Push-forward of Hopf-Galois extensions: the non central case

Chiara Pagani
University of Naples, Federico II, Italy

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Quantum principal bundle:

- a Hopf algebra H (structure group);
- an H-comodule algebra A (total space) with coaction $\delta : A \to A \otimes H$, $a \to a_{(0)} \otimes a_{(1)}$
- $B := A^{coH} = \{b \in A \,|\, \delta(b) = b \otimes 1_H\}$ the subalgebra of coinvariants (base space);
- A an H–Galois algebra extension of B: the canonical map $\chi : A \otimes_B A \to A \otimes H$, $a \otimes_B a' = a\delta(a')$ is bijective.

In classical geometry, given a principal G–bundle $P \to Y$ and a map $\mathsf{F} : X \to Y$ then the induced bundle (pull-back) $\mathsf{F}_*(P)$

$$\begin{array}{ccc} \mathsf{F}^*(P) := \{(x,p) \in X \otimes P \,|\, \mathsf{F}(x) = \pi(p)\} & & P \\ \downarrow{\scriptstyle \mathrm{id}\otimes\pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\quad \mathsf{F} \quad} & Y \end{array}$$

is a principal G–bundle. The fiber over a point $x \in X$ is $\pi^{-1}(\mathsf{F}(x))$.

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Push-forwards of central Hopf–Galois extensions along morphisms of commutative algebras

D. Rumynin, *Hopf-Galois extensions with central invariants and their geometric properties*, Algebra Repr. Th. (1998)
C. Kassel, *Quantum principal bundles up to homotopy equivalence*, The legacy of Niels Henrik Abel, Springer (2004)

For any morphism of algebras $\mathsf{F} : B \to C$ and any left B-module algebra A, the push-forward F_*A of A is the covariant F-extension of A:

$$C \otimes_B A = C \otimes A / \langle c \otimes ba - c\,\mathsf{F}(b) \otimes a\ ,\ a \in A, b \in B, c \in C \rangle$$

Problem: Defining an algebra structure on $C \otimes_B A$.

The most natural way to equip F_*A with a multiplication is to define it to be $m_{C\otimes A}$

$\to$ For $m_{C\otimes A}$ to be well-defined, the algebra B has to be central in A, and thus in particular commutative, and $\mathsf{F}(B)$ central in C.

Theorem [Kassel]. The push-forward algebra $C \otimes_B A$ of a (faithfully flat) central H–Galois extension $B \subset A$ is a (faithfully flat) central H–Galois extension of C.

Pull-back of principal bundles on noncommutative spaces

Joint work in progress with Giovanni Landi (Trieste)

Aim of the research project: To go beyond the central case and study the pull-back of principal bundles on not necessarily commutative algebras

Idea: To approach the problem by using Twisted tensor products of algebras

- Let A and C be two algebras and $\psi : A \otimes C \to C \otimes A$, $a \otimes c \mapsto c^{[\psi]} \otimes a^{[\psi]}$ be a linear map,

$$m^{\psi} := (m_C \otimes m_A)(\mathrm{id}_C \otimes \psi \otimes \mathrm{id}_A) : (C \otimes A) \otimes (C \otimes A) \to C \otimes A$$

$$(c \otimes a) \otimes (c' \otimes a') \mapsto c\psi(a \otimes c')a' = cc'^{[\psi]} \otimes a^{[\psi]}a'$$

defines an associative product on the vector space $C \otimes A$ if and only if

$$(\mathrm{id}_C \otimes m_A) \circ (\psi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes m_C \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \mathrm{id}_C \otimes \psi) =$$
$$(m_C \otimes \mathrm{id}_A) \circ (\mathrm{id}_C \otimes \psi) \circ (\mathrm{id}_C \otimes m_A \otimes \mathrm{id}_C) \circ (\psi \otimes \mathrm{id}_A \otimes \mathrm{id}_C).$$

Examples

- A and C two algebras and $\psi = \mathit{flip} : A \otimes C \to C \otimes A,\, a \otimes c \mapsto c \otimes a$ the flip map. Then $C \otimes_\psi A$ is the ordinary tensor product of algebras;
- A a left K-comodule algebra and C a left K-module algebra for K a Hopf algebra,

$$\psi : A \otimes C \to C \otimes A, \quad a \otimes c \mapsto a_{(-1)} \triangleright c \otimes a_{(0)}.$$

  Then $C \otimes_\psi A$ is Takeuchi's smash product (1980).

Dually, one can define the twisted tensor product of two coalgebras, with twisted coproduct and also the twisted tensor product of bialgebras.

- Let H and A be two Hopf algebras such that A is a right H-module algebra and H is a left A-comodule coalgebra. Then

$$\psi : A \otimes H \to H \otimes A, \quad a \otimes h \mapsto h_{(1)} \otimes a \triangleleft h_{(2)} \qquad (\rightsquigarrow \text{ twisted } m_\psi)$$

$$\Phi : H \otimes A \to A \otimes H, \quad h \otimes a \mapsto h_{(-1)} a \otimes h_{(0)} \qquad (\rightsquigarrow \text{ twisted } \Delta_\Phi)$$

  Then $H \otimes_\psi^\Phi A$ is Majid's bicrossproduct (1990).

Twisted tensor products of algebras $C \otimes^{\psi} A := (C \otimes A, m^{\psi})$

S. Caenepeel, B. Ion , G. Militaru, S. Zhu, *The factorization problem and the smash biproduct of algebras and coalgebras.* Algebr. Represent. Theory 3 (2000).

- A sufficient condition for the associativity of m^{ψ}:

$$\psi(m_A \otimes \mathrm{id}_C) = (\mathrm{id}_C \otimes m_A) \circ (\psi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \psi), \qquad \psi(aa' \otimes c) = \psi(a \otimes c^{[\psi]})a'^{[\psi]} \quad (1)$$

$$\psi(\mathrm{id}_A \otimes m_C) = (m_C \otimes \mathrm{id}_A) \circ (\mathrm{id}_C \otimes \psi) \circ (\psi \otimes \mathrm{id}_C), \qquad \psi(a \otimes cc') = c^{[\psi]}\psi(a^{[\psi]} \otimes c') \quad (2)$$

- The element $1_C \otimes 1_A$ is a unit for $C \otimes^{\psi} A$ if and only if ψ is normal:

$$\psi(1_A \otimes c) = c \otimes 1_A \quad \text{(right normal)}, \qquad \psi(a \otimes 1_C) = 1_C \otimes a \quad \text{(left normal)}$$

- For ψ normal, conditions (1) and (2) are equivalent to the associativity of the product.

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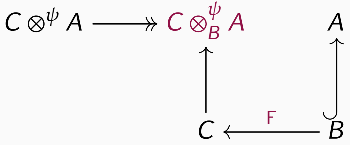

From twisted tensor product algebras to push-forward of H–Galois extensions

Consider A, C, $A \otimes C$, $C \otimes A \in {}_B\mathcal{M}_B$ with obvious B-bimodule structures.

Proposition

Let $\psi : A \otimes C \to C \otimes A$ be a normal twist map and $C \otimes^{\psi} A = (C \otimes A, m^{\psi})$ be the corresponding twisted tensor product algebra. The multiplication

$$m^{\psi} = (m_C \otimes m_A)(\mathrm{id}_C \otimes \psi \otimes \mathrm{id}_A)$$

on $C \otimes^{\psi} A$ descends to a well-defined product on the push-forward $C \otimes_B A$ if and only if ψ is a B-bimodule morphism:

$$\psi(b \triangleright (a \otimes c) \triangleleft b') = \psi(ba \otimes c\,\mathrm{F}(b')) = \mathrm{F}(b)\psi(a \otimes c)b', \qquad a \in A,\ b, b' \in B,\ c \in C.$$

$\rightsquigarrow$ the twisted push-forward algebra of A along F: $\quad C \otimes_B^{\psi} A := (C \otimes_B A, m^{\psi})$

Suppose A is an H-comodule algebra with coaction $\delta : A \to A \otimes H$ and $B = A^{coH}$.

- The vector space $C \otimes A$ is a right H-comodule with coaction $\mathrm{id}_C \otimes \delta$. Since $B = A^{coH}$, this descends to a coaction $\mathrm{id}_C \otimes_B \delta$ on $C \otimes_B A$.
- Also $A \otimes C$ is a right H-comodule with coaction $(\mathrm{id}_A \otimes \tau) \circ (\delta \otimes \mathrm{id}_C)$.

Proposition

Let $\psi : A \otimes C \to C \otimes A$ be a normal twist and a B-bimodule morphism.
Suppose ψ is an H-comodule morphism:

$$(a^{[\psi]})_{(0)} \otimes c^{[\psi]} \otimes (a^{[\psi]})_{(1)} = (a_{(0)})^{[\psi]} \otimes c^{[\psi]} \otimes a_{(1)}\,, \qquad a \in A,\, b \in B,\, c \in C$$

then the twisted push-forward algebra $C \otimes_B^{\psi} A$ is an H-comodule algebra with coaction $\mathrm{id}_C \otimes \delta$ and $C \subseteq \left(C \otimes_B^{\psi} A\right)^{coH}$.

Theorem

Let $B \subset A$ be H–Galois with A faithfully flat as a B-module. Let $F : B \to C$ be a morphism of algebras and $\psi : A \otimes C \to C \otimes A$ a normal twist map that is compatible with F, that is a B-bimodule map, and is an H-comodule morphism. Then the twisted push-forward algebra $C \otimes_B^{\psi} A$ of $B \subset A$ along $\mathsf{F} : B \to C$ is a faithfully flat H–Galois extension of C.

Proof.
The left C-module $C \otimes_B A$ is faithfully flat since A is faithfully flat as a left B-module.
Under the isomorphism $(C \otimes_B A) \otimes_C (C \otimes_B A) \simeq C \otimes_B (A \otimes_B A)$

$$\chi^{\psi} : (C \otimes_B A) \otimes_C (C \otimes_B A) \to (C \otimes_B A) \otimes H \,, \; (c \otimes_B a) \otimes_C (c' \otimes_B a') \mapsto (c \otimes_B a) \cdot^{\psi} (\mathrm{id}_C \otimes_B \delta)(c' \otimes_B a')$$

is simply $\chi^{\psi} = \mathrm{id}_C \otimes_B \chi$, where $\chi : A \otimes_B A \to A \otimes H$ is the canonical map of $B \subset A$.
Then χ^{ψ} is surjective. Since the left C-module $C \otimes_B A$ is faithfully flat and χ^{ψ} is surjective, $C = (C \otimes_B^{\psi} A)^{coH}$ [Takeuchi 1977]. Then the algebra extension $C \subset C \otimes_B^{\psi} A$ is H–Galois. □

$\rightsquigarrow$ $C \otimes_B^{\psi} A$ is the twisted push-forward H–Galois extension of $B \subset A$ along F

On $*$-structures

Lemma

Suppose A and C are $*$-algebras. The twisted tensor product algebra $C \otimes^{\psi} A$ is a $*$-algebra with $*_C \otimes *_A \iff \psi = \text{flip}$.

Proposition

The twisted tensor product algebra $C \otimes^{\psi} A$ is a $*$-algebra with

$$* := \psi(*_A \otimes *_C) \circ \text{flip}, \qquad (c \otimes a)^* := \psi(a^* \otimes c^*)$$

$\iff$ ψ is invertible with inverse $\psi^{-1} = (*_A \otimes *_C) \circ \text{flip} \circ \psi \circ (*_A \otimes *_C) \circ \text{flip}$. In this case, we can form the twisted tensor product algebra $A \otimes^{\psi^{-1}} C$. It is a $*$-algebra with $(a \otimes c)^* := \psi^{-1}(c^* \otimes a^*)$ and $\psi : A \otimes^{\psi^{-1}} C \to C \otimes^{\psi} A$ is a $*$-algebra isomorphism.

Lemma

Suppose $\mathrm{F} : B \to C$ is a $*$-algebra morphism and B a $*$-subalgebra of A. The $*$-structure $(c \otimes a)^* = \psi(a^* \otimes c^*)$ on $C \otimes^{\psi} A$ descends to a well-defined $*$-structure on $C \otimes^{\psi}_B A$

$$\iff \quad \psi(ab \otimes c) = \psi(a \otimes \mathrm{F}(b)c), \quad \forall\, a \in A,\ b \in B,\ c \in C.$$

Example: The push-forward to the total algebra along $i : B \to A$

The push-forward of a Hopf-Galois extension to the total algebra is trivial (the pull-back of a principal bundle to the total space is trivial).

Let $B \subset A$ be a faithfully flat H-Galois extension. [Schauenburg-Schneider (2005)]: there always exists a strong connection: a linear map $\ell : H \to A \otimes A,\ h \mapsto \ell(h) := h^{\{1\}} \otimes h^{\{2\}}$ with properties

$$\ell(1) = 1 \otimes 1, \quad \pi_B \circ \ell = \tau, \quad (id_A \otimes \delta) \circ \ell = (\ell \otimes \mathrm{id}_H) \circ \Delta, \quad (\delta_l \otimes id_A) \circ \ell = (\mathrm{id}_H \otimes \ell) \circ \Delta$$

where

- $\pi_B : A \otimes A \to A \otimes_B A$ the canonical projection,
- $\delta_l : A \to H \otimes A,\ a \mapsto S^{-1}(a_{(1)}) \otimes a_{(0)}$ the induced left coaction of H on A
- $\tau = \chi^{-1}|_{1 \otimes H} : H \to A \otimes_B A,\ h \mapsto h^{<1>} \otimes_B h^{<2>}$ the translation map.

Proposition

Let $B \subset A$ be a faithfully flat H-Galois extension with strong connection $\ell : H \to A \otimes A$, $h \mapsto \ell(h) := h^{\{1\}} \otimes h^{\{2\}}$. The map

$$\psi_\ell : A \otimes A \to A \otimes A\,, \quad a \otimes c \mapsto a_{(0)} c {a_{(1)}}^{\{1\}} \otimes {a_{(1)}}^{\{2\}}$$

is a normal twist map. It is left and right B-linear (being $\ell(h)b = b\,\ell(h), \forall\, b \in B$):

$$\psi_\ell(ba \otimes a'b') = b\psi_\ell(a \otimes a')b'\,, \quad \forall\, a, a' \in A,\, b, b' \in B.$$

⇝ we can construct the algebra $A \otimes^{\psi_\ell} A$ with twisted product

$$(a \otimes c) \cdot_{\psi_\ell} (a' \otimes c') := (a\psi_\ell(c \otimes a')c') = a{a'}_{(0)} c({a'}_{(1)})^{\{1\}} \otimes ({a'}_{(1)})^{\{2\}} c'$$

and this descends to a well-defined product

$$(a \otimes_B c) \cdot_{\psi_\ell} (a' \otimes_B c') = \pi_B\Big(a{a'}_{(0)} c({a'}_{(1)})^{\{1\}} \otimes ({a'}_{(1)})^{\{2\}} c'\Big) = a{a'}_{(0)} c({a'}_{(1)})^{<1>} \otimes_B ({a'}_{(1)})^{<2>} c'.$$

on the twisted push-forward $A \otimes_B^{\psi_\ell} A$ of $B \subset A$ along the algebra inclusion $i : B \to A$.

The twisted push-forward algebra $A \otimes_B^{\psi_\ell} A$ is $A \otimes_B A$ with algebra structure induced by that of the tensor product algebra $A \otimes H$ via the canonical map $\chi : A \otimes_B A \to A \otimes H$ (which becomes then an algebra map):

$$m_{\psi_\ell} = \chi^{-1} \circ m_{A \otimes H} \circ (\chi \otimes \chi).$$

Proposition

The twisted push-forward algebra $A \otimes_B^{\psi_\ell} A$ of $B \subset A$ along the algebra inclusion $i : B \to A$ is a trivial H-Galois extension.

Observation: The twist map ψ_ℓ is the lift to $A \otimes A$ of the Durdević braiding

$$\sigma : A \otimes_B A \to A \otimes_B A\,, \quad a \otimes_B c \mapsto a_{(0)} c (a_{(1)})^{<1>} \otimes_B (a_{(1)})^{<2>}.$$

Final remarks

- An alternative construction consists in constructing

$$m^{\psi} = (m_C \otimes m_A)(\mathrm{id}_C \otimes \psi \otimes \mathrm{id}_A)$$

directly on the covariant F-extension $C \otimes_B A$, without assuming it comes from a twisted product on $C \otimes A$. Here

$$\psi : A \otimes C \to C \otimes_B A\,, \qquad a \otimes c \mapsto c^{[\psi]} \otimes_B a^{[\psi]}$$

cf. [Durdević, Quantum classifying spaces and universal quantum characteristic classes, Banach Publ. 1997]

- For each n, the Hopf fibering $S^{2n+1} \to \mathbb{C}P^n$ pull-backs via the classifying map

$$\mathsf{F}_n : \mathbb{C}P^1 \simeq S^3/\mathrm{U}(1) \to \mathbb{C}P^n \simeq S^{2n+1}/\mathrm{U}(1), \quad [z_0, z_1] \mapsto \left[z_0^n, \dots, \binom{n}{j}^{\frac{1}{2}} z_0^{n-j} z_1^j, \dots, z_1^n \right].$$

to a $\mathrm{U}(1)$-bundle on $\mathbb{C}P^1$ whose total space is the Lens space $L(n,1)$.

$\rightsquigarrow$ its quantum version as an example of a twisted push-forward algebra.