Gauged Matrix Models

Denjoe O'Connor

School of Theoretical Physics
Dublin Institute for Advanced Studies
Dublin, Ireland

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Outline:

I will review recent progress on gauged matrix models including a sketch of the derivation of the Molien-Weyl formula from path integrals.

Outline

- Black holes in AdS spacetime.
- Small black holes have negative specific heat.
- Gauged Matrix Models
- Microcanonical View of Matrix Models
- Negative specific heat natural in matrix models at low energy.
  Based on DOC, S. Ramgoolam, arXiv:2312.12397, arXiv:2312.12398, arXiv:2405.13150, arXiv:2506.18813

Black holes in AdS

Schwarzschild-AdS metric

$$ds^2 = -(1 - \frac{2GM}{r} + \frac{r^2}{L^2})dt^2 + \frac{dr^2}{(1 - \frac{2GM}{r} + \frac{r^2}{L^2})} + r^2 d^2\Omega$$

$G = \frac{1}{m_p^2}$ and $L = \sqrt{-\frac{3}{\Lambda}}$

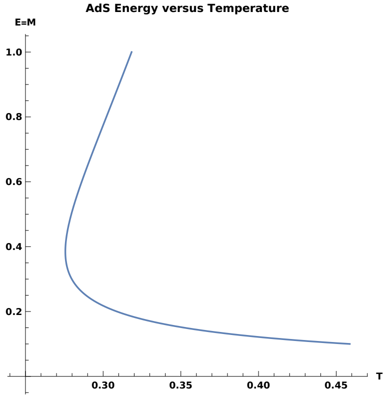
Thermodynamics—horizon, energy and temperature

$$\frac{r_h}{L} - \frac{2GM}{L} + \frac{r_h^3}{L^3} = 0$$

$$\beta_{AdS} = 4\pi L \frac{\frac{r_h}{L}}{1 + 3\frac{r_h^2}{L^2}} \qquad \text{with} \quad M = \frac{1}{2G} r_h (1 + \frac{r_h^2}{L^2})$$

r_h is a monotonic increasing function of M.
β_{AdS} has a maximum at $r_h = \frac{L}{\sqrt{3}}$ at $M = \frac{2L}{3\sqrt{3}}$.

Energy vs Temperature in AdS



Large black holes:

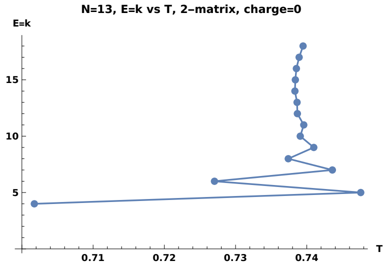
$r_h \gg L$ $r_h \sim (2GML^2)^{1/3}$

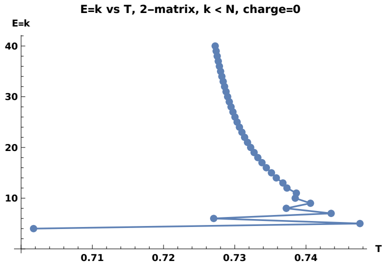
As M (the internal energy E) increases, temperature, T, increases—a normal system—the specific heat $C_v = \frac{dE}{dT} > 0$.

Small Black Holes:

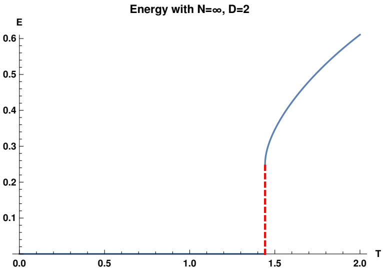
$r_h \ll L \implies r_h \sim 2GM$— asymptotically flat Schwarzschild. As M increase, temperature, T, decreases $\implies$ the specific heat $C_v < 0$.

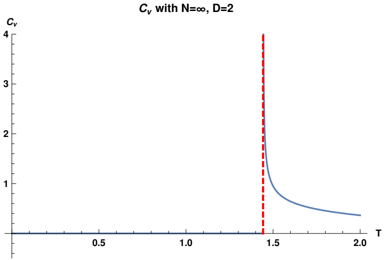
Result of Charge zero 2-Matrix Model





Canonical Ensemble





Gauged Matrix Models

Dimensional Reduction of Yang-Mills

Compactifying $SU(N)$ Yang-Mills from $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{T}^3$:

$$S_{YM} = \frac{1}{4g^2} \int dtd^3x \mathbf{tr} F_{\mu\nu} F^{\mu\nu} \xrightarrow[V_{\mathbb{T}^3} \to 0]{} \frac{V_{\mathbb{T}^3}}{4g^2} \int \mathbf{tr} F_{\mu\nu}(t) F^{\mu\nu}(t)$$

The spatial gauge fields become $N \times N$ matrices $A_a \to X_a$ and only $A_0 = A$ remains as a gauge field.

Reduced Hamiltonian

Lagrangian

$$L = \int_{\mathbb{T}^3} d^3x \frac{1}{2}\mathbf{tr}(\vec{E}^2 - \vec{B}^2) = \frac{V_{\mathbb{T}^3}}{4g^2}\mathbf{tr}(\frac{1}{2}[D_t, X_a]^2 + \frac{1}{4}[X_a, X_b][X^a, X^b])$$

Hamiltonian

$$H = \int_{\mathbb{T}^3} d^3x \frac{1}{2}\mathbf{tr}(\vec{E}^2 + \vec{B}^2) = \frac{V_{\mathbb{T}^3}}{4g^2}\mathbf{tr}(\frac{1}{2}[D_t, X_a]^2 - \frac{1}{4}[X_a, X_b][X^a, X^b])$$

This is now a quantum mechanical system of matrices. The gauge invariance is
$X_a \to gX_ag^{-1}$, $A \to gAg^{-1} + ig\partial_t g^{-1}$.

Quantization in a Thermal Bath

- The gauge field, A, is non-dynamical—**the Lagrangian has no $\partial_t A$ dependence**.
- A is a Lagrange multiplier for a constraint—the Gauss law constraint.
- The constraint requires that the only physical degrees of freedom are gauge invariant observables.

Canonical Quantization

$$Z = \mathbf{Tr}_{\mathbf{Inv}}(\mathrm{e}^{-\beta \mathbf{H}})$$

The physical degrees of freedom are the invariants of the matrices X_a and $\Pi^a = E^a$, Note $[X_a, X_b] \neq 0$.

Path Integral Quantization

Since this is a quantum mechanical system we can follow the usual Feynman route to a path integral treatment and perform a Wick rotation to Euclidean (imaginary) time.

Path Integral Quantization in a Thermal Bath

$$Z = \int [dX][dA] \mathrm{e}^{-N \int_0^\beta d\tau \, \mathrm{Tr}(\frac{1}{2}(D_\tau X^a)^2 - \frac{1}{4}[X^a, X^b]^2)}$$

One can the evaluate observables with the path integral by standard techniques.

Hamiltonian Quantization

The residual gauge field A is not dynamical and appears only in

$$D_\tau X^a = \partial_\tau X^a - i[A, X^a].$$

It leads to a constraint on the dynamics.

Gauss law constraint

The Lagrange multiplier field, A, multiplies the Gauss law constraint and forces $SU(N)$ invariant physical states.

From the action we can obtain the Hamiltonian and once we have the Hamiltonian H we can equally consider thermal ensembles whose partition function is given by

$$Z = \mathbf{Tr}_{\mathbf{Inv}}(\mathrm{e}^{-\beta H}) = \sum_E \boldsymbol{\Omega}(\mathbf{E})\mathrm{e}^{-\beta E}.$$

Inv means $SU(N)$ singlets and $\boldsymbol{\Omega}(\mathbf{E})$ the energy degeneracy.

Simple Model:The gauged Gaussian 2-Matrix Model

$$S[X] = \int_0^\beta d\tau \mathbf{tr}\left(|\mathcal{D}_\tau X|^2 + m^2|X|^2\right)$$

$\mathcal{D}_\tau = \partial_\tau - i[A, \cdot] - iA_e\mathbb{1}$ is the covariant derivative with the gauge field $A(\tau)$ being an $N \times N$ hermitian matrix and A_e an abelian gauge field.

Lattice Version

$$\mathcal{D}_\tau X \to \frac{\mathrm{e}^{iaA_e(n,n+1)} g_{n,n+1} X_{n+1} g_{n+1,n} - X_n}{a} = \frac{\mathrm{e}^{a\mathcal{D}_\tau} - 1}{a^2} X_n$$

$g_{n,n+1}$ are the link parallel transporters from $n+1$ back to that at n and $g_{n+1,n} = g^{-1}_{n,n+1}$ and $\mathrm{e}^{iaA_e(n,n+1)}$ transports the Abelian phase.

Continuum limit

The partition function is then

$$Z_{N,\Lambda} = \int_{U(1)xU(N)} \int_{\mathbb{R}^{N^2\Lambda}} \mathrm{e}^{-S_{\Lambda,g}}$$

where

$$S_{\Lambda,g} = \sum_{n,n'=1}^{\Lambda} a\mathbf{tr}(X^\dagger_{n'}(\Delta_{\Lambda,g} + m^2)_{n',n} X_n)\,.$$

$$Z_{N,\Lambda} = \int \mu(g)\mathbf{Det}^{-1} M_{\Lambda,g} \quad \text{where} \quad M_{\Lambda,g} := 2+\mu^2 - e^{a\mathcal{D}_\tau} - e^{-a\mathcal{D}_\tau}$$

$$\mathrm{Det}\, M_{\Lambda,g} = z_+^\Lambda + z_-^\Lambda - g\otimes g^{-1} - g^{-1}\otimes g = z_+^\Lambda(\mathbf{1} - z_-^\Lambda g\otimes g^{-1})(\mathbf{1} - z_-^\Lambda g^{-1}\otimes g)\,.$$

A continuum limit, $\Lambda \to \infty$, then leads to the Molien Weyl formula.

$$Z_{N,\infty} = \mathrm{e}^{-N^2\beta m} \int \frac{d\theta}{2\pi} \int \mu(g) \frac{1}{|\det(1 - \mathrm{e}^{i\theta} x g \otimes g^{-1})|^2}$$

Generalizations

Discrete Gauge Group

$$Z_N(x) = \mathrm{e}^{-N^2\beta m}\hat{Z}_N(x) \quad \text{with} \quad \hat{Z}_N(x) = \sum_{g\in G} \frac{1}{|\det(1 - xR(g)|^2}$$

Fermions

$$Z_N^F(x) = \mathrm{e}^{N^2\beta m}\hat{Z}_N^F \quad \text{with} \quad \hat{Z}_N^F(x) = \sum_{g\in G} |\det(1 + xR(g)|^2$$

Expanding $\hat{Z}_N(x)$ or $Z_N^F(x)$ in x gives integer coefficients— the $\Omega_N(n)$ we wish to calculate. Typically for $n \leq N$ these dimensions are independent of N.

Molien Weyl

Molien Weyl: Two matrices with $U(1)$ singlet constraint

$$Z_{U(N)}(t_1, \cdots, t_d) = \frac{1}{N!}\int \frac{d\theta}{2\pi}\int \prod_{l=1}^{N} \frac{dz_l}{2\pi i z_l}\Delta(z)\Delta(\frac{1}{z})\prod_{l,m=1}^{N}\frac{1}{1-2\cos(\theta)xz_l z_m^{-1}}$$

with $\Delta(z)$ the Vandermonde determinant. For small N and small d the integrals can be performed exactly and some results are known.

A large N analysis

Expanding in the exponential the partition function becomes

$$Z(x) = \int \mu(g) \exp[\sum_{n=1}^{\infty} \frac{a_n}{n} \mathbf{tr}(g^n) \mathbf{tr}(g^{-n})]$$

Keeping only the $n = 1$ term gives the a_1 model

The a_1 model.

$$Z(a_1) = \int \mu(g) \mathrm{e}^{a_1 \mathbf{tr}(g) \mathbf{tr}(g^{-1})}$$

The Hagedorn (confining/deconfining) Phase Transition.

High Temperature (small β)

$$S[X,A] = \frac{1}{2}\int_0^\beta d\tau \,\mathrm{Tr}\left\{(D_\tau X)^2 + X^2\right\} \qquad D_\tau = \partial_\tau + i[A,\cdot]$$

for β small becomes the random matrix model

$$S[X,A] \simeq \frac{\beta}{2}\,\mathrm{Tr}\left\{-[A,X]^2 + X^2\right\}$$

The eigenvalues of βA, the θ_i, are distributed roughly with a Wigner semi-circle distribution.

For $\beta \to 0$

$$Z_N(t,d) \sim \beta^{(d-1)N^2} = \mathrm{e}^{(d-1)N^2 \ln(-\ln t)}$$

$$\dim_n(N,d) \sim \mathrm{e}^{N^2(d-1)\ln n}$$

The transition Point

From

$$S_{GG}(\theta, d) \simeq N^2 \sum_{n=1}^{\infty} \frac{(1-a_n)}{n} |u_n|^2 \, ,$$

we see that the transition occurs at $a_1 = 1$ where the coefficient of $|u_1|^2$ changes sign. For $a_1 = \sum_{i=1}^{d} x_i = d\mathrm{e}^{-\beta}$ the transition occurs at $T_H = \frac{1}{\beta_H} = \frac{1}{\ln d}$.

If we integrate over u_n (Aharoney et al arXiv:hep-th/0310285) and set $Z_\infty = 1$ for $a_n = 0$, we obtain

$$Z_\infty = \prod_{n=1}^{\infty} \frac{1}{1-a_n} = \prod_{n=1}^{\infty} \frac{1}{1-\sum_{i=1}^{d} x_i^n}$$

F. Dolan arXiv:0704.1038 obtained this for $d = 2$ by exact methods. Though the result is exact for $d = 1$ it breaks down for $a_1 \to 1$, but still allows us to count low energy states count states at large N.

There is a phase trasition when the eigenvalue distribution covers the unit circle.

Gauge/gravity duality $\implies$ the transition should be dual to a Hawking-Page transition in a dual AdS spacetime.

Stable regime–low energy

Small n generating function

For two Hermitian matrices gauged under $U(N)$ and different masses one gets

$$\hat{Z}_\infty(x) = \prod_{n=1}^{\infty} \frac{1}{1 - x^n - y^n}$$

Charge neutral is equivalent to $x \to zx$ $y \to z^{-1}x$ and a contour integral over z.

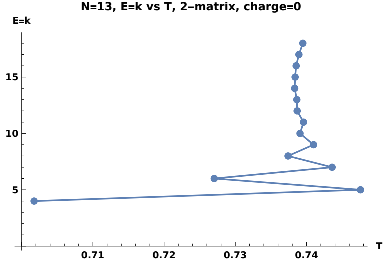
$$Z^0_\infty(x) = \oint \frac{dz}{2\pi i z} \prod_{n=1}^{\infty} \frac{1}{1-(z^n+z^{-n})x^n} = \int \frac{d\theta}{2\pi} \prod_{n=1}^{\infty} \frac{1}{1-2\cos(n\theta)x^n}$$

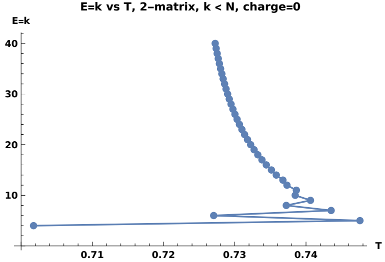
This expression is well approximated by the first term in the product and on doing the integral one finds

$$Z^0_\infty(x) \simeq \frac{1}{\phi(\frac{1}{2})} \frac{1}{\sqrt{1-4x^2}}$$

with $\phi(x) = (x,x)_\infty$ the Euler function and $(a,q)_\infty$ is the q-Pochhammer symbol.

Result of Charge zero 2-Matrix Model





Thanks for Your Attention!