# Path integral for the closed string and the matrix model

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20 Sep, 2025 @Corfu 2025

Based on JHEP10 (2024) 082 [arXiv: 2408.04000]

# Overview

**Euclidean perturbative superstring** → (equivalent) → **Minkowskian perturbative superstring**

**exhibits stringy causality**

↓ Matrix regularisation of the type IIB string

**Minkowskian NBI-type IKKT matrix model**

**Theory of "causal fuzzy world-sheet"**

A "causal" matrix model is obtained by matrix regularisation that respects the stringy causality.

# Introduction

## *Perturbative string theory*

(Euclidean)
$S_{\mathrm{P}}^{(\mathrm{E})}$: Polyakov-type action

An S-matrix is described as

$$A_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi} \int DX\, D\theta\, Dg\, V_{j_1}(k_1)\cdots V_{j_n}(k_n)\, \exp[-S_{\mathrm{P}}^{(\mathrm{E})}]$$

[Polyakov '81]

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  ➡ $\det h_{ab} > 0$ does not contribute

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## *Euclidean v. Minkowskian*

We start with the Minkowski signature but at some point,
Wick-rotate the theory to the Euclidean signature.

⋯ Because Euclidean theory is usually well-defined

But we should NOT naively Wick-rotate it;
otherwise, we might arrive at a different theory.

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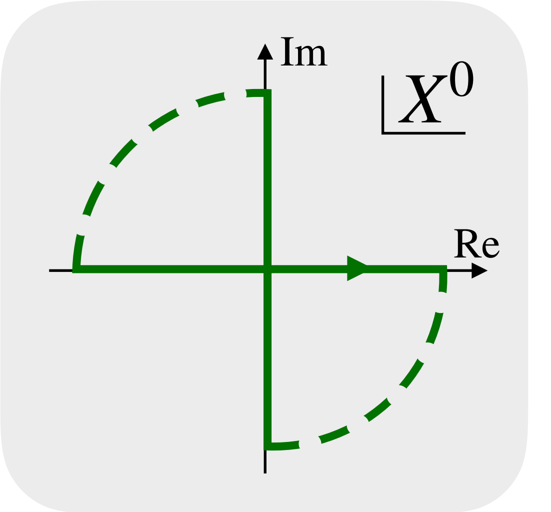


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$$\int DX\, Dg \, \exp\left[-i \int d^2\sigma \sqrt{-g}\, g^{ab} \partial_a X^\mu \partial_b X_\mu\right] \qquad g^{ab} = {e_\alpha}^a \eta^{\alpha\beta} {e_\beta}^b$$

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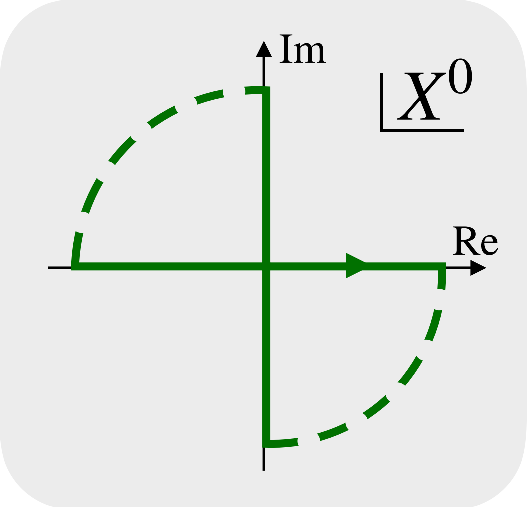


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$$- e^{-3i\theta} e_1{}^a e_1{}^b \partial_a X^D \partial_b X^D + e^{-i\theta} e_2{}^a e_2{}^b \partial_a X^D \partial_b X^D)\Big]$$

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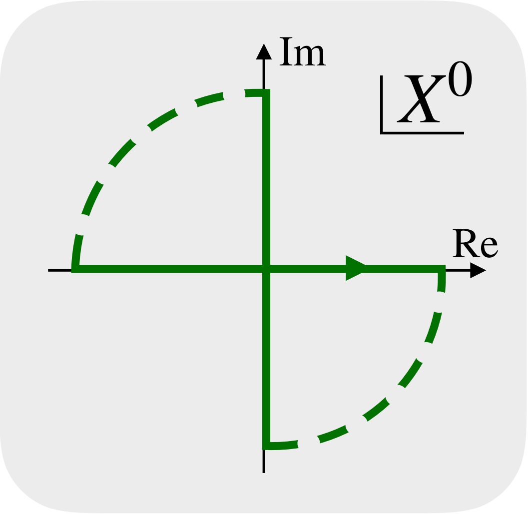


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Cauchy's integral thm. cannot be applied.

# Introduction

## 3 different types of world-sheet formulation

*Nambu-Goto type*

$$S_{\rm NG} = -\int d^2\sigma \sqrt{-h}$$

$$h_{ab} = \Pi^\mu_a \Pi_{b\mu}, \quad h = \det h_{ab}$$

$\Pi^\mu_a = \partial_a X^\mu$ for the bosonic str.

*Schild type*

$$S_{\rm Schild} = -\frac{1}{2}\int d^2\sigma \left( \frac{-h}{e_g} + e_g \right)$$

*Polyakov type*

$$S_{\rm P} = -\frac{1}{2}\int d^2\sigma \sqrt{-g}\, g^{ab} h_{ab}$$

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Quantum mechanically
Equivalent?

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The Euclidean case was discussed long time ago.

[Polyakov '87; Yoneya '97]

# Path integral — Euclidean to Minkowskian

Let's start with Polyakov's Euclidean path integral
in the case of critical bosonic string theory for simplicity.

※ The equivalences hold for critical type IIB and IIA string and critical bosonic string theory on the flat target space.

$$Z = \int DX\,Dg\;\exp\left[-\frac{1}{2}\int d^2\sigma\sqrt{g}\,g^{ab}h_{ab}\right]$$

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$$Dg = D\phi \prod_{\sigma} \frac{2e^{\phi}\, d\Lambda_1 d\Lambda_2}{(\Lambda_2)^2} \qquad g^{ab} = e^{-\phi}\begin{pmatrix} \frac{\Lambda_1^2+\Lambda_2^2}{\Lambda_2} & -\frac{\Lambda_1}{\Lambda_2} \\ -\frac{\Lambda_1}{\Lambda_2} & \frac{1}{\Lambda_2} \end{pmatrix}$$

$$\| \delta g \|^2 = \frac{1}{2}\int d^2\sigma \sqrt{g}\, g^{ab}\delta g_{bc} g^{cd} \delta g_{da} = \int d^2\sigma\, e^{\phi}\left(\frac{\delta\Lambda_1^2 + \delta\Lambda_2^2}{(\Lambda_2)^2} + \delta\phi^2\right)$$

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$$= \int DX \left[\prod_\sigma \int_0^\infty \frac{2d\Lambda_2}{\sqrt{\Lambda_2^3 h_{11}}}\right] \exp\left[-\frac{1}{2}\int d^2\sigma \left\{\frac{h_{11}h_{22}-h_{12}^2}{\Lambda_2 h_{11}} + \Lambda_2 h_{11}\right\}\right]$$

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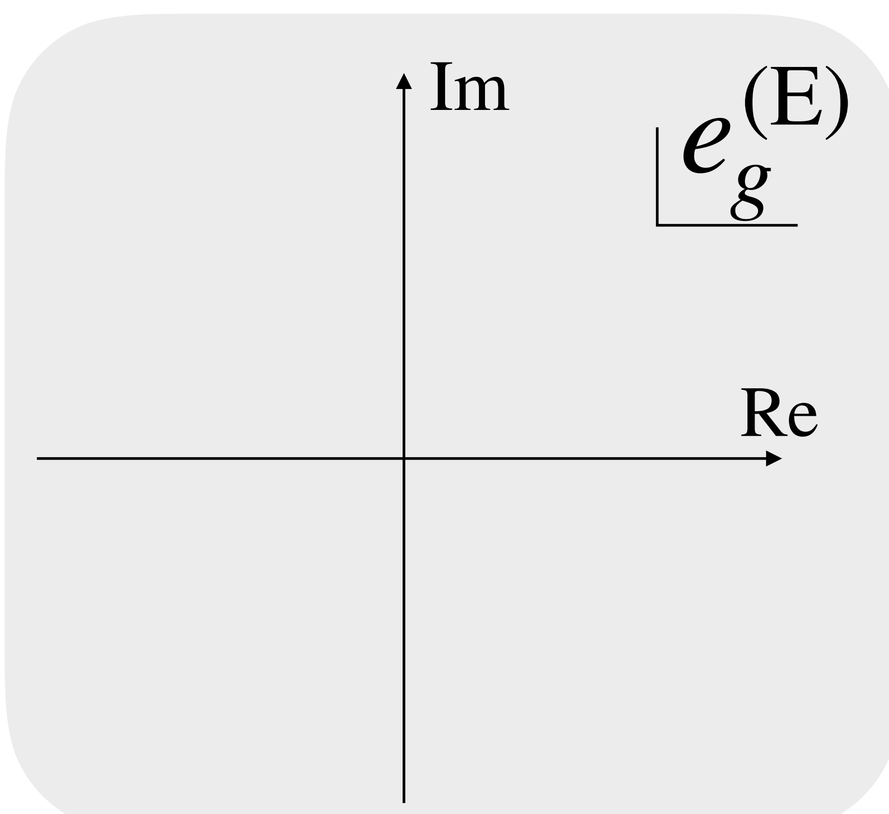
$$Z = \int DX \left[ \prod_\sigma \int_0^\infty \frac{2de_g^{(\mathrm{E})}}{e_g^{(\mathrm{E})\,3/2}} \right] \exp\left[ -\frac{1}{2} \int d^2\sigma \left\{ \frac{h_{11}h_{22} - h_{12}^2}{e_g^{(\mathrm{E})}} + e_g^{(\mathrm{E})} \right\} \right] \quad \Lambda_2 h_{11} =: e_g^{(\mathrm{E})}$$



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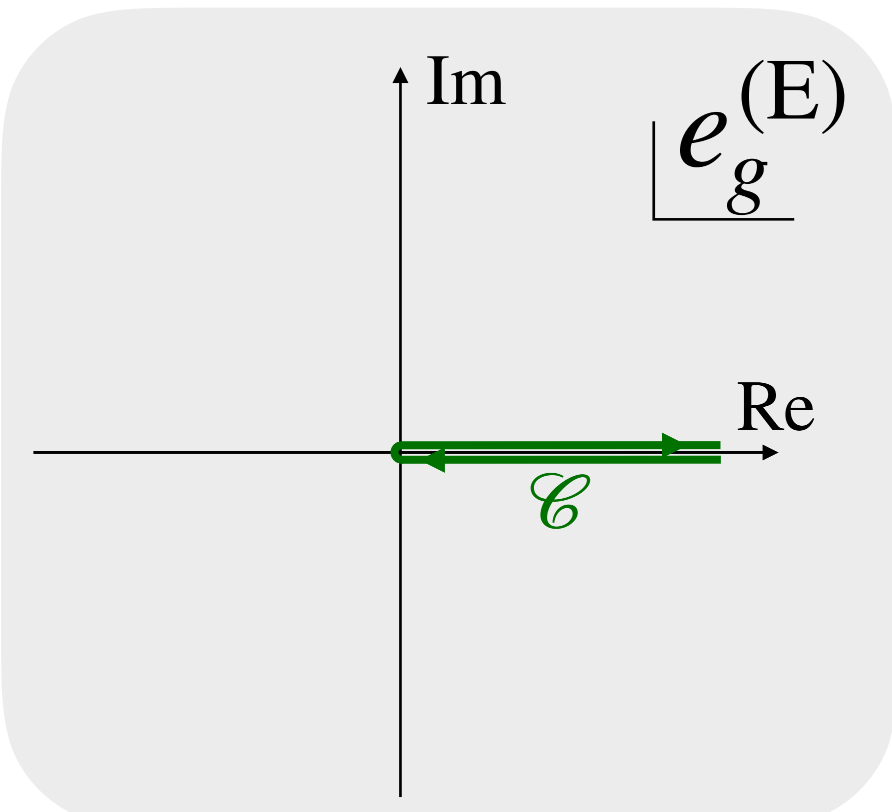
$$\mathscr{C} : +\infty - i0 \to -0 \to +\infty + i0$$



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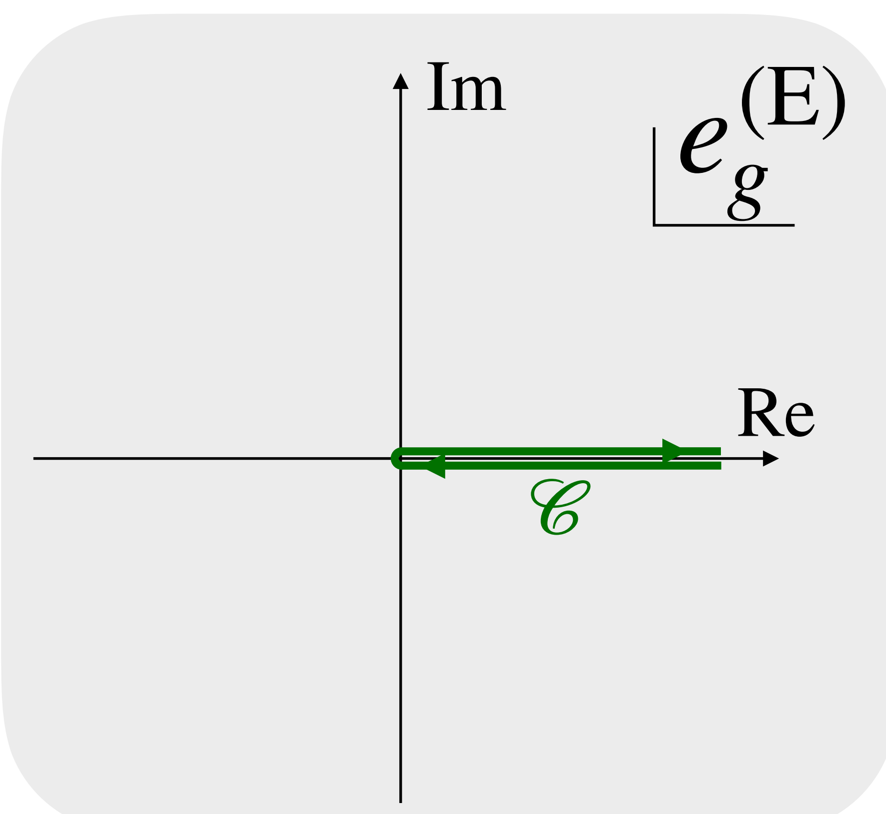

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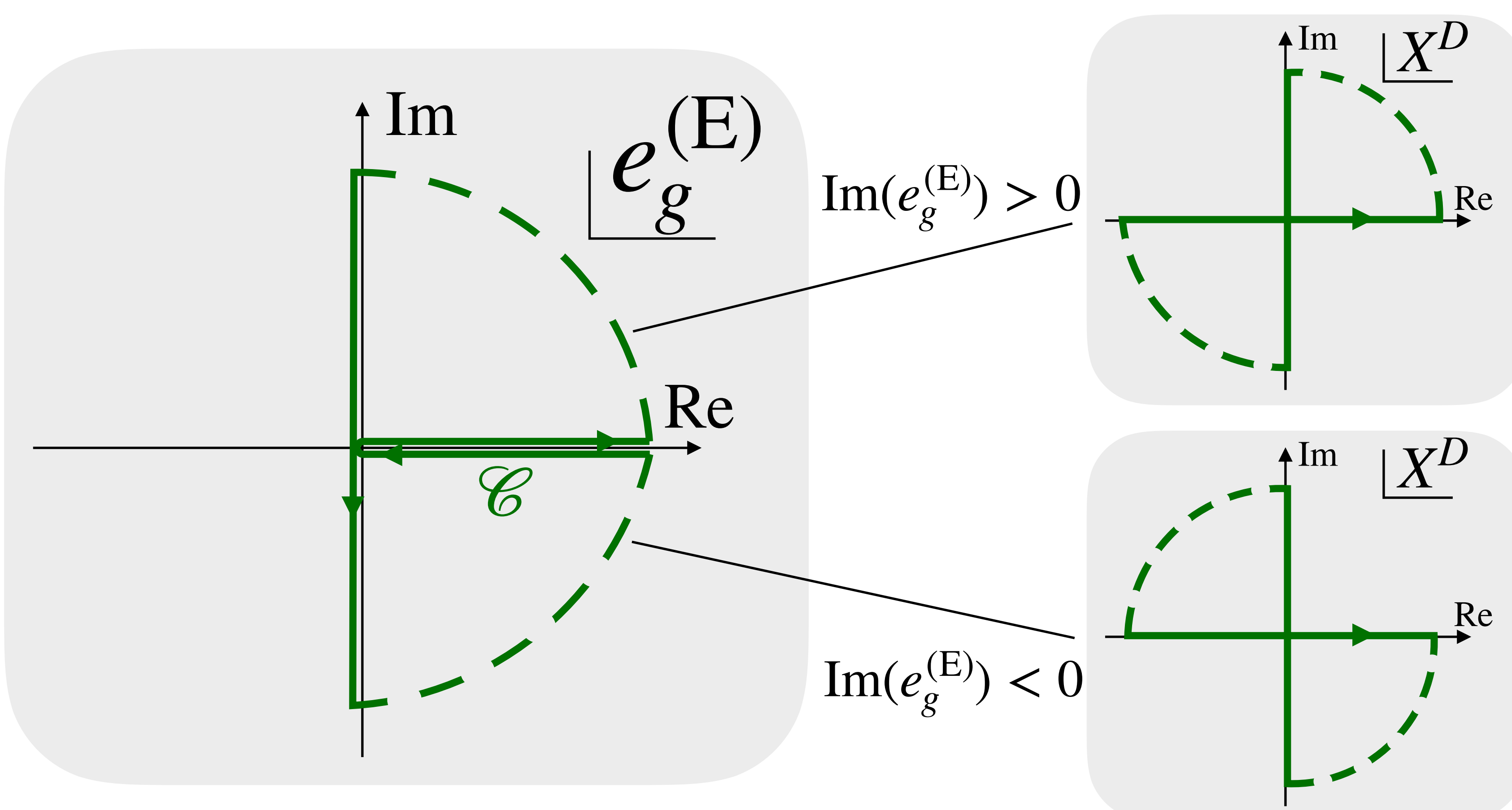


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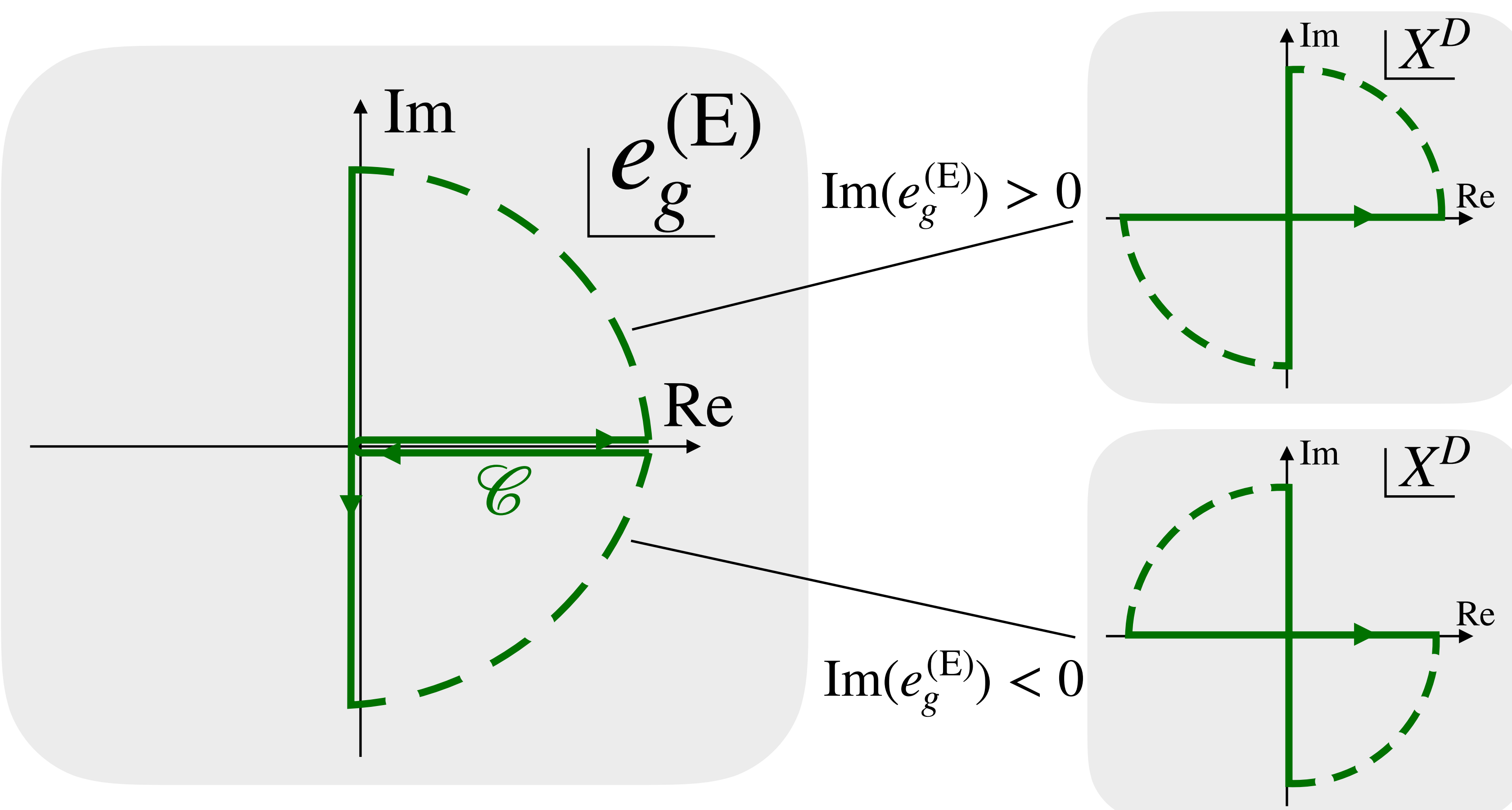


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Cauchy's integral thm. equates the Euclidean path integral to its Minkowskian version by this deformation of contour.

$$\ldots)^2\} > 0$$

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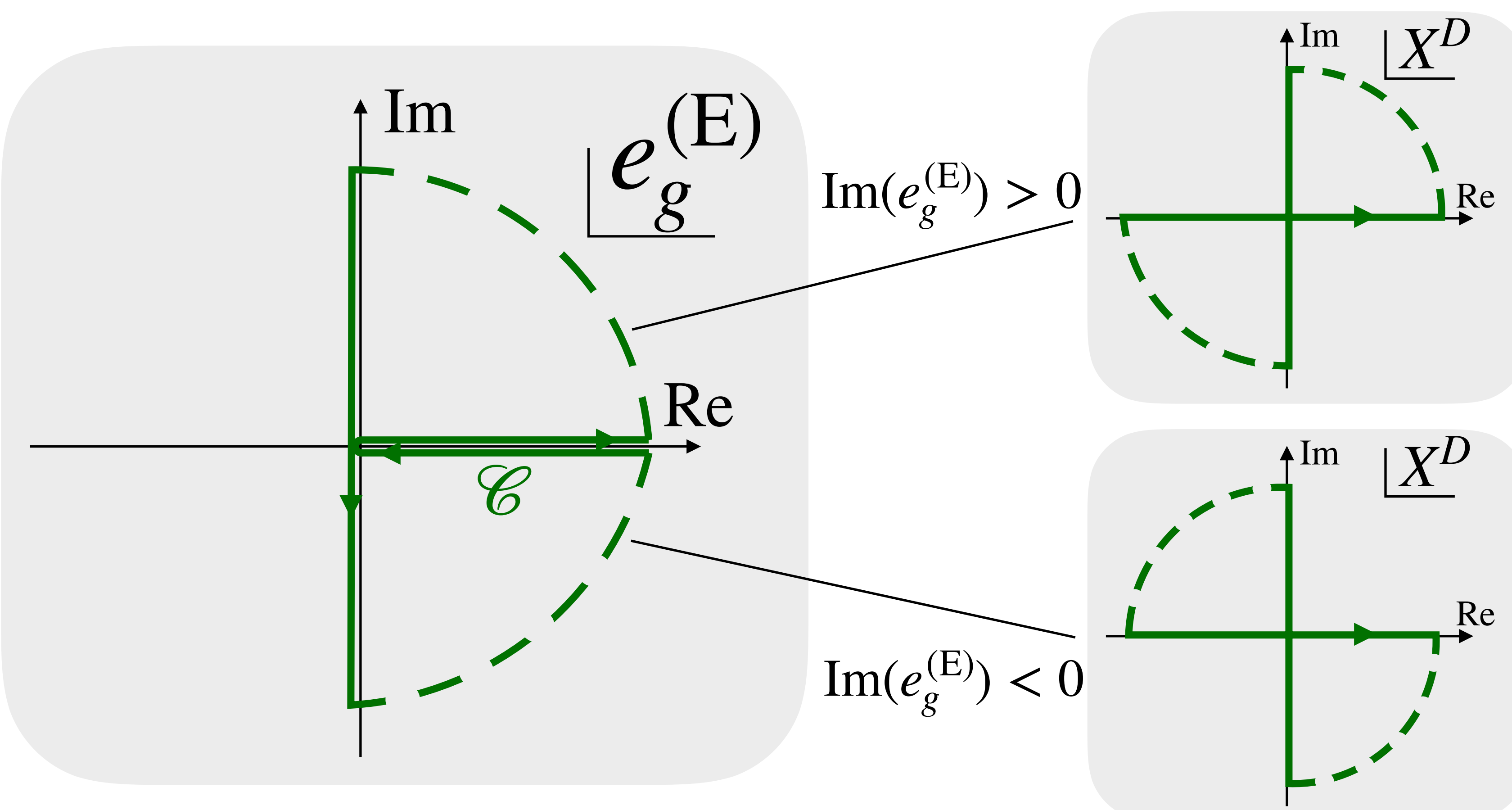


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$$1 = \left[ \prod_{\sigma} \int_{-\infty}^{\infty} \frac{d\Lambda_1}{(i\Lambda_0/h_{11})^{1/2}} \right] \exp \left[ \frac{i}{2} \int d^2\sigma \frac{h_{11}}{\Lambda_0} \left( \Lambda_1 - \frac{h_{10}}{h_{11}} \right)^2 \right]$$

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$$1 = \left[\prod_\sigma \int_{-\infty}^{\infty} \frac{d\Lambda_1}{(i\Lambda_0/h_{11})^{1/2}}\right] \exp\left[\frac{i}{2}\int d^2\sigma \frac{h_{11}}{\Lambda_0}\left(\Lambda_1 - \frac{h_{10}}{h_{11}}\right)^2\right]$$

$$e_g = \Lambda_0 h_{11}$$

$$g^{ab} = e^{-\phi}\begin{pmatrix} -\frac{1}{\Lambda_0} & \frac{\Lambda_1}{\Lambda_0} \\ \frac{\Lambda_1}{\Lambda_0} & \frac{-\Lambda_1^2 + \Lambda_0^2}{\Lambda_0} \end{pmatrix}$$

$$= \int DX \left[\prod_\sigma \frac{d\Lambda_0 d\Lambda_1}{(\Lambda_0)^2}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \sqrt{-g}\, g^{ab} h_{ab}\right]$$

Polyakov-type

Polyakov's Euclidean path int. is equivalent to its Minkowskian ver.

[Y.A. '24]

# Path integral—Polyakov to Nambu-Goto

The obtained Minkowskian Schild-type path integral effectively contains $i\epsilon$ terms:

$$Z = \int DX \left[ \prod_{\sigma} \int_{-\infty}^{\infty} \frac{-de_g}{(ie_g)^{3/2}} \right] \exp \left[ -\frac{i}{2} \int d^2\sigma \left\{ \frac{-h}{e_g} + e_g \qquad \right\} \right]$$

Schild-type

$e_g = \Lambda_0 h_{11}$

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Schild-type

$e_g = \Lambda_0 h_{11}$

Regulators for the convergence of the path integral:

- $i\epsilon$ terms are regarded as terms from the ground state wave function
- gauge invariant
- $\Lambda_a$ correspond to constraints: $\delta(\chi) = \int_{-\infty}^{\infty} \frac{d\Lambda}{2\pi} e^{i\Lambda\chi - \epsilon|\Lambda|}$ $\rightarrow e_g \in (-\infty, \infty)$

# Path integral—Polyakov to Nambu-Goto

Schild-type

$$\left[\prod_\sigma \int_{-\infty}^{\infty} \frac{-de_g}{(ie_g)^{3/2}}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \left\{ \frac{-h}{e_g} + e_g - i\epsilon\,|\,e_g\,| - i\frac{\tilde{\epsilon}}{|\,e_g\,|} \right\}\right]$$

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$$=\prod_{\sigma}\left(\frac{\sqrt{2\pi}}{\sqrt{-h-i\epsilon}}e^{-i\Delta\Sigma\sqrt{-h-i\epsilon}}+\frac{\sqrt{2\pi}}{\sqrt{-h+i\epsilon}}e^{i\Delta\Sigma\sqrt{-h+i\epsilon}}\right)$$

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$$= \left[\prod_\sigma \sum_{s(\sigma)=\pm 1} \frac{\sqrt{2\pi}}{\sqrt{-h-i\epsilon s}}\right] \exp\left[-i\int d^2\sigma\, s\sqrt{-h-i\epsilon s}\right]$$

Nambu-Goto-type

The Polyakov, Schild and Nambu-Goto types are quantum mechanically equivalent.

# Path integral—Polyakov to Nambu-Goto

Schild-type

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$$=\prod_{\sigma}\left(\frac{\sqrt{2\pi}}{\sqrt{-h-i\epsilon}}e^{-i\Delta\Sigma\sqrt{-h-i\epsilon}}+\frac{\sqrt{2\pi}}{\sqrt{-h+i\epsilon}}e^{i\Delta\Sigma\sqrt{-h+i\epsilon}}\right)$$

cancel if $h > 0$ → No space-like propagation

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Nambu-Goto-type

$\begin{cases} s=1\text{: F1} \\ s=-1\text{: anti-F1}\end{cases}$

The Polyakov, Schild and Nambu-Goto types are quantum mechanically equivalent.

The causality is realised by an anti-F1. [Y.A. '24]

# Toward the non-pert. definition

The path integral of perturbative string theory:

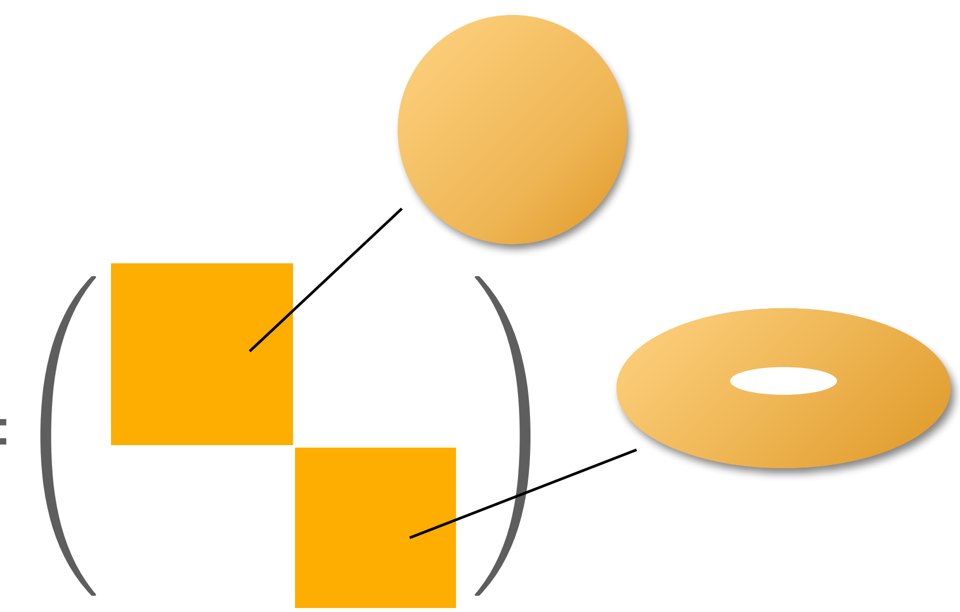
$$A(k_1, \cdots, k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi} \int DX\, D\theta\, De_g\, V(k_1)\cdots V(k_n)\ \exp\left[iS_{\mathrm{Schild}}\right]$$

… This is merely perturbation theory around the 10D flat spacetime.

Matrix regularisation

$$A(k_1, \cdots, k_n) = \int d\mu\, V(k_1)\cdots V(k_n)\ \exp\left[iS_{\mathrm{MM}}\right]$$

$X^\mu =$



We expect the matrices describe multi-body systems of superstrings.

# Matrix regularisation

A map of functions on a compact space to matrices [Hoppe '82]

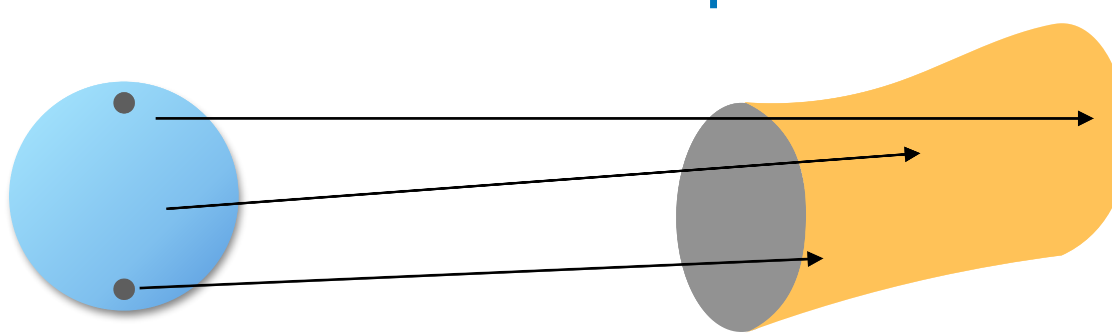
$$f(\sigma) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} \underline{Y_{lm}(\sigma)} \longmapsto \sum_{l=0}^{N-1} \sum_{m=-l}^{l} f_{lm} (Y_{lm})_{ij} = f_{ij}$$

fn. on $S^2$ — spherical harmonics — matrix

*Matrix regularisation of the Schild-type theory*

There are some ways to apply matrix regularisation.
In this talk, we consider matrix regularisation w/o Wick rotation:

Though the target space is Lorentzian, the worldsheet coordinates are just parameters. Consider compact worldsheet with punctures.



# Toward the non-pert. definition

We fix the fermionic gauge of the Schild-type theory by

$$\varphi = \theta^1 + i\theta^2 = 0 \qquad \psi = \theta^1 - i\theta^2$$

$$\{f, g\}_{\hat{\mathrm{P}}} := \varepsilon^{ab}\partial_a f \partial_b g$$

Then we obtain

$$S_{\mathrm{Schild}} = \frac{1}{2\pi}\int d^2\sigma \left[\frac{1}{4e_g}\{X^\mu, X^\nu\}_{\hat{\mathrm{P}}}^2 + 2i\psi^T\Gamma_\mu\{X^\mu, \psi\}_{\hat{\mathrm{P}}} - \frac{e_g}{2}\right]$$

[Ishibashi, Kawai, Kitazawa, Tsuchiya '96]

# Toward the non-pert. definition

We fix the fermionic gauge of the Schild-type theory by

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[Ishibashi, Kawai, Kitazawa, Tsuchiya '96]

By matrix regularisation, $\{\,\cdot\,,\,\cdot\,\}_{\hat{\mathrm{P}}} \mapsto \frac{N}{i}[\,\cdot\,,\,\cdot\,]$, $\quad \frac{1}{\pi}\int d^2\sigma \mapsto \frac{1}{N}\mathrm{tr}$,

with $e_g \mapsto -Y$, without Wick rotation

$$\int DX\, D\psi\, De_g\, e^{iS_{\mathrm{Schild}}}$$

# Toward the non-pert. definition

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with $e_g \mapsto -Y$, without Wick rotation

$$\int DX\, D\psi\, De_g\, e^{iS_{\mathrm{Schild}}} \longmapsto \int DX\, D\psi\, DY\, e^{iS_{\mathrm{NBI}}}$$

$$S_{\mathrm{NBI}} = N\,\mathrm{tr}\left(\frac{1}{4}Y^{-1}[X^\mu, X^\nu]^2 + \frac{1}{2}\psi^T\Gamma_\mu[X^\mu, \psi] + Y + \frac{i}{N}(N+\tfrac{1}{2})\ln(-iY)\right)$$

cf. [Fayyazuddin, Makeenko, Olesen, Smith, Zarembo '96]

※ The Euclidean IKKT is obtained by MR after Wick rot. w/ $e_g^2 = 1$

# Minkowskian “dielectric” NBI IKKT model

$$S_{\mathrm{NBI}} = N\,\mathrm{tr}\left(\frac{1}{4}Y^{-1}[X^\mu, X^\nu]^2 + \frac{1}{2}\psi^T\Gamma_\mu[X^\mu,\psi] + Y + \frac{i}{N}(N+\tfrac{1}{2})\ln(-iY)\right)$$

cf. [Fayyazuddin, Makeenko, Olesen, Smith, Zarembo ’96]

Unlike the IKKT model,
this NBI-type IKKT model explicitly holds the “causality” property:

$$\int DY\,\exp\left[iN\,\mathrm{tr}\left(\frac{1}{4}Y^{-1}M + \frac{1}{N^2}Y + \frac{i}{N}\left(N+\frac{1}{2}\right)\ln(-iY) + i\epsilon Y^2 + i\tilde{\epsilon}Y^{-2}\right)\right]$$

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$$\propto \Delta(m)^{-1}\det_{i,j}\left[\left(iN\frac{\partial}{\partial\alpha}\right)^{j-1}\left(\frac{e^{-i\sqrt{m_i - i\epsilon'}\sqrt{\alpha}}}{\sqrt{m_i - i\epsilon'}} + \frac{e^{i\sqrt{m_i + i\epsilon'}\sqrt{\alpha}}}{\sqrt{m_i + i\epsilon'}}\right)\right]_{\alpha\to 1}$$

$M := [X^\mu, X^\nu]^2$, $m_i$: the $i$th eigenvalue of $M$

This is zero if at least one eigenvalue of $M$ is negative.

⋯ similar to the cancellation in perturbative string theory

# Summary

- The Minkowskian perturbative superstring theory is quantum mechanically equivalent to its Euclidean version in terms of path integration.
- The Polyakov, Schild and Nambu-Goto-type formulations are quantum mechanically equivalent in the case of critical string theory (bosonic & type II).
- Full integration over the worldsheet metric provides the stringy causality. Since configs. with $\det h_{ab} > 0$ don't contribute to the path integral, string propagation with a space-like area is prohibited.
- We obtained the Minkowskian NBI-type IKKT model as a causal matrix model by matrix regularisation of the type IIB string.

  This partially answers how we define the IKKT model in the path-integral formalism, but it doesn't uniquely determine the matrix model (even whether it's Euclidean or Minkowskian).

# Backups

# Overview

## *Matrix model*

Matrix model is proposed as a non-perturbative formulation.

- c=1 matrix model: 1D matrix Q.M. (bosonic)

  ⋯ 2D bosonic/0B string theory

- BFSS model: 1D matrix Q.M. w/ SUSY

  ⋯ DLCQ M-theory

- IKKT model: 0D matrix model w/ SUSY

  ⋯ type IIB string theory

E.g. One-matrix model (0D matrix model)

$$S_{1\mathrm{M}} = N \operatorname{tr}\left(\frac{1}{2}M^2 + \frac{\lambda}{4}M^4\right)$$

$N \to \infty$ with

$g_s = N^{-1}(\lambda + \frac{1}{12})^{-5/4}$ fixed

correlator:

$$A(N,\lambda) = A_0 + \frac{(\lambda + \frac{1}{12})^{-\frac{5}{2}} A_1}{N^2} + \frac{(\lambda + \frac{1}{12})^{-5} A_2}{N^4} + \cdots$$

$$\to A_{\mathrm{DSL}}(g_s) = \sum_{h=0}^{\infty} g_s^{2h}\left(A_h + e^{-\alpha/g_s} A_h^{(1)} + \cdots\right)$$

This model describes 2D pure gravity/1D critical bosonic string

# Overview

## *The IKKT matrix model*

[Ishibashi, Kawai, Kitazawa, Tsuchiya ’96]

$$S[X,\psi] = N\,\mathrm{tr}\left[\frac{1}{4}[X^\mu,X^\nu][X_\mu,X_\nu] + \frac{1}{2}\psi^T\Gamma^\mu[X_\mu,\psi]\right]$$

$X^\mu$: bosonic $N \times N$ matrices ($\mu = 0,\cdots,9$)  $\psi$: Majorana-Weyl fermionic $N \times N$ matrices

This 0-dimensional theory is considered to describe type IIB superstring theory non-perturbatively. We believe this because it:

- has supersymmetry identical to that of type IIB string: $\mathcal{N} = (2{,}0)$ in (9+1)D
- reproduces perturbative results
  (graviton-exchange potential, D-brane scattering amplitudes, etc. )
- can reproduce the light-cone string field theory by the Schwinger-Dyson eq.
  [Fukuma, Kawai, Kitazawa, Tsuchiya ’97]
- has potential to dynamically realise (3+1)D space-time at large $N$
  - Dynamics of the diagonal elements of $X^\mu$ forms 4D [Aoki, Iso, Kawai, Kitazawa, Tada ’98]
  - SSB to SO(3) is observed [Anagnostopoulos, Azuma, Ito, Nishimura, Okubo, Papadoudis ’20; Kumar, Joseph, Kumar ’22]

# Overview

*Problem: How is the 0D theory defined?*

The IKKT action:

$$S[X,\psi;G_{\mu\nu}] = N\,\mathrm{tr}\left[\frac{1}{4}G_{\mu\rho}G_{\nu\sigma}[X^\mu,X^\nu][X^\rho,X^\sigma] + \frac{1}{2}\psi^T G_{\mu\nu}\Gamma^\mu[X^\nu,\psi]\right]$$

However, we don't really know how the IKKT action enters in the partition fn.

$$Z = \int [dX][d\psi]\, e^{iS[X,\psi;\eta_{\mu\nu}]} \text{ ?} \qquad \left(\eta_{\mu\nu} = \mathrm{diag}(-1,1,\cdots,1)_{\mu\nu}\right)$$

| | | metric in the action | |
|---|---|---|---|
| | | Euclidean | Minkowski |
| weight | Euclidean | $e^{-S[X,\psi;\delta_{\mu\nu}]}$ | $e^{-S[X,\psi;\eta_{\mu\nu}]}$ |
| | Minkowski | $e^{iS[X,\psi;\delta_{\mu\nu}]}$ | $e^{iS[X,\psi;\eta_{\mu\nu}]}$ |

"Euclidean IKKT model" ($e^{-S[X,\psi;\delta_{\mu\nu}]}$)

(Lorentzian) "Minkowskian IKKT model" ($e^{iS[X,\psi;\eta_{\mu\nu}]}$)

We will try to answer this by revisiting perturbative string theory.

# Overview

## *Ambiguity of the Minkowskian IKKT model*

The Minkowskian IKKT w/ $e^{iS}$ w/o regulators is conditionally convergent.

→ There are various definitions
depending on how to make it finite.

E.g.

1. Mass term + Lorentz-sym. breaking cutoff

$$S = N\,\mathrm{tr}\left[\frac{1}{4}[X^\mu, X^\nu]^2 + \gamma\,\mathrm{tr}(e^{i\epsilon}X^iX^i - e^{-i\epsilon}X^0X^0)\right]$$

- $\gamma \to 0^-$: equiv. to the Euclidean IKKT $\left(X^i = e^{-\frac{i}{4}\theta}\tilde{X}^i,\, X^0 = e^{\frac{3i}{4}\theta}\tilde{X}^{10}, \quad \theta:\ 0 \to \frac{\pi}{2}\right)$
- $\gamma \to 0^+$: a different theory

[Y.A., Nishimura, Piensuk, Yamamori, to appear]

2. Lorentz symmetry "gauge-fixed" model

$$Z = \int DX\,D\psi\,\Delta_{\mathrm{FP}}[X]\,\prod_{i=1}^{9}\delta(\mathrm{tr}(X^0X^i))\,e^{iS[X,\psi;\eta_{\mu\nu}]}$$

[Y.A., Nishimura, Piensuk, Yamamori '24;
Chou, Nishimura, Tripathi '25]

We do not discuss such ambiguity in this talk.

# Overview of the equivalences

$$A_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi} \int DX\,D\theta\,Dg\; V_{j_1}(k_1)\cdots V_{j_n}(k_n)\ \exp[-S_{\rm P}^{({\rm E})}]$$

Polyakov's Euclidean path int.

⇕ ※ equiv. at least after gauge-fixing $\kappa$ sym.

Minkowskian path int. w/ the Polyakov-type action

⇕

Schild-type action

⇕

Nambu-Goto-type action

The equivalences hold for critical type IIB and IIA string and critical bosonic string theory on the flat target space.

# Minkowskian "dielectric" NBI IKKT model

The large-$N$ limit reproduces the perturbative string theory up to a measure factor.

$$\int DY \exp\left[iN\,\mathrm{tr}\left(\frac{1}{4}Y^{-1}M+\frac{1}{N^2}Y+\frac{i}{N}\left(N+\frac{1}{2}\right)\ln(-iY)+i\epsilon Y^2+i\tilde{\epsilon}Y^{-2}\right)\right]$$

$$\propto \Delta(m)^{-1}\det_{i,j}\left[\left(iN\frac{\partial}{\partial\alpha}\right)^{j-1}\left(\frac{e^{-i\sqrt{m_i-i\epsilon'}\sqrt{\alpha}}}{\sqrt{m_i-i\epsilon'}}+\frac{e^{i\sqrt{m_i+i\epsilon'}\sqrt{\alpha}}}{\sqrt{m_i+i\epsilon'}}\right)\right]_{\alpha\to 1}$$

$$=\left(\frac{N}{2}\right)^N\left[\prod_{i=1}^{N}\sum_{s_i=\pm 1}\right]\frac{1}{\prod_{i,j<i}(s_i\sqrt{m_i-i\epsilon' s_i}+s_j\sqrt{m_j-i\epsilon' s_j})}\prod_{i=1}^{N}\frac{e^{-is_i\sqrt{m_i-i\epsilon' s_i}}}{\sqrt{m_i-i\epsilon' s_i}}$$

$$\to\left[\prod_{\sigma}\sum_{s(\sigma)=\pm 1}\frac{1}{\sqrt{-h-i\epsilon s}}\right]\mathscr{M}[-h(\sigma),s(\sigma)]\exp\left[-i\int d^2\sigma\, s\sqrt{-h-i\epsilon s}\right]$$

cf. [Fayyazuddin, Makeenko, Olesen, Smith, Zarembo '96]

※ $\sum_i \sqrt{m_i}\to\int d^2\sigma\sqrt{-\frac{1}{2}\{X^\mu,X^\nu\}^2}=\int d^2\sigma\sqrt{-h}$ as $N\to\infty$

⋯ the inverse of matrix regularisation

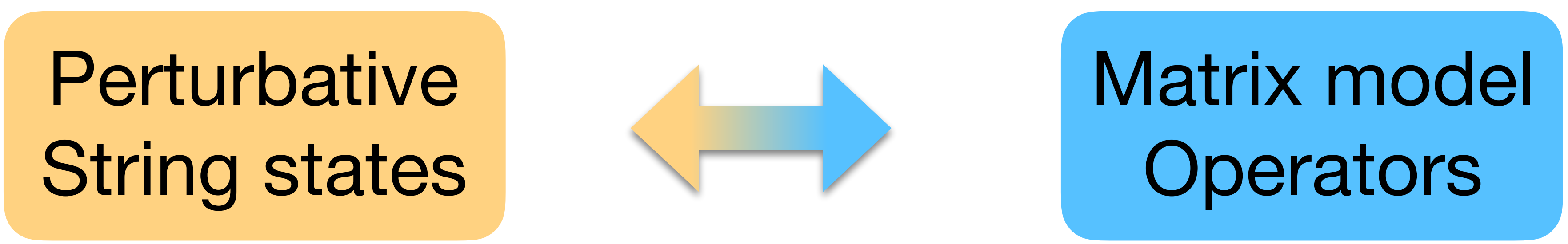
# Future work

- It's important to establish the exact relationship between perturbative superstring d.o.f. and matrix d.o.f. cf. SFT: [Fukuma, Kawai, Kitazawa, Tsuchiya '97] string states: [Iso, Kawai, Kitazawa '00; Steinacker '16]

In the matrix model, we have a vertex op. $V^{\Phi} = \mathrm{tr}\, e^{ik_\mu X^\mu}$

This forms a massless multiplet of type-IIB SUGRA by acting the supercharge operator $Q$ onto this vertex.

[Kitazawa '02; Iso, Terachi, Umetsu '04; Kitazawa, Mizoguchi, Saito '07]




Amplitudes computed by the vertex operators in the NBI model are expected to reproduce the genus expansion via the matrix reg.
Is it also obtained by a 1/N expansion?

- A mass term may be essential to define the IKKT model.

The Polarised IKKT model is singular and subtle. [Hartnoll, Liu '24; Komatsu et al. '24]
[Benelli '02]

➡ Double scaling limit as $\Omega \to 0$ (massless lim.)?

# Path integral—point particle

$$S_{\text{pp}} = -m\int d\tau\sqrt{-\dot{X}_\mu \dot{X}^\mu} \qquad P_\mu = \frac{m\dot{X}_\mu}{\sqrt{-\dot{X}^2}} \qquad P_\mu P^\mu = -m^2$$

$$H = \frac{1}{2}\Lambda(P^2 + m^2) \qquad \dot{X}^\mu = \Lambda P^\mu$$

$$S_{\text{Schild}} = \int d\tau \frac{1}{2}\left(\frac{\dot{X}^2}{\Lambda} - m^2\Lambda\right) \qquad S_{\text{P}} = \int d\tau \frac{\sqrt{-g_{00}}}{2}\left(-g_{00}^{-1}\dot{X}^2 - m^2\right)$$

$$\sqrt{-g_{00}} = \Lambda \qquad \|\delta g\|^2 = \frac{1}{2}\int d\tau \sqrt{-g_{00}}\, g^{00}\delta g_{00} g^{00}\delta g_{00} = \int d\tau\, \frac{2\delta\Lambda^2}{\Lambda}$$

$$Z_{\text{b}} = \int DX\left[\prod_\tau \int_{-\infty}^{\infty} \frac{d\Lambda}{(i\Lambda)^{\frac{1}{2}}}\right] e^{iS_{\text{Schild}}}$$

$$= \int DX \prod_\tau \frac{-i\sqrt{2\pi}}{m\sqrt{\Delta\tau}}\left(e^{-im\Delta\tau\sqrt{-\dot{X}^2 - i\epsilon}} - e^{im\Delta\tau\sqrt{-\dot{X}^2 + i\epsilon}}\right)$$

$$\propto \int DX\left[\prod_\tau \sum_{s(\tau)=\pm 1}\right]\exp\left[-im\int d\tau\, s\sqrt{-\dot{X}_\mu\dot{X}^\mu - is\epsilon}\right]$$

signed line element $d\tau\, s = d\tau' \frac{d\tau}{d\tau'}$ w/ better parameter $\tau'$