Generalised Cartan Geometry

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EISA
European Institute for Sciences and Their Applications

Motivation: Generalised Geometry & Curvatures

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  - generalisation of Riemannian geometry geometrising some duality group $\mathcal{G}$ (for example from string/M-theory)
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$\mathcal{G} = E_{d(d)} \quad R_1 M = (T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \dots) M$

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  - generalised Lie bracket, generalised metric, generalised connection, generalised curvature/torsion [Coimbra/Strickland-Constable/Waldram, Hohm/Zwiebach, Cederwall/Edlund/Karlsson, Garcia-Fernandez, ...]

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- systematic construction
- non-uniqueness problem of generalised Levi-Civita connection

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CHALLENGES: – systematic construction → CARTAN GEO? [Poláček/Siegel '13]
– non-uniqueness problem of generalised Levi-Civita connection

How to generalise Cartan geometry?

Standard Cartan geometry (rephrased in language that admits gen.)

IDEA: model tangent space as homogeneous space

- Cartan connection: fibrewise isomorphism: $\theta_p : T_pP \to \mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{g}/\mathfrak{h})$,

  s.t. $\theta_p|_{T_pH} : T_pH \to \mathfrak{h} = \mathrm{id}$, 'H-equivariance'

  $\alpha, \ldots, \mu, \ldots$ indices on $\mathfrak{h}, T_pH$

  $a, \ldots, m, \ldots$ indices on $\mathfrak{g}/\mathfrak{h}, T_pM$

$$\Rightarrow \theta_M{}^A = \begin{pmatrix} \delta_\mu{}^\alpha & 0 \\ \omega_m{}^\alpha & e_m{}^a \end{pmatrix}$$

spin connection

frame/vielbein

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- ingredients: – model algebra $\mathfrak{g}$ / model space G/H

  – principle H-bundle
  $P \to M$

examples:
– flat model space: $\frac{ISO(d)}{SO(d)}$
– sphere $\frac{SO(d+1)}{SO(d)}$
– ...

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- Cartan curvature:

  $$\boxed{\Theta = d\theta|_{T_pM} + \tfrac{1}{2}[\theta|_{T_pM}, \theta|_{T_pM}] \in \mathfrak{g}}$$

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How to generalise?
$\to$ gen. of TP
$\to$ gen. for $[\,,\,]$ on T_1P
$\to$ gen. for model algebra $\mathfrak{g}$

Overview

0. Motivation / How to generalise Cartan geometry?
1. basics of generalised geometry
2. extending generalised geometry by a gauge symmetry
3. generalised Cartan geometry (gen model algebra, gen. Cartan connection, gen. Cartan curvature)

① Basics of generalised geometry

duality group G with hierarchy of representations:

$$\mathcal{T} = R_0 \oplus R_1 \oplus R_2 \oplus R_3 \oplus \dots$$ (graded vector space)

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[Cederwall/Palmkvist; Lavau/Palmkvist; Bonezzi/Hohm]

compatibility conditions for $\bullet, \partial$,

here: η, D-symbols!

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notation: $V \in R_p$: V^{M_p}

section condition:

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why this algebraic structure? $\quad (\bullet, \partial)$

$\hookrightarrow$ in order to define generalised Lie bracket of $V \in R_1$, $W \in R_p$:

$$\mathcal{L}_V W = V \bullet \partial W + \partial(V \bullet W)$$

$\longrightarrow$ $\mathcal{G}$-covariance

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in addition to duality group $\mathcal{G}$?

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  $v \in T_p P$, $v = (v_{\mathfrak{h}}, v_{TM})$, $v_{\mathfrak{h}} \in \mathfrak{h} = T_e H$, $v_{TM} \in T_p M$
- algebra of symmetries:
  Lie bracket on TP

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$H \subset \mathcal{G}$ max. compact.

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$$\Rightarrow \quad V \in \mathfrak{R}_1 P := \mathfrak{h} \oplus R_1 M \oplus (R_2 M \otimes \mathfrak{h}^*) \oplus (R_3 M \otimes \Lambda^2 \mathfrak{h}^*) \oplus \dots$$

$$V = (V^\alpha, V_{M_1}, V_{M_2}^{\alpha}, V_{M_3}^{\alpha_1 \alpha_2}, \dots)$$

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extension necessary, s.t. one can define a 'generalised' Lie derivative on $\mathcal{R}_1 P$

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$\mathcal{V} \in \mathfrak{R}_1 P := \mathfrak{h} \oplus R_1 M \oplus (R_2 M \otimes \mathfrak{h}^*) \oplus (R_3 M \otimes \Lambda^2 \mathfrak{h}^*) \oplus \ldots$

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- closure for $\mathcal{V}^{(i)} = (v^{\alpha (i)}, V_1^{(i)}, 0, 0, \ldots)$ & $\mathcal{W} \in \mathfrak{R}_1 P$

$$[\mathcal{L}_{\mathcal{V}^{(1)}}, \mathcal{L}_{\mathcal{V}^{(2)}}]\mathcal{W} = \mathcal{L}_{\frac{1}{2}(\mathcal{L}_{\mathcal{V}^{(1)}}\mathcal{V}^{(2)} - \mathcal{L}_{\mathcal{V}^{(2)}}\mathcal{V}^{(1)})}\mathcal{W}$$

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(- realisation on phase space of p-branes)

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for consistency → hierarchy of „gauged" representations

$$\mathcal{R}_2 P := R_2 M \oplus (R_3 M \otimes \mathfrak{h}^*) \oplus (R_4 M \otimes \Lambda^2 \mathfrak{h}^*) \oplus \ldots$$

$\vdots$

same structure as for gen. geometry:

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$$\mathcal{T} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3 \oplus \ldots$$

graded vector space with $\bullet$, ∂,

inherited from G- gen. geometry and H-gauge symmetry / comp.

③ generalised Cartan geometry

reminder:

standard Cartan geometry:

- model Lie algebra $\mathfrak{g}$
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- $\Theta = d\theta + \frac{1}{2}[\theta, \theta]$

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model algebra ℓ_1: Leibniz algebra

$\ell_1 = \mathrm{span}(t_{\alpha_1}, t_{A_1}, t^{\alpha}_{A_2}, \ldots)$

generators of $\mathfrak{h}$

③ generalised Cartan geometry

reminder:

standard Cartan geometry:

- model Lie algebra $\mathfrak{g}$
- $\theta_p : T_pP \longrightarrow \mathfrak{g}$
- $\Theta = d\theta + \frac{1}{2}[\theta, \theta]$

generalisation:

model algebra ℓ_1: Leibniz algebra

$\ell_1 = \mathrm{span}(t_\alpha, t_{A_1}, t^\alpha_{A_2}, \ldots)$

t_α: generators of $\mathfrak{h}$

example: 'gen Poincaré'

$[t_\alpha, t_{A_1}] = f_{\alpha A_1}{}^{B_1} t_{B_1}$

$[t_{A_1}, t_{B_1}] = 0$

others: non-vanishing

contains Poincaré as subalgebra

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others: non-vanishing

contains Poincaré as subalgebra

for consistency: $\ell = \ell_0 \oplus \ell_1 \oplus \ell_2 \oplus \dots$
dgLa?

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generalised Cartan connection

$\theta_p : \tilde{T}_p P \longrightarrow \mathfrak{l} \; (= \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \dots)$

$= \mathcal{R}_0 P \oplus \mathcal{R}_1 P \oplus \dots$

model dgLa

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generalised Cartan connection

$$\theta_p : \mathcal{T}_pP \longrightarrow \mathfrak{l} \; (= \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \dots)$$

$= T_0P \oplus K_1P \oplus \dots$ model dgLa

vector space isomorphism of dgLa's, G-invariance, (leaving $\circ$ inv)

3 generalised Cartan geometry

reminder:

standard Cartan geometry:

- model Lie algebra $\mathfrak{g}$
- $\theta_p : T_pP \longrightarrow \mathfrak{g}$
- $\Theta = d\theta + \frac{1}{2}[\theta,\theta]$

generalised Cartan connection

$$\boxed{\begin{array}{ccc} \theta_p : \mathcal{T}_p P & \longrightarrow & \mathfrak{l} \; (= \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \dots) \\ = \mathcal{R}_0 P \oplus \mathcal{R}_1 P \oplus \dots & & \text{model dgLa} \end{array}}$$

vector space isomorphism, of dgLa's

$\mathcal{G}$-invariance, (leaving $\circ$ inv)

luckily: sufficient to specify:

$$\boxed{\theta_p : \mathcal{R}_1 P \longrightarrow \mathfrak{l}_1}$$

3 generalised Cartan geometry

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standard Cartan geometry:

- model Lie algebra $\mathfrak{g}$
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generalised Cartan connection

$$\boxed{\theta_p : \underbrace{\mathcal{T}_pP}_{= \mathcal{T}_0P \oplus \mathcal{T}_1P \oplus \dots} \longrightarrow \underbrace{\ell \, (= \ell_0 \oplus \ell_1 \oplus \dots)}_{\text{model dgLa}}}$$

vector space isomorphism, of dgLa's

G-invariance, (leaving $\circ$ inv)

luckily: sufficient to specify:

$$\boxed{\theta_p : \mathcal{T}_1P \longrightarrow \ell_1}$$

(otherwise, similar assumptions as in standard Cartan geometry)

3 generalised Cartan geometry

generalised Cartan connection

$$\boxed{\theta_P : \mathcal{R}_1 P \longrightarrow \mathfrak{e}_1}$$

$$V^M = \left(v^\mu, V^{M_1}, V_\mu^{M_2}, V_{\mu_1\mu_2}^{M_3}\right) \longmapsto X^{A} = \left(X^\alpha, X^{A_1} \ldots\right) \in \mathfrak{e}_1$$

$$\in \mathfrak{h} \oplus \mathcal{R}_{1P}M \oplus (\mathcal{R}_{2P}M \otimes \mathfrak{h}^*) \oplus \ldots$$

③ generalised Cartan geometry

generalised Cartan connection

$$\boxed{\theta_\rho : \mathcal{R}_1 \rho \longrightarrow \mathfrak{e}_1}$$

$$V^M = (v^\mu, V^{M_1}, V^{M_2}{}_\mu, V^{M_3}{}_{\mu_1\mu_2}) \longmapsto X^{\mathcal{A}} = (X^\alpha, X^{A_1} \dots) \in \mathfrak{e}_1$$

$$\in \mathfrak{h} \oplus \mathcal{R}_{1\rho} M \oplus (\mathcal{R}_{2\rho} M \otimes \mathfrak{h}^*) \oplus \dots$$

from standard assumptions (H-equivariance, $\theta_\rho|_{\mathfrak{h}} = id, \dots$) + $\mathcal{G}$-invariance

$$\theta_M{}^{\mathcal{A}} = \begin{pmatrix} \delta_\mu^\alpha & 0 & 0 & 0 & \cdots \\ \Omega_{M_1}{}^\alpha & E_{M_1}{}^{A_1} & 0 & 0 & \cdots \\ S^{\mu\alpha}{}_{M_2} & & \delta^\mu_\alpha E_{M_2}{}^{A_2} & 0 & \cdots \\ S^{\mu_1\mu_2\alpha}{}_{M_3} & & & \ddots & \\ \vdots & & & & \ddots \end{pmatrix}$$

fixed by 1st column

generalisation of [Poláček, Siegel '13] (O(d,d))

③ generalised Cartan geometry

generalised Cartan connection

$$\boxed{\theta_p : R_1 P \longrightarrow e_1}$$

$V^M = (v^\mu, v^{M_1}, v^{M_2}_{\mu}, v^{M_3}_{\mu_1\mu_2}) \longmapsto X^{\mathcal{A}} = (X^\alpha, X^{A_1} \ldots) \in e_1$

$\in h \oplus R_{1p}M \oplus (R_{2p}M \otimes h^*) \oplus \ldots$

from standard assumptions (H-equivariance, $\theta_p|_h = id, \ldots$) + G-invariance

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fixed by 1st column

$\longmapsto$ independent comp.:

- gen. frame $E_{M_1}{}^{A_1}$
- gen. spin connection $\Omega_{M_1}{}^\alpha$
- higher auxiliary connections: $\varsigma^{\alpha_1\ldots\alpha_p}_{M_p}$

generalisation of [Poláček, Siegel '13] (O(d,d))

③ generalised Cartan geometry

generalised Cartan curvature

consider: $\Theta_M{}^{\mathcal{A}}(p)$ as $\boxed{\Theta_M \in \mathcal{R}_{1,p}P}$

then define: using generalised Lie derivative on $\mathcal{R}_1 P$

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consider: $\theta_\mu{}^{\mathcal{A}}(p)$ as $\boxed{\theta_\mu \in \mathfrak{R}_{1,p} P}$

then define: using generalised Lie derivative on $\mathfrak{R}_1 P$

$$\boxed{\Theta_{\mu\nu} = \mathcal{L}_{\theta_\mu} \theta_\nu \in \mathfrak{R}_1 P}$$

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$V_M = (v_\mu, v_{M_1}, \ldots)$
decomposition

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decomposition

- $\Theta_{\mu\nu}{}^K$, $\Theta_{\mu M_p}{}^{N_p}$, $\Theta_\mu \dots$ model algebra
- $\Theta_{M_1 N_p}{}^{K_p}$ generalised torsion
- hierarchy of curvatures: $\Theta^{\nu_1 \dots \nu_p}_{M_1 N_p}$:

schematically, linearised

$\Theta^\alpha_{M_1 N_1} \sim d\Omega + \hat{\partial} \rho_2$

$\Theta_{M_1 N_p} \sim d\rho_p + \hat{\partial} \rho_{p+1}$

③ generalised Cartan geometry

generalised Cartan curvature

consider : $\theta_\mu{}^{\mathcal{A}(p)}$ as $\boxed{\theta_\mu \in R_1 P}$

then define ; using generalised Lie derivative on $R_1 P$

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unifies, & extends known results in gen. geo.

Summary & outlook

- generalisation of Cartan geometry

summary & outlook

- generalisation of Cartan geometry

$$TP \longrightarrow \mathcal{R}_1 P$$

model algebra $\longrightarrow$ Leibniz algebra

conn. $\quad e_m{}^a, \omega_m{}^\alpha \longrightarrow E_{M_1}{}^{A_1}, \rho_{M_1}{}^\alpha, \rho_{M_2}{}^{\alpha\beta}, \ldots$

torsion / curvature linearised

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torsion / curvature linearised

- what next?

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torsion / curvature linearised

- what next?
  - metric-comp. connection $\longrightarrow$ Maxwell-∞
  - exotic brane (gauged generalised geometry is the natural language)

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conn. $e_m{}^a, \omega_m{}^\alpha \longrightarrow E_{M_1}{}^{A_1}, \ell_{M_1}{}^\alpha, \beta_{M_2}{}^{\alpha\beta}, \ldots$

torsion / curvature linearised

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  - metric-comp. connection $\longrightarrow$ Maxwell-∞
  - exotic brane (gauged generalised geometry is the natural language)

Thank you for your attention!