.

Nevertheless I feel that the semigroup we stumbled across is very interesting, and can have applications even outside of the quantum group context.

Thermal states are a first application, but there can be others.