The L_∞ structure of Free Differential Algebras and d=11 Supergravity

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1. Introduction and motivations

- Supergravity theories can be naturally formulated in a super-Lie group-geometric framework.

- Tangent vectors t_A on the G group manifold , defined by the infinitesimal action of G on itself, satisy the Lie commutation relations
$$[t_A, t_B] = C^C{}_{AB} t_C$$

- The dual cotangent basis of one-forms σ^A (with $\sigma^A(t_B) = \delta^A_B$) satisfies the Cartan-Maurer eq.s.
$$d\sigma^A + \frac{1}{2} C^A{}_{BC}\, \sigma^B \wedge \sigma^C = 0$$

- The basic fields correspond to the σ^A . For *p*-form fields: generalize Cartan-Maurer eq.s to Free Differential Algebras (FDA). Their dual formulation is given by L_∞ algebras, a generalization of Lie algebras with higher brackets.

2. Group geometric approach

to (super) gauge and gravity theories

- To interpret all (local) symmetries as coordinate transformations

- Thus diff.s, supersymmetry, gauge transformations are all diffeomorphisms in the (super)group manifold G

- They are invariances of an action invariant under group manifold diff.s

Dynamical fields: Vielbein (components) on G

Group geometric construction of supergravity theories, Torino group 80's

originates from Ne'eman and Regge (1978), then D'Adda, D' Auria, Fré, LC, van Nieuwenhuizen, Townsend, …

Reviews: D' Auria Fré LC 1991, LC 2018, D' Auria 2019

Related approaches: Chamseddine, West 1977

Basic steps

- Lie (super)algebra G

$$[T_A, T_B] = C^C{}_{AB} T_C$$

On the group manifold G: basis of tangent vectors $\mathbf{t}_A$ closes on the same Lie algebra.

- Cartan-Maurer eq.s

Dual (cotangent) basis: left-invariant one-forms σ^A , Vielbeine of the group manifold

$$d\sigma^A + \frac{1}{2} C^A{}_{BC}\, \sigma^B \wedge \sigma^C = 0$$

Jacobi id.s $\quad d^2 = 0 \Longleftrightarrow C^A{}_{B[C} C^B{}_{DE]} = 0$

- fundamental fields $\longleftrightarrow$ σ^A

More precisely the dynamical fields are the vielbeins of $\tilde{G}$ a smooth deformation of G , with curvature

$$R^A \equiv d\tilde{\sigma}^A + \frac{1}{2} C^A{}_{BC} \, \tilde{\sigma}^B \wedge \tilde{\sigma}^C \neq 0$$

measuring the deformation

- Bianchi identities

$$dR^A - C^A{}_{BC} R^B \tilde{\sigma}^C = 0$$

Example: N=1, D=4 supergravity

- Lie algebra (superPoincaré)

$$[P_a, P_b] = 0 \qquad [M_{ab}, M^{cd}] = 4\delta^{[c}_{[a} M_{b]}{}^{d]}$$

$$\{Q_\alpha, Q_\beta\} = i(C\Gamma^a)_{\alpha\beta} P_c \qquad [M_{ab}, P^c] = 2\delta^c_{[a} P_{b]}$$

$$[Q_\alpha, P_b] = 0 \qquad [M_{ab}, Q_\alpha] = \frac{1}{4}(\Gamma_{ab})_\alpha{}^\beta Q_\beta$$

- Dynamical fields

G-coordinates

σ^A, $A = (a, ab, \alpha)$:

$V^a \longleftrightarrow P_a$		vierbein	x
$\omega^{ab} \longleftrightarrow M_{ab}$		spin connection	y^{ab}
$\psi^\alpha \longleftrightarrow Q_\alpha$		gravitino	θ^α

- Curvatures

$$R^a = dV^a - \omega^a{}_b V^b - \frac{i}{2}\bar{\psi}\Gamma^a\psi$$ Torsion

$$R^{ab} = d\omega^{ab} - \omega^a{}_c\omega^{cb}$$ Lorentz curvature

$$\rho = d\psi - \frac{1}{4}\omega^{ab}\Gamma_{ab}\psi$$ gravitino curvature

- Bianchi identities

$$dR^a - \omega^a{}_b R^b + R^a{}_b V^b - i\bar{\psi}\Gamma^a\rho = 0$$

$$dR^{ab} + 2\omega^{[a}{}_c R^{b]c} = 0$$

$$d\rho - \frac{1}{4}\omega^{ab}\Gamma_{ab}\psi + \frac{1}{4}\Gamma_{ab}\psi R^{ab} = 0$$

Geometric action

Task: construct an action, invariant under the diffeomorphisms on the G-group manifold

Then the symmetries of the theory are given by the G-diffeomorphisms

The group-geometric approach provides a systematic and algorithmic procedure to construct locally supersymmetric actions

N=1 supergravity in d=4

Ferrara, Friedman, van Nieuwenhuizen 1976
Deser, Zumino 1976

Action

$$I_{SG} = \int_{M^4} (R\sqrt{-g} + \bar{\psi}_\mu \gamma_5 \gamma_a \mathcal{D}_\nu \psi_\rho V^a_\sigma \epsilon^{\mu\nu\rho\sigma}) d^4x$$

in form language:

$$I_{SG} = \int_{M^4} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} + 4\ \bar{\psi} \wedge \gamma_5 \gamma_d \rho \wedge V^d$$

with $R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}$

$\rho = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi$

vierbein, spin connection, gravitino

$$V^a = V^a_\mu\ dx^\mu,\quad \omega^{ab} = \omega^{ab}_\mu\ dx^\mu,\quad \psi = \psi_\mu dx^\mu$$

$$R^{ab} = R^{ab}{}_{\mu\nu} dx^\mu \wedge dx^\nu = (\partial_{[\mu}\omega^{ab}_{\nu]} - \omega^{ac}_{[\mu}\omega_{\nu]c}{}^{b}) dx^\mu \wedge dx^\nu$$

Invariances

- diffeomorphisms
- local Lorentz rotations
- local supersymmetry

$$\delta_\epsilon V^a = i\ \bar{\epsilon}\gamma^a\psi$$

$$\delta_\epsilon \psi = d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon$$

2025: ~ 50 years of supergravity

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Progress toward a theory of supergravity*

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As a new approach to supergravity, an action containing only vierbein and Rarita-Schwinger fields ($V_{a\mu}$ and ψ_μ) is presented together with supersymmetry transformations for these fields. The action is explicitly shown to be invariant except for a ψ^5 term in its variation. This term may also vanish, depending on a complicated calculation. (Added note: This term has now been shown to vanish by a computer calculation, so that the action presented here does possess full local supersymmetry.)

Even early in the development of the Fermi-Bose supersymmetry concept, it was thought that the new fermionic symmetry transformation might be important for the theory of gravitation.[1] Two similar but apparently inequivalent approaches to this theory of "supergravity" have been formulated by Arnowitt, Nath, and Zumino[2] and by Zumino.[3] These approaches exploit the geometry of "superspace,"[4] a manifold parametrized by four anticommuting spinor coordinates θ_α in addition to the normal Riemannian coordinates x^μ. The theories are formulated in terms of superfields which contain a very large number of ordinary fields—i.e., vectors, tensors, spinors, etc. Although it is expected that some component fields are merely generalized gauge excitations and not true physical fields, the physical content of the Arnowitt-Nath-Zumino theories has never been spelled out, but there are indications[2] that, as is perhaps desirable, the approaches necessarily bring in gauge vector and spin-$\frac{1}{2}$ particles in addition to tensor and spin-$\frac{3}{2}$ particles.

In this note we report on progress in a very different approach to supergravity in which we commit ourselves from the start to a formulation without superspace in which the only fields in the gravitational supermultiplet are the metric tensor $g_{\mu\nu}(x)$ [or, equivalently, the vierbein field $V_{a\mu}(x)$] and a Rarita-Schwinger field $\psi_\mu(x)$. If fully successful, we would then expect to adjoin matter supermultiplets of lower-spin fields in much the same way as matter fields are treated in conventional gravitation.

There is a theorem[5] in the usual theory of global supersymmetry which demonstrates the existence of irreducible representations of the graded Lie algebra of supersymmetry charges and Poincaré group generators. Some of these representations act in the Hilbert space of helicity states of two massless particles, one neutral boson and one Majorana fermion of adjacent spins J and $J+\frac{1}{2}$ (for any $J=0,\frac{1}{2},1,\ldots$). It is therefore known that a representation exists containing states of massless spin-$\frac{3}{2}$ and spin-2 particles, and it was suggested[6] earlier that these particles form the gravitational supermultiplet. The theorem does not guarantee that there exists a corresponding interacting quantum field theory, but it is reasonable to hope that it exists, and this is the basic mathematical motivation for our approach. Many questions can be asked about the physical motivation and consistency of both this treatment and the entire concept of supergravity. We shall discuss some of them at the end of this note, and we proceed now to the formulation.

The starting point of our approach is the generally covariant action[7]

$$I=\int d^4x(\mathcal{L}_2+\mathcal{L}_{3/2})$$
$$=\int d^4x[\tfrac{1}{4}\kappa^{-2}\sqrt{-g}\,R-\tfrac{1}{2}\epsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda(x)\gamma_5\gamma_\mu D_\nu\psi_\rho(x)] \quad (1)$$

describing the interaction of vierbein fields and Rarita-Schwinger[8] fields subject to the Majorana constraint $\psi_\rho(x)=C\bar{\psi}_\rho(x)^T$. The covariant derivative[9]

$$D_\nu\psi_\rho(x)=\partial_\nu\psi_\rho(x)-\Gamma^\sigma_{\nu\rho}\psi_\sigma+\tfrac{1}{2}\omega_{\nu ab}\sigma^{ab}\psi_\rho \quad (2)$$

involves the standard Christoffel symbol (although it cancels in $\mathcal{L}_{3/2}$ because of the tensor density $\epsilon^{\lambda\rho\mu\nu}$) and the vierbein connection

$$\omega_{\nu ab}=\tfrac{1}{2}[V_a{}^\mu(\partial_\nu V_{b\mu}-\partial_\mu V_{b\nu})+V_a{}^\rho V_b{}^\sigma(\partial_\sigma V_{c\rho})V^c{}_\nu]-(a\leftrightarrow b), \quad (3)$$

while

$$\sigma_{ab}=\tfrac{1}{4}[\gamma_a,\gamma_b].$$

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CONSISTENT SUPERGRAVITY

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A combined spin 2 – spin 3/2 extension of general relativity is given which is both free of the usual higher spin inconsistencies and invariant under local supersymmetry transformations.

The unification of the gravitational field with a spin 3/2 system is a natural goal within the framework of supersymmetry [1]. In constructing such a theory one faces obvious problems due to the highly non-linear nature of general relativity [2] and to the well-known difficulties encountered in coupling higher spin fields in a consistent way. We shall show here that the simplest candidate, namely the sum of the Einstein action and that for a massless, minimally coupled, Rarita-Schwinger [3] Majorana field fulfils the consistency criteria. As we shall also see, this fact is related to the invariance of the theory under local supersymmetry transformations[‡1].

The key to our results lies in the use of the first order formalism for gravitation, in which vierbeins and connection coefficients are treated independently (a convenient description of the first order formalism can be found in a paper by Kibble [5]). Minimal coupling in this sense implies the existence of torsion, or of non-minimal contact interactions in second order language. The first order formulation with torsion is closely related to the description of supergravity in superspace [1, 6].

The combined Lagrangian has the form[‡2]

$$L=-\tfrac{1}{2}eR-\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\,\bar{\psi}_\lambda\gamma_5\gamma_\mu D_\nu\psi_\rho \quad (1)$$

where

$$e=\det e_{\mu a},\qquad R=e_a{}^\mu e_b{}^\nu R_{\mu\nu,}{}^{ab}. \quad (2)$$

The covariant derivative on ψ_μ is defined according to its spin 1/2 content only

$$D_\mu=\partial_\mu-\tfrac{1}{2}\omega_{\mu,ab}\Sigma^{ab},\qquad \Sigma^{ab}=\tfrac{1}{4}[\gamma^a,\gamma^b] \quad (3)$$

and satisfies

$$[D_\mu,D_\nu]=-\tfrac{1}{2}R_{\mu\nu,ab}\Sigma^{ab}. \quad (4)$$

This form for D_μ is strongly suggested by the Maxwell-like gauge invariance of the flat space action uncer $\delta\psi_\mu=2\partial_\mu\alpha$ and preserves the simplest definition of the curl. This results in a particularly simple form of the torsion. The vierbeins $e_{\mu a}$, the connection coefficients $\omega_{\mu,ab}$ and the Majorana vector-spinor ψ_μ are to be varied independently. Note that we have not introduced any auxiliary fields. They will be useful, however, when coupling to matter is included. The equations of motion are

$$R^\lambda\equiv\epsilon^{\lambda\mu\nu\rho}(\gamma_\mu D_\nu\psi_\rho-\tfrac{1}{4}\gamma_\tau C_{\mu\nu}{}^\tau\psi_\rho)=0 \quad (5)$$

$$C_{\mu\nu}{}^\tau=\frac{1}{2}\bar{\psi}_\mu\gamma^\tau\psi_\nu \quad (6)$$

and

$$G^{\tau\mu}=\frac{i}{2}\epsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma^\tau D_\nu\psi_\rho. \quad (7)$$

Here

$$G^\tau{}_a=R^\tau{}_a-\tfrac{1}{2}e_a{}^\tau R,\qquad R_{\tau a}=R_{\tau\lambda,a}{}^\lambda \quad (8)$$

is the (non-symmetric)Einstein tensor and

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‡1 Local supersymmetry transformations, with parameters depending arbitrarily upon the space-time co-ordinates were first discussed in ref. [4], where their existence was stated for two space-time dimensions and conjectured for four dimensions. The commutator of two local supersymmetry transformations contains a general co-ordinate transformation.

‡2 We choose units in which $\kappa=1$. Our gamma matrices satisfy $(\gamma_0)^2=(\gamma_5)^2=-1$. If we use the Majorana representation the matrices γ_μ and γ_5 are real and the field ψ_μ Hermitean, with $\bar{\psi}_\mu=\tilde{\psi}_\mu\gamma^0$. We also take $\epsilon^{0123}=-\epsilon_{0123}=1$.

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3. Free differential algebras (FDA)

Sullivan, 1977
D'Auria, Fré 1982

- convenient algebraic setting for field theories with antisymmetric tensors (*p*-forms)
- generalize Cartan-Maurer eq.s of group manifold G 1-form vielbeins σ^A, by including *p*-forms B^i
- example : ordinary Cartan-Maurer 1-forms σ^A supplemented by a single *p*-form B^i in a representation $D^i{}_j$ of G

$$d\sigma^A + \frac{1}{2} C^A{}_{BC}\ \sigma^B \wedge \sigma^C = 0$$

$$\underbrace{dB^i + C^i{}_{Aj}\ \sigma^A B^j}_{\nabla B^i} + \frac{1}{(p+1)!} C^i{}_{A_1 \ldots A_{p+1}} \sigma^{A_1} \ldots \sigma^{A_{p+1}} = 0$$

- taking d of l.h.s. and requiring $d^2 = 0$ →

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- example : ordinary Cartan-Maurer 1-forms σ^A supplemented by a single *p*-form B^i in a representation $D^i{}_j$ of G

suggests a multibracket

$$d\sigma^A + \frac{1}{2} C^A{}_{BC}\, \sigma^B \wedge \sigma^C = 0$$

$$\underbrace{dB^i + C^i{}_{Aj}\, \sigma^A B^j}_{\nabla B^i} + \frac{1}{(p+1)!} C^i{}_{A_1 \ldots A_{p+1}} \sigma^{A_1} \ldots \sigma^{A_{p+1}} = 0$$

- taking d of l.h.s. and requiring $d^2 = 0$ →

Generalized Jacobi identities

$$C^A{}_{B[C} C^B{}_{DE]} = 0$$ usual Jacobi id.s

$$C^i{}_{Aj} C^j{}_{Bk} - C^i{}_{Bj} C^j{}_{Ak} = C^C{}_{AB} C^i{}_{Ck}$$ representation condition

$$2\ C^i{}_{A_1 j} C^j{}_{A_2 \ldots A_{p+2}]} - (p+1) C^i{}_{B[A_1 \ldots A_p} C^B{}_{A_{p+1} A_{p+2}]} = 0$$ cocycle condition

- $C^i = C^i{}_{A_1 \ldots A_{p+1}} \sigma^{A_1} \ldots \sigma^{A_{p+1}}$ is a (p+1)-cocycle $(\nabla C^i = 0)$

- given a FDA, there is a well-defined procedure to construct a Lagrangian with the *p*-forms as fundamental fields

- To extend a Lie algebra to a FDA: need a covariantly closed (p+1)-form C^i

- given such a form C^i, C^i + covariantly closed (p+1)-form still yields a FDA. But if this cov. closed form is cov. exact (= $\nabla \Phi^i$) $C^i + \nabla \Phi^i$ leads to an equivalent FDA via the redefinition

$$B^i \to B^i + \Phi^i$$

- Thus inequivalent FDA's are classified by nontrivial cohomology classes of the covariant derivative ∇ , i.e. by Chevalley cohomology

4. Example: FDA of d=11 supergravity

D'Auria, Fré 1982

$$d\omega^{ab} - \omega^{a}{}_{c}\omega^{cb} = 0 \qquad [= R^{ab}]$$

$$dV^a - \omega^{a}{}_{b}V^b - \frac{i}{2}\bar{\psi}\Gamma^a\psi = 0 \qquad [= R^a]$$

$$d\psi - \frac{1}{4}\omega^{ab}\Gamma_{ab}\psi = 0 \qquad [= \rho]$$

$$dA - \frac{1}{2}\bar{\psi}\Gamma_{ab}\psi V^a V^b = 0 \qquad [= R(A)]$$

nontrivial 4-cocycle

- the d=11 Fierz identity $\bar{\psi}\Gamma_{ab}\psi\bar{\psi}\Gamma^a\psi = 0$ ensures FDA closure ($d^2 = 0$)

- extends the superPoincaré Lie algebra in d=11 with a 3-form A in the identity representation

- $C^{i}{}_{A_1 \ldots A_{p+1}} \longrightarrow \qquad C_{\;\alpha\beta ab} = -12(C\Gamma_{ab})_{\alpha\beta}$

- The lagrangian of d =11 supergravity can be written as a 11- form, made out of (exterior) products of fields and curvatures, therefore invariant by construction under diff.s. D'Auria, Fré 1982

- Original construction (in components) by Cremmer, Julia, Scherk 1978

d=11 supergravity Lagrangian

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{9} R^{a_1 a_2} V^{a_3} \wedge \cdots \wedge V^{a_{11}} \epsilon_{a_1 \ldots a_{11}} + \frac{7}{30} T^a \wedge V_a \wedge \bar{\psi} \Gamma^{b_1 \ldots b_5} \wedge \psi \, V^{b_6 \ldots b_{11}} \epsilon_{b_1 \ldots b_{11}} \\
& + 2 \bar{\rho} \Gamma_{c_1 \ldots c_8} \psi \wedge V^{c_1 \ldots c_8} - 84 F^{(4)} \wedge \left(\mathrm{i} \bar{\psi} \Gamma_{b_1 \ldots b_5} \wedge \psi \, V^{b_1 \ldots b_5} - 10 A^{(3)} \wedge \bar{\psi} \gamma_{ab} \psi V^{ab} \right) \\
& + \frac{1}{4} \bar{\psi} \Gamma^{a_1 a_2} \psi \wedge \bar{\psi} \Gamma^{a_3 a_4} \psi \wedge V^{a_5 \ldots a_{11}} \epsilon_{a_1 \ldots a_{11}} - 210 \, \bar{\psi} \Gamma^{a_1 a_2} \psi \wedge \bar{\psi} \Gamma^{a_3 a_4} \psi \wedge V^{a_1 \ldots a_4} \wedge A^{(3)} \\
& - 840 \, F^{(4)} \wedge F^{(4)} \wedge A^{(3)} - \frac{1}{330} F_{a_1 \ldots a_4} \, F^{a_1 \ldots a_4} V^{c_1 \ldots c_{11}} \epsilon_{c_1 \ldots c_{11}} + 2 F_{a_1 \ldots a_4} \, F^{(4)} V^{a_5 \ldots a_{11}} \epsilon_{a_1 \ldots a_{11}} .
\end{aligned}
$$

5. The general structure of FDA's

- Use simplified notations:

p - forms: t^a with $t^a(t_b) = \delta^a_b$

p - vectors: t_a

- Generalization of Cartan-Maurer equations:

$$dt^a + \sum_{k=1}^{\infty} \frac{1}{k!} \, C^a_{a_1 \cdots a_k} \, t^{a_1} \wedge \cdots \wedge t^{a_k} = 0$$

- Closure of exterior differential *d*

$$d^2 t^a = \sum_{k,j=1}^{\infty} \frac{1}{(k-1)!} \frac{1}{j!} \, C^a_{a_1 a_2 \cdots a_k} C^{a_1}_{b_1 \cdots b_j} \, t^{b_1} \wedge \cdots \wedge t^{b_j} \wedge t^{a_2} \wedge \cdots \wedge t^{a_k}$$

- From $d^2 = 0$:

$$\sum_{k,j=1}^{\infty} \frac{1}{(k-1)!} \frac{1}{j!} \sum_{\sigma} \chi(\sigma, t)\; C^{a}_{a_1 \sigma(a_2)\cdots\sigma(a_k)} C^{a_1}_{\sigma(b_1)\cdots\sigma(b_j)}\; t^{b_1} \wedge \cdots \wedge t^{b_j} \wedge t^{a_2} \wedge \cdots \wedge t^{a_k} = 0$$

permutations that shuffle separately $a_2, \cdots a_k$ and $b_1, \cdots b_j$ give identical terms, then need to consider only "unshuffles"

$$\sum_{k,j=1}^{\infty} \sum_{\sigma \in \mathrm{Unsh}} \chi(\sigma, t)\; C^{a}_{a_1 \sigma(a_2)\cdots\sigma(a_k)} C^{a_1}_{\sigma(b_1)\cdots\sigma(b_j)}\; t^{b_1} \wedge \cdots \wedge t^{b_j} \wedge t^{a_2} \wedge \cdots \wedge t^{a_k} = 0$$

- For a fixed *n = j+k-1* (number of p-forms in the wedge products):

$$\boxed{\sum_{k+j=n+1}^{\infty} \sum_{\sigma \in \mathrm{Unsh}} \chi(\sigma, t)\; C^{a}_{a_1 \sigma(a_2)\cdots\sigma(a_k)} C^{a_1}_{\sigma(b_1)\cdots\sigma(b_j)} = 0}$$

generalized Jacobi identities

5. The duality between FDA's and L_∞

- based on the duality of derivations on cotangent space V^* and tangent space V

exterior derivative d	$\longleftrightarrow$	exterior derivative D
acts on *p*-forms		acts on *p*-vectors
degree +1		degree -1

- *p*-vectors belong to the antisymmetric tensor space

$$V \vee \cdots \vee V$$

(space of antisymmetric tensor products of tangent vectors)

- The duality d $\longleftrightarrow$ D is defined by

$$d\Theta\ (v_1 \vee \cdots \vee v_n) = \Theta\ D\,(v_1 \vee \cdots \vee v_n)$$

- Definition:

$$(v_1 \vee \cdots \vee v_n) \equiv \sum_{\sigma} \chi(\sigma, v)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

where the sign $\chi(\sigma, v)$ takes into account

- the parity of the permutation σ
- the gradings of the (multi)vectors v_i

(analogous to wedge products of *p* - forms)

- From the definition of duality we deduce the following properties of *D*:

1) *D* can be written as a sum of differential operators ℓ_n

$$D = \ell_1 + \ell_2 + \ell_3 + \cdots$$

FDA

2) the action of ℓ_i on *n*-plets

- if $i = n$, $\ell_n(t_{a_1} \vee \cdots \vee t_{a_n}) = -C^b_{a_1 \cdots a_n} t_b$
- if $i > n$, ℓ_i vanishes
- if $i < n$, ℓ_i acts as a coderivation:

$\ell_i(v_1 \vee \cdots \vee v_n)$ =

$$\sum_\sigma \chi(\sigma, v)\, \ell_i(v_{\sigma(1)} \vee \cdots \vee v_{\sigma(i)}) \vee v_{\sigma(i+1)} \vee \cdots \vee v_{\sigma(n)}$$

- From the definition of duality we deduce the following properties of *D*:

1) *D* can be written as a sum of differential operators ℓ_n

$$D = \ell_1 + \ell_2 + \ell_3 + \cdots$$

FDA

2) the action of ℓ_i on *n*-plets

generalization of Lie bracket

- if $i = n$, $\ell_n(t_{a_1} \vee \cdots \vee t_{a_n}) = -C^b_{a_1 \cdots a_n} t_b$
- if $i > n$, ℓ_i vanishes
- if $i < n$, ℓ_i acts as a coderivation:

$$\ell_i(v_1 \vee \cdots \vee v_n) =$$

$$\sum_\sigma \chi(\sigma, v)\; \ell_i(v_{\sigma(1)} \vee \cdots \vee v_{\sigma(i)}) \vee v_{\sigma(i+1)} \vee \cdots \vee v_{\sigma(n)}$$

- Then $D^2 = (\ell_1 + \ell_2 + \ell_3 + \cdots)(\ell_1 + \ell_2 + \ell_3 + \cdots) = 0$

  implies, when acting on a *n*-plet of (multi)vectors :

$$\sum_{k+j=n+1} \sum_{\sigma \in \mathrm{Unsh}} \chi(\sigma,t)\ \ell_k(\ell_j(t_{\sigma(a_1)} \vee \cdots \vee t_{\sigma(a_j)}) \vee t_{\sigma(a_{j+1})} \vee \cdots \vee t_{\sigma(a_{j+k-1})}) = 0$$

(“strong homotopy Jacobi identity”)

- reproduces the generalized Jacobi id.s of FDA, after using

$$\ell_n(t_{a_1} \vee \cdots \vee t_{a_n}) = -C^b_{a_1 \cdots a_n} t_b$$

- In multibracket notation $\ell_n(t_{a_1} \vee \cdots \vee t_{a_n}) = [t_{a_1}, \cdots, t_{a_n}]$

$$\sum_{k+j=n+1} \sum_{\sigma \in \mathrm{Unsh}} \chi(\sigma,t)\ [[t_{\sigma(a_1)}, \cdots, t_{\sigma(a_j)}], t_{\sigma(a_{j+1})}, \cdots, t_{\sigma(a_{j+k-1})}] = 0$$

characterizes L_∞ algebras

- In the multibracket formulation, L_∞ algebras were first introduced in Stasheff 1992, Lada and Stasheff 1993 and later recognized to be dual to FDA's by Fiorenza, Sati and Schreiber 2014

- Two alternative routes to relate FDA's to algebras of tangent vectors:

  - "resolution" of p-form fields as products of (new)1-forms, satisfying the FDA Cartan-Maurer eqs., leading to a larger Lie algebra (for ex. the M algebra of d=11 SG) D' Auria Fré 1982

  - introducing a generalized Lie derivative along antisymmetrized multivectors, with p-form parameters. The algebra of Lie derivatives becomes non-associative Perotto and LC 1996, LC 2011, 2014.

7. The L_∞ structure of FDA1

1-forms σ^A

2-forms B^i

1-vectors t_A

2-vectors t_i

$$d\sigma^A + \frac{1}{2} C^A{}_{BC} \, \sigma^B \sigma^C = 0$$

$$dB^i + C^i{}_{Aj} \sigma^A \wedge B^j + \frac{1}{3!} C^i_{A_1 A_2 A_3} \sigma^{A_1} \wedge \sigma^{A_2} \wedge \sigma^{A_3} = 0$$

Satisfy Jacobi id.s of FDA1

Satisfy Jacobi id.s of L_∞

$$[t_A, t_B] = C^A{}_{BC} T_C$$

$$[t_A, t_i] = C^j{}_{AC} T_j$$

$$[t_A, t_B, t_C] = C^i{}_{ABC} T_i$$

8. The L_∞ structure of d=11 supergravity

Cartan-Maurer eq.s | dual tangent vectors

Cartan-Maurer eq.s		dual tangent vectors
$d\omega^{ab} - \omega^a{}_c\omega^{cb} = 0$	$[= R^{ab}]$	M_{ab}
$dV^a - \omega^a{}_b V^b - \frac{i}{2}\bar{\psi}\Gamma^a\psi = 0$	$[= R^a]$	P_a
$d\psi - \frac{1}{4}\omega^{ab}\Gamma_{ab}\psi = 0$	$[= \rho]$	$\bar{Q}_\alpha$
$dA - \frac{1}{2}\bar{\psi}\Gamma_{ab}\psi V^a V^b = 0$	$[= R(A)]$	$t(A)$

L_∞ structure

d=11 superPoincaré Lie algebra + 4- bracket

$$[\bar{Q}_\alpha, \bar{Q}_\beta, P_a, P_b] = \frac{1}{2}(\Gamma_{ab})_{\alpha\beta}\ t(A)$$

with also a 6-form B, add a 5- and a 7- bracket

$$[\bar{Q}_\alpha, \bar{Q}_\beta, P_a, P_b, t(A)] = \frac{15}{2}(\Gamma_{ab})_{\alpha\beta}\ t(B)$$

$$[\bar{Q}_\alpha, \bar{Q}_\beta, P_{a_1}, \cdots, P_{a_5}] = \frac{i}{2}(\Gamma_{a_1 \cdots a_5})_{\alpha\beta}\ t(B)$$

9. Conclusions and outlook

- Need to understand symmetry structure induced by L_∞
- use of L_∞ structure for double copy formulation of SG ? (cf L. Jonke talk)
- Include 0-forms in the L_∞ structure
- Relate the resolved Lie algebra directly to the L_∞ algebra , without the "bridge" of FDA.

Thank you !