4D Fuzzy Gravity on a Covariant Noncommutative Space and Unification with Internal Interactions

Stelios Stefas

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Introduction

- So far in the gauge theoretic approach of gravity, general relativity is described by gauging the symmetry of the tangent manifold in four dimensions.
- Usually the dimension of the tangent space is considered to be equal to the dimension of the curved manifold. However, the tangent group of a manifold of dimension d is not necessarily SO_d. *Weinberg '84*
- It has been suggested that by gauging an enlarged symmetry of the tangent space in four dimensions one could unify gravity with internal interactions. *Chamseddine, Mukhanov '10*
- We aim to unify FG as a gauge theory with internal interactions under one unification gauge group.

4D Conformal Gravity as a Gauge Theory

- Group parameterizing the symmetry: $SO(2,4)$
- 15 generators:

$$[\hat{M}_{AB}, \hat{M}_{CD}] = \eta_{AC}\hat{M}_{DB} - \eta_{BC}\hat{M}_{DA} - \eta_{AD}\hat{M}_{CB} + \eta_{BD}\hat{M}_{CA}$$

- Indices splitting $\rightarrow$ 6 LT M_{ab}, 4 translations P_a, 4 conformal boosts K_a and 1 dilatation D
- Action is taken of $SO(2,4)$ invariant quadratic form
- Initial symmetry breaks spontaneously by introducing a scalar in the adjoint rep fixed in the dilatation direction *Roumelioti, S, Zoupanos '24*

SSB by using a scalar in the adjoint representation

Gauge connection:

$$A_\mu = \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a + b_\mu{}^a K_a + \tilde{a}_\mu D,$$

Field strength tensor:

$$F_{\mu\nu} = \frac{1}{2}R_{\mu\nu}{}^{ab}M_{ab} + \tilde{R}_{\mu\nu}{}^a P_a + R_{\mu\nu}{}^a K_a + R_{\mu\nu}D,$$

where

$$\begin{aligned}
R_{\mu\nu}{}^{ab} &= \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b - 8e_{[\mu}{}^{[a}b_{\nu]}{}^{b]} \\
&= R^{(0)ab}_{\mu\nu} - 8e_{[\mu}{}^a b_{\nu]}{}^{b]}, \\
\tilde{R}_{\mu\nu}{}^a &= \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^{ab}e_{\nu b} - \omega_\nu{}^{ab}e_{\mu b} - 2\tilde{a}_{[\mu}e_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu}(e) - 2\tilde{a}_{[\mu}e_{\nu]}{}^a, \\
R_{\mu\nu}{}^a &= \partial_\mu b_\nu{}^a - \partial_\nu b_\mu{}^a + \omega_\mu{}^{ab}b_{\nu b} - \omega_\nu{}^{ab}b_{\mu b} + 2\tilde{a}_{[\mu}b_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu}(b) + 2\tilde{a}_{[\mu}b_{\nu]}{}^a, \\
R_{\mu\nu} &= \partial_\mu\tilde{a}_\nu - \partial_\nu\tilde{a}_\mu + 4e_{[\mu}{}^a b_{\nu]a},
\end{aligned}$$

▷ $R^{(0)ab}_{\mu\nu}$, $T^{(0)a}_{\mu\nu}(e)$: *Curvature and Torsion of 4D Poincaré grav.*

We start with the following action, which is quadratic in terms of the field strength tensor and introduce a scalar in the adjoint rep.

$$\mathcal{S}_{SO(2,4)} = a_{CG} \int d^4x \left[\operatorname{tr} \epsilon^{\mu\nu\rho\sigma} m\phi F_{\mu\nu} F_{\rho\sigma} + \left(\phi^2 - m^{-2}\mathbb{1}_4\right)\right],$$

The scalar expanded on the generators is:

$$\phi = \phi^{ab} M_{ab} + \tilde{\phi}^a P_a + \phi^a K_a + \tilde{\phi} D,$$

We pick the specific gauge in which ϕ is only in the direction of the dilatation generator D:

$$\phi = \phi^0 = \tilde{\phi} D \xrightarrow{\phi^2 = m^{-2}\mathbb{1}_4} \phi = -2m^{-1} D.$$

The resulting broken action is (after employing anticommutator relations and the traces over the generators):

$$\mathcal{S}_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd}$$

The $\tilde{a}_\mu$ and $R_{\mu\nu}$ are not present in the action, so we can set both equal to zero.

$$R_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu + 4e_{[\mu}{}^a b_{\nu]a} = 0 \xrightarrow{\tilde{a}_\mu = 0}$$

$$e_\mu{}^a b_{\nu a} - e_\nu{}^a b_{\mu a} = 0$$

We examine two possible solutions of the above equation:

- $b_\mu{}^a = a e_\mu{}^a$, *Chamseddine '03*
- $b_\mu{}^a = -\frac{1}{4}\left(R_\mu{}^a + \frac{1}{6} R e_\mu{}^a\right)$ *Kaku, Townsend, van Nieuwenhuizen, 78*

Freedman, Van Proyen 'Supergravity' '12

The first choice leads to the Einstein-Hilbert action, while the second leads to Weyl action.

Einstein-Hilbert action

- When $b_\mu{}^a = a e_\mu{}^a$, the broken action becomes:

$$\mathcal{S}_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4}\int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd} \implies$$

$$\mathcal{S}_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4}\int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big[R^{(0)ab}_{\mu\nu}R^{(0)cd}_{\rho\sigma} - 16m^2 a R^{(0)ab}_{\mu\nu}e_\rho{}^c e_\sigma{}^d + \\ + 64m^4a^2 e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d\Big]$$

This action consists of three terms: one G-B topological term, the E-H action, and a cosmological constant. For $a < 0$ describes GR in AdS space.

Weyl action

- When $b_\mu{}^a = -\frac{1}{4}(R_\mu{}^a + \frac{1}{6}Re_\mu{}^a)$, the broken action becomes

$$\mathcal{S} = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big[R^{(0)ab}_{\mu\nu} - \frac{1}{2}\left(\tilde{e}_\mu{}^{[a}R_\nu{}^{b]} - \tilde{e}_\nu{}^{[a}R_\mu{}^{b]}\right) + + \frac{1}{3}R\tilde{e}_\mu{}^{[a}\tilde{e}_\nu{}^{b]}\Big] \Big[R^{(0)cd}_{\rho\sigma} - \frac{1}{2}\left(\tilde{e}_\rho{}^{[c}R_\sigma{}^{d]} - \tilde{e}_\sigma{}^{[c}R_\rho{}^{d]}\right) + + \frac{1}{3}R\tilde{e}_\rho{}^{[c}\tilde{e}_\sigma{}^{d]}\Big],$$

where $\tilde{e}_\mu{}^a = me_\mu{}^a$ is the rescaled vierbein. The above action is equal to

$$\mathcal{S} = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}C_{\mu\nu}{}^{ab}C_{\rho\sigma}{}^{cd},$$

where $C_{\mu\nu}{}^{ab}$ is the Weyl conformal tensor.

The NC framework & gauge theories

- Noncommutative space → replace coordinates with operators X^i $(\in \mathcal{A})$ satisfying: $[X^i, X^j] = i\Theta^{ij}(X)$

Connes '94, Madore '99

- Antisymmetric tensor $\Theta^{ij}(X)$ - defines the NC of the space
- Introduction of *covariant NC coordinate*:

$$\mathcal{X}_\mu = X_\mu + A_\mu$$

Madore, Schraml, Schupp, Wess '00

- Obeys a covariant gauge transformation rule: $\delta\mathcal{X}_\mu = i[\epsilon, \mathcal{X}_\mu]$
- Definition of a NC *covariant field strength tensor*:

$$F_{ab} = [\mathcal{X}_a, \mathcal{X}_b] - i\Theta_{ab}$$

Non-Abelian case

- Let us consider the commutator of two elements of an algebra:

$$[\epsilon, A] = [\epsilon^A T^A, A^B T^B] = \frac{1}{2}\{\epsilon^A, A^B\}[T^A, T^B] + \frac{1}{2}[\epsilon^A, A^B]\{T^A, T^B\}$$

- *Not* possible to restrict to a matrix algebra:
  - ▷ last term neither *vanishes* in NC nor is an *algebra element*

- There are two options to overpass the difficulty:
  - ▷ Consider the universal enveloping algebra
  - ▷ Fix the rep and expand algebra so that the anticommutators close

Aschieri, Castellani '09

Ćirić, Gočanin, Konjik, Radovanović '18

▷ *We will later employ the second option*

The 4d covariant noncommutative space

- Constructing field theories on NC spaces is non-trivial: NC deformations break Lorentz invariance
- Such an example is the Fuzzy Sphere (2d space) - coords are identified as rescaled SU(2) generators *Madore '92, Hammou, Lagraa, Sheikh Jabbari '02*
  Vitale, Wallet '13, Vitale '14, Jurman, Steinacker '14
  Chatzistavrakidis, Jonke, Jurman, Manolakos, Manousselis, Zoupanos '18
- We will need a 4d covariant NC space to construct a gravity gauge theory
- We will aim for a NC version of dS_4, described by the embedding $\eta^{AB} X_A X_B = R^2$ into M_5

Snyder's Model '47

- The SO(1,4) generators, $J_{mn}, m, n = 0, \dots, 4$, satisfy the commutation relation:

$$[J_{mn}, J_{rs}] = i(\eta_{mr} J_{ns} + \eta_{ns} J_{mr} - \eta_{nr} J_{ms} - \eta_{ms} J_{nr})$$

- Consider decomposition of $SO(1,4)$ to maximal subgroup, $SO(1,3)$
- Introduce a length parameter λ and convert the generators to physical quantities by identifying $\Theta_{ij} = \hbar J_{ij}$, $X_i = \lambda J_{i4}$
- Thus, the commutation relations regarding the operators $\Theta_{\mu\nu}$ and X_μ are:

$$[\Theta_{ij}, \Theta_{kl}] = i\hbar \left(\eta_{ik}\Theta_{jl} + \eta_{jl}\Theta_{ik} - \eta_{jk}\Theta_{il} - \eta_{il}\Theta_{jk}\right),$$
$$[\Theta_{ij}, X_k] = i\hbar \left(\eta_{ik} X_j - \eta_{jk} X_i\right),$$
$$[X_i, X_j] = \frac{i\lambda^2}{\hbar}\Theta_{ij}$$

- The noncommutativity of coordinates becomes manifest

Yang's Model '47

- Extending covariance, also including momenta as generators → use a group with larger symmetry → minimum extension: $SO(1,4) \subset SO(1,5)$

Yang '47
Kimura '02, Heckman, Verlinde '15
Steinacker '16
Sperling, Steinacker '17,'19
Burić-Madore '14,'15
Manousselis, Manolakos, Zoupanos '19,'21

- The SO(1,5) generators, $J_{MN}, M, N = 0, \ldots, 5$, satisfy the commutation relation:

$$[J_{MN}, J_{P\Sigma}] = i(\eta_{MP}J_{N\Sigma} + \eta_{N\Sigma}J_{MP} - \eta_{NP}J_{M\Sigma} - \eta_{M\Sigma}J_{NP})$$

- Employ a *2-step* decomposition $SO(1,5) \supset SO(1,4) \supset SO(1,3)$
- Introducing a length parameter λ (like in Snyder's case) we convert the generators to physical quantities by identifying
  $\Theta_{ij} = \hbar J_{ij}, \;\; X_i = \lambda J_{i5}, \;\; P_i = \frac{\hbar}{\lambda} J_{i4}, \;\; h = J_{45}$

Yang's Model '47 (Continued)

- Thus, the commutation relations regarding all the operators $\Theta_{\mu\nu}, X_\mu, P_\mu, h$ are:

$$
\begin{aligned}
[\Theta_{\mu\nu}, \Theta_{\rho\sigma}] &= i\hbar(\eta_{\mu\rho}\Theta_{\nu\sigma} + \eta_{\nu\sigma}\Theta_{\mu\rho} - \eta_{\nu\rho}\Theta_{\mu\sigma} - \eta_{\mu\sigma}\Theta_{\nu\rho}) \,, \\
[\Theta_{\mu\nu}, X_\rho] &= i\hbar(\eta_{\mu\rho}X_\nu - \eta_{\nu\rho}X_\mu) \\
[\Theta_{\mu\nu}, P_\rho] &= i\hbar(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[P_\mu, P_\nu] &= i\frac{\hbar}{\lambda^2}\Theta_{\mu\nu} \,, \qquad [X_\mu, X_\nu] = i\frac{\lambda^2}{\hbar}\Theta_{\mu\nu} \,, \\
[P_\mu, h] &= -i\frac{\hbar}{\lambda^2}X_\mu \,, \qquad [X_\mu, h] = i\frac{\lambda^2}{\hbar}P_\mu \,, \\
[P_\mu, X_\nu] &= i\hbar\eta_{\mu\nu}h \,, \qquad [\Theta_{\mu\nu}, h] = 0
\end{aligned}
$$

- Momenta are seamlessly included in algebra
  - ▷ Momentum space becomes quantized
  - ▷ Heisenberg type CR between momenta and coords
- The above relations describe the noncommutative space

The 4d covariant noncommutative space (Continued)

- We begin by considering the isometry group of $dS_4 \rightarrow SO(1,4)$
- Extending covariance $\rightarrow$ extension of $SO(1,4)$ to $SO(1,5)$
- Following Yang's example $\rightarrow$ minimal extension of $SO(1,5)$ to $SO(1,6)$ looking for interesting results
- Perform three step decomposition by indices splitting to reach 4d language:

$$SO(1,6) \supset SO(1,5) \supset SO(1,4) \supset SO(1,3)$$

- Introduce length parameter and convert generators to physical quantities.

The commutation relations regarding all the operators $\Theta_{ij}, X_i, P_i, Q_i, q, p, h$ are:

$$[\Theta_{ij}, \Theta_{kl}] = i\hbar\left(\eta_{ik}\Theta_{jl} + \eta_{jl}\Theta_{ik} - \eta_{jk}\Theta_{il} - \eta_{il}\Theta_{jk}\right), \quad [Q_i, Q_j] = i\frac{\hbar}{\lambda^2}\Theta_{ij},$$
$$[\Theta_{ij}, Q_k] = \frac{i}{\hbar}\left(\eta_{ik}Q_j - \eta_{jk}Q_i\right), \quad [\Theta_{ij}, X_k] = \frac{i}{\hbar}\left(\eta_{ik}X_j - \eta_{jk}X_i\right),$$
$$[\Theta_{ij}, P_k] = \frac{i}{\hbar}\left(\eta_{ik}P_j - \eta_{jk}P_i\right), \quad [Q_i, X_j] = -i\frac{\hbar}{\lambda^2}\eta_{ij}q, \quad [Q_i, P_j] = -i\frac{\hbar^2}{\lambda^2}\eta_{ij}p,$$
$$[Q_i, q] = i\frac{\hbar}{\lambda^2}X_i, \quad [Q_i, p] = iP_i, \quad [X_i, X_j] = i\frac{\lambda^2}{\hbar}\Theta_{ij},$$
$$[X_i, P_j] = -i\hbar\eta_{ij}h, \quad [X_i, q] = -i\frac{\lambda^2}{\hbar}Q_i, \quad [X_i, h] = i\frac{\lambda^2}{\hbar}P_i,$$
$$[P_i, P_j] = i\frac{\hbar}{\lambda^2}\Theta_{ij}, \quad [P_i, p] = -iQ_i, \quad [P_i, h] = -i\frac{\hbar}{\lambda^2}X_i,$$
$$[q, p] = -ih, \quad [q, h] = ip, \quad [p, h] = -iq$$

They closely resemble conformal algebra!

- ▷ On top of NC coords and momenta, as well as Heisenberg type relation between them, we also get bonus info regarding group that shall be gauged

Noncommutative gauge theory of 4d gravity

- We want to formulate gravitation theory on the above space
- We make use of NC gauge theory toolbox combined with the procedure described in the 4d conformal gravity case

  Kimura '02, Heckman, Verlinde '15

- Begin by gauging the isometry group of the space, $\mathrm{SO}(1,4)$
- Anticommutators do not close $\rightarrow$ fix the representation $+$ enlargement of the algebra *Aschieri, Castellani '09*

  Chatzistavrakidis, Jonke, Jurman, Manolakos, Manousselis, Zoupanos '18

- Noncommutative gauge theory of $\mathrm{SO}(2,4) \times \mathrm{U}(1)$

  Manolakos, Manousselis, Zoupanos '19, '21

  Roumelioti, S, Zoupanos '24

- The generators of $\mathrm{SO}(2,4) \times \mathrm{U}(1)$ are represented by combinations of the 4×4 gamma matrices:
  - six Lorentz rotation generators: $M_{ab} = -\frac{i}{4}\left[\gamma_a, \gamma_b\right]$
  - four generators for conformal boosts: $K_a = \frac{1}{2}\gamma_a(1+\gamma_5)$
  - four generators for translations: $P_a = -\frac{1}{2}\gamma_a(1-\gamma_5)$
  - one generator for special conformal transformations: $\mathrm{D} = -\frac{1}{2}\gamma_5$
  - one $\mathrm{U}(1)$ generator: $\mathbb{1}$

- The above expressions of the generators allow the calculation of the algebra they satisfy:

$$[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} + \eta_{ad} M_{bc} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac},$$
$$[K_a, P_b] = -2\left(\eta_{ab} D + M_{ab}\right), \ [P_a, D] = P_a, \ [K_a, D] = -K_a,$$
$$[M_{ab}, K_c] = \eta_{bc} K_a - \eta_{ac} K_b, \ [M_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b$$

- Generators satisfy the following anticommutation relations:

Smolin '03

$$\{M_{ab}, M_{cd}\} = \frac{1}{2}\left(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad}\right) - i\epsilon_{abcd} D,$$
$$\{M_{ab}, P_c\} = +i\epsilon_{abcd} P^d,$$
$$\{M_{ab}, K_c\} = -i\epsilon_{abcd} K^d,$$
$$\{M_{ab}, D\} = 2M_{ab} D,$$
$$\{P_a, K_b\} = 4M_{ab} D + \eta_{ab},$$
$$\{K_a, K_b\} = \{P_a, P_b\} = -\eta_{ab},$$
$$\{P_a, D\} = \{K_a, D\} = 0.$$

Noncommutative Gauge Theory

- Since the gauge group is determined to be $SO(2,4) \times U(1)$, we can move on with the gauging procedure.

Manolakos, Manousselis, Zoupanos '21

- Consider the *covariant coordinate* $\mathcal{X}_\mu = X_\mu + A_\mu$
- Determine appropriate *covariant field strength tensor*
  $\mathcal{R}_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\frac{\lambda^2}{\hbar}\hat{\Theta}_{\mu\nu}$,
  where $\hat{\Theta}_{\mu\nu} = \Theta_{\mu\nu} + \mathcal{B}_{\mu\nu}$, the *covariant noncommutative tensor*

For the SSB to take place we:

$\hookrightarrow$ Introduce scalar field $\Phi(X)$ belonging in the 2nd rank antisym. of $SO(4)$, *charged* under $U(1)$ $\rightarrow$ $U(1)$ breaks and doesn't appear in final action

$\hookrightarrow$ Gauge fix $\Phi(X)$ in the direction that leads to Lorentz group

Gauge connection and field strength tensor decompose as:

$$A_\mu(X) = e_\mu{}^a \otimes P_a + \omega_\mu{}^{ab} \otimes M_{ab} + b_\mu{}^a \otimes K_a + \tilde{a}_\mu \otimes D + a_\mu \otimes \mathbb{1}_4 \,.$$

$$\mathcal{R}_{\mu\nu}(X) = \tilde{R}_{\mu\nu}{}^a \otimes P_a + R_{\mu\nu}{}^{ab} \otimes M_{ab} + R_{\mu\nu}{}^a \otimes K_a + \tilde{R}_{\mu\nu} \otimes D + R_{\mu\nu} \otimes \mathbb{1}_4 \,.$$

The component curvatures:

$$\begin{aligned}
R_{\mu\nu} &= [X_\mu, a_\nu] - [X_\nu, a_\mu] + [a_\mu, a_\nu] + [b_\mu{}^a, b_{\nu a}] + [\tilde{a}_\mu, \tilde{a}_\nu] + \frac{1}{2}[\omega_\mu{}^{ab}, \omega_{\nu ab}] \\
&\quad + [e_{\mu a}, e_\nu{}^a] - \frac{i\hbar}{\lambda^2} B_{\mu\nu} \\
\tilde{R}_{\mu\nu} &= [X_\mu, \tilde{a}_\nu] + [a_\mu, \tilde{a}_\nu] - [X_\nu, \tilde{a}_\mu] - [a_\nu, \tilde{a}_\mu] - i\{b_{\mu a}, e_\nu{}^a\} + i\{b_{\nu a}, e_\mu{}^a\} \\
&\quad + \frac{1}{2}\epsilon_{abcd}[\omega_\mu{}^{ab}, \omega_\nu{}^{cd}] - \frac{i\hbar}{\lambda^2}\tilde{B}_{\mu\nu} \\
R_{\mu\nu}{}^a &= [X_\mu, b_\nu{}^a] + [a_\mu, b_\nu{}^a] - [X_\nu, b_\mu{}^a] - [a_\nu, b_\mu{}^a] + i\{b_{\mu b}, \omega_\mu{}^{ab}\} - i\{b_{\nu b}, \omega_\mu{}^{ab}\} \\
&\quad + i\{\tilde{a}_\mu, e_\nu{}^a\} - i\{\tilde{a}_\nu, e_\mu{}^a\} + \epsilon_{abcd}([e_\mu{}^b, \omega_\nu{}^{cd}] - [e_\nu{}^b, \omega_\mu{}^{cd}]) - \frac{i\hbar}{\lambda^2} B_{\mu\nu}{}^a \\
\tilde{R}_{\mu\nu}{}^a &= [X_\mu, e_\nu{}^a] + [a_\mu, e_\nu{}^a] - [X_\nu, e_\mu{}^a] - [a_\nu, e_\mu{}^a] + i\{b_\mu{}^a, \tilde{a}_\nu\} - i\{b_\nu{}^a, \tilde{a}_\mu\} \\
&\quad - ([b_\mu{}^b, \omega_\nu{}^{cd}] - [b_\nu{}^b, \omega_\mu{}^{cd}])\epsilon_{abcd} - i\{\omega_\mu{}^{ab}, e_{\nu b}\} + i\{\omega_\nu{}^{ab}, e_{\mu b}\} - \frac{i\hbar}{\lambda^2}\tilde{B}_{\mu\nu}{}^a \\
R_{\mu\nu}{}^{ab} &= [X_\mu, \omega_n{}^{ab}] + [a_\mu, \omega_\nu{}^{ab}] - [X_\nu, \omega_\mu{}^{ab}] - [a_\nu, \omega_m{}^{ab}] + 2i\{b_\mu{}^a, b_\nu{}^b\} + ([b_\mu{}^c, e_\nu{}^d] \\
&\quad - [b_\nu{}^c, e_\mu{}^d])\epsilon_{abcd} + \frac{1}{2}([\tilde{a}_\mu, \omega_\nu{}^{cd}] - [\tilde{a}_\nu, \omega_\mu{}^{cd}])\epsilon_{abcd} + 2i\{\omega_\mu{}^{ac}, \omega_\nu{}^b{}_c\} \\
&\quad + 2i\{e_\mu{}^a, e_\nu{}^b\} - \frac{i\hbar}{\lambda^2} B_{\mu\nu}{}^{ab}
\end{aligned}$$

Symmetry Breaking

Introduction of auxiliary field $\Phi(X)$ charged under $U(1)$:

$$\Phi = \tilde{\phi}^a \otimes P_a + \phi^{ab} \otimes M_{ab} + \phi^a \otimes K_a + \phi \otimes \mathbb{1}_4 + \tilde{\phi} \otimes D$$

into the action:

$$\mathcal{S} = \mathrm{Trtr}_G\, \lambda \Phi(X) \mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} + \eta(\Phi(X)^2 - \lambda^{-2} \mathbb{1}_N \otimes \mathbb{1}_4)\,,$$

when the auxiliary field is gauge fixed as:

$$\Phi(X) = \tilde{\phi}(X) \otimes D|_{\tilde{\phi}=-2\lambda^{-1}} = -2\lambda^{-1} \mathbb{1}_N \otimes D$$

it induces a symmetry breaking:

$$\boxed{\mathcal{S}_{br} = \mathrm{Tr}\left(\frac{\sqrt{2}}{4} \varepsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd} - 4 R_{\mu\nu} \tilde{R}_{\rho\sigma} \right) \varepsilon^{\mu\nu\rho\sigma}}$$

Residual symmetry: $SO(1,3) \times U(1)$

The following components do not appear in the action, so we can take the constraints:

$R_{\mu\nu}^{\ a} = \frac{i}{2} \tilde{R}_{\mu\nu}^{\ a} = 0$ leading to $\tilde{a}_\mu = 0$, $b_\mu{}^a = \frac{i}{2} e_\mu{}^a$ and $B_{\mu\nu}{}^a = \frac{i}{2} \tilde{B}_{\mu\nu}{}^a$ *Chamseddine '02*

Unification of FG with Internal Interactions

- Fuzzy gravity is based on gauging $SO(2,4) \times U(1)$.
- Internal Interactions by $SO(10)$ (GUT).
- Spontaneous symmetry breaking is used to reach wanted gauge groups.

In order to have a chiral theory we need an $SO(4n+2)$ group. The smallest unification group in which we can accommodate chiral fermions is $SO(2,16)$ from which:

$$SO(2,16) \xrightarrow{SSB} SO(2,4) \times SO(12)$$

and

$$SO(12) \xrightarrow{SSB} SO(10) \times [U(1)].$$

Breakings and branching rules

- We start from $SO(2,16) \sim SO(18)$ (Euclidean signature)

$$
\begin{aligned}
SO(18) &\supset SU(4) \times SO(12) \\
\mathbf{18} &= (\mathbf{6},\mathbf{1}) + (\mathbf{1},\mathbf{12}) && \text{vector} \\
\mathbf{153} &= (\mathbf{15},\mathbf{1}) + (\mathbf{6},\mathbf{12}) + (\mathbf{1},\mathbf{66}) && \text{adjoint} \\
\mathbf{256} &= (\mathbf{4},\overline{\mathbf{32}}) + (\overline{\mathbf{4}},\mathbf{32}) && \text{spinor} \\
\mathbf{170} &= (\mathbf{1},\mathbf{1}) + (\mathbf{6},\mathbf{12}) + (\mathbf{20'},\mathbf{1}) + (\mathbf{1},\mathbf{77}) && \text{2nd rank symmetric}
\end{aligned}
$$

Giving VEV in the $\langle \mathbf{1},\mathbf{1} \rangle$ component of a scalar in **170** leads to $SU(4) \times SO(12)$.

Breakings and branching rules (Continued)

- Moving on with the $SO(12)$:

$$\begin{aligned} SO(12) &\supset SO(10) \times U(1) \\ \mathbf{66} &= (\mathbf{1})(0) + (\mathbf{10})(2) + (\mathbf{10})(-2) + (\mathbf{45})(0) \end{aligned}$$

  we break it down to $SO(10) \times U(1)$ by giving VEV to the $\langle(\mathbf{1})(0)\rangle$ of the **66** rep.

- Lastly, regarding $SU(4)$:

$$\begin{aligned} SU(4) &\supset SU(2) \times SU(2) \times U(1) \\ \mathbf{4} &= (\mathbf{2},\mathbf{1})(1) + (\mathbf{1},\mathbf{2})(-1) \\ \mathbf{15} &= (\mathbf{1},\mathbf{1})(0) + (\mathbf{2},\mathbf{2})(2) + (\mathbf{2},\mathbf{2})(-2) + (\mathbf{3},\mathbf{1})(0) + (\mathbf{1},\mathbf{3})(0), \end{aligned}$$

  we break it down to $SU(2) \times SU(2) \times U(1)$ by giving VEV to a scalar in the $\langle(\mathbf{1},\mathbf{1})\rangle$ direction of the **15** rep.

Fermions in Fuzzy Gravity and Unification with Internal Interactions

- Fermions should be chiral in the original theory to have a chance to survive in low energies and also appear in a matrix representation since FG is a matrix model

▷ Instead of using fermions in fundamental, spinor or adjoint reps of an $SU(N)$, we can use bi-fundamental reps of cross product of gauge groups.

Chatzistavrakidis, Steinacker, Zoupanos '10

Interesting example $N = 1$, $SU(N)^k$ models:

$$SU(N)_1 \times SU(N)_2 \times ... \times SU(N)_k$$

with matter content

$$(N, \overline{N}, 1, ..., 1) + (1, N, \overline{N}, ..., 1) + ... + (\overline{N}, 1, 1, ..., N)$$

with successful phenomenology, $N = 1$, $SU(3)^3$.

Ma, Mondragon, Zoupanos '04

Fermions in Fuzzy Gravity and Unification with Internal Interactions (Continued)

- ▷ In FG choosing to start with the $SU(4) \times SO(12)$ as the initial gauge theory with fermions in the $(\mathbf{4}, \overline{\mathbf{32}}) + (\overline{\mathbf{4}}, \mathbf{32})$ we satisfy the criteria to obtain chiral fermions in tensorial representation.
- ▷ The gauge $U(1)$ of FG due to the anticommutation relations, is identified with the one appearing in the $SO(12) \supset SO(10) \times U(1)$.

Fermions

We start with fermions in the $(\mathbf{4}, \overline{\mathbf{32}}) + (\bar{\mathbf{4}}, \mathbf{32})$ of the $SU(4) \times SO(12)$. Then

$$\begin{gathered} SO(12) \supset SO(10) \times U(1) \\ \mathbf{32} = (\overline{\mathbf{16}})(1) + (\mathbf{16})(-1) \end{gathered}$$

On the other hand

$$\begin{gathered} SU(4) \supset SU(2) \times SU(2) \times U(1) \\ \mathbf{4} = (\mathbf{2}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{2})(-1). \end{gathered}$$

Following the full sequence of symmetry breakings, by imposing the Weyl condition, we will be left with four families of fermions

$$4 \times \mathbf{16}_L(-1)$$

Finally, it is noted that the corresponding $U(1)$ gauge boson will in turn vanish using the recipe presented in the 4d conformal case.

Thank you for your attention!