Convex orders and quantum tangent spaces

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CaLiSTA meeting Corfù-

(Joint work with Christophe Hohlweg and Paolo Papi)

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- This work is a collaboration with P. Papi and C. Hohlweg. It is a continuation of the work with the (quantum) group of Prague on quantum homogeneous spaces of $U_q(\mathfrak{g})$.
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- It follows the line of research on the building up of a theory of Noncommutative differential geometry for quantum flag manifolds.
- A question in Noncommutative Differential Geometry inspired a problem in combinatorial geometry.
- Convex orders on the set of positive roots of a semisimple Lie algebra play an important role when one wants to understand better the interplay between quantum root vector and the coproduct in $U_q(\mathfrak{g})$.

Quantum tangent spaces and Luzstig quantum root vectors

- $\mathfrak{g}$ is a complex, semisimple Lie algebra. The quantized enveloping algebra $U_q\mathfrak{g}$ is generated by $\langle E_i, F_i, K_i^{\pm 1} \mid i = 1, \ldots, \mathrm{rank}(\mathfrak{g})\rangle$.

$$\Delta(K_i) = K_i \otimes K_i,$$
$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$$

- We define dually $\mathcal{O}_q(G)$ the **quantum coordinate algebra** of G (the unique simple, simply connected, complex Lie group associated to $\mathfrak{g}$.)

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- We are looking for a first order differential calculus on $A = \mathcal{O}_q(G)$
  - Ω^1 is a A-bimodule
  - $\mathrm{d} : A \to \Omega^1$ is a derivation (**the exterior derivative**).
  - Ω^1 is generated as a left B-module by those elements of the form $\mathrm{d}b$, for $b \in B$.

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- We can describe it dually by means of a quantum tangent space: that is $T \subseteq U_q(\mathfrak{g})$ such that

$$\Delta(T) \subseteq (T \oplus \mathbb{C}) \otimes U_q(\mathfrak{g})$$

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- $\mathcal{B}_{\mathfrak{g}}$ the **braid group of** $\mathfrak{g}$ is the group generated by $T_i \quad 1 \leq i \leq l$ with relations

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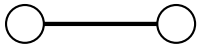
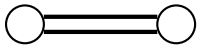
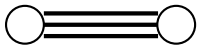
$T_i T_j T_i = T_j T_i T_j$	
$T_j T_i T_j T_i = T_i T_j T_i T_j$	
$T_j T_i T_j T_i T_j T_i = T_i T_j T_i T_j T_i T_j$	

- $\mathcal{W}_{\mathfrak{g}}$, the **Weyl group of** $\mathfrak{g}$, is generated by w_i with the same relations and additionally

$$w_i^2 = 1.$$

- During the '90 Lusztig gave a representation $\mathcal{B}_{\mathfrak{g}} \times U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$
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- For $X, Y \in U_q(\mathfrak{g})$ one has $T(XY) = T(X)T(Y)$
- The action is given explicitely on the generators and can then be extended. For example we have

$$\begin{aligned}
T_i(F_j) &= F_j && \text{when} \quad a_{ij} = 0\\
T_i(F_j) &= [F_j, F_i]_q && \text{when} \quad a_{ij} = -1\\
T_i(F_j) &= [F_j, [F_j, F_i]_{q^0}]_{q^2} && \text{when} \quad a_{ij} = -2\\
T_i(F_j) &= [F_j, [F_j, [F_j, F_i]_{q^{-1}}]_{q^1}]_{q^3} && \text{when} \quad a_{ij} = -3
\end{aligned}$$

(Where $[X, Y]_q = XY - qYX$.)

- Notice that this representation **does not give a coalgebra homeomorphism.**

- In the universal enveloping $U(\mathfrak{g})$ we have root vectors for every root of $\mathfrak{g}$.
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- Let $w = w_{i_1} \dots w_{i_n}$ be a reduced decomposition of the longest element of $\mathcal{W}_{\mathfrak{g}}$, denote by α_i the simple roots of $\mathfrak{g}$. The list

$$\beta_1 = \alpha_{i_1} \quad \beta_k = w_{i_1} \dots w_{i_{k-1}}(\alpha_{i_k})$$

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- The elements

$$E_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}), \quad F_{\beta_k} = T_{i_1} \dots T_{i_{k-1}}(F_{i_k})$$

from $U_q(\mathfrak{g})$ are the **quantum root vectors** of $U_q(\mathfrak{g})$ corresponding to the root β and $-\beta$ respectively.

- For each $\mathfrak{g}$ we can look for a reduced decomposition $w_0 = w_{i_1} \dots w_{i_n}$ such that the corresponding $\{F_{\beta_i} K_{\beta_i}\}_{i=1}^n$ span a quantum tangent space in $U_q(\mathfrak{g})$.

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- **Non-Example**: For $\mathfrak{g} = \mathfrak{sl}_4$, consider the decomposition

$$w_0 = w_2 w_1 w_2 w_3 w_2 w_1$$

we have

$$\Delta F_{\beta_4} = F_{\beta_4} \otimes 1 + (q - q^{-1}) F_{\beta_1} F_{\beta_3} K_6^{-1} \otimes F_{\beta_6} + K_{\beta_4}^{-1} \otimes F_{\beta_4}.$$

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- **Example [R. Ó Buachalla–P. Somberg]**: Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, then the two reduced decompositions

$$w_1 w_2 \dots w_n w_1 \dots w_{n-1} w_1 \dots w_1 w_2 w_1 \quad \text{and}$$

$$w_n w_{n-1} \dots w_1 w_n \dots w_2 w_n \dots w_n w_{n-1} w_n$$

give rise to two tangent spaces on $U_q(\mathfrak{g})$ spanned by their respective $\{F_\beta K_\beta\}$.

Convex orderings on the set of positive roots

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- Denote by Δ^+ the set of positive roots of $\mathfrak{g}$.
- We say that a total ordering $\leq$ on Δ^+ is a **convex ordering** when the following holds:

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- It turns out that the coproduct of quantum root vectors can see the convex ordering!

Proposition (A.C., P. Papi, C. Hohlweg)

Let $w_0 = w_{i_1} \dots w_{i_n}$ be a reduced decomposition of w_0. The coproduct of the negative quantum root vectors F_{β_k} reads:

$$\Delta(F_{\beta_k}) = \sum c_{i_{p_1},\dots i_{p_r}}^{i_{q_1},\dots i_{q_s}} F_{\beta_{i_{p_1}}} \dots F_{\beta_{i_{p_r}}} \otimes F_{\beta_{i_{q_1}}} \dots F_{\beta_{i_{q_s}}}$$

where $c_{i_{p_1},\dots i_{p_r}}^{i_{q_1},\dots i_{q_s}}$ are polynomials in q, q^{-1} and, with respect to the convex order induced by **i** *we have*

$$\beta_{i_{p_1}} \leq \dots \beta_{i_{p_r}} \leq \beta_k \leq \beta_{i_{q_1}} \leq \dots \beta_{i_{q_s}}.$$

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- The coproduct of a quantum root vector separates the quantum root vectors with respect to the convex order!
- The reduced decomposition $\mathbf{i} = (i_1, \dots i_n)$ span a left quantum tangent space if the only nonzero coeeficients are the ones $c^{i_{q_1},\dots i_{q_s}}_{i_{p_1}}$.

The combinatorial description of quantum tangent spaces

- We want to know exactly which terms appear in the corpoduct of F_{β_k}. This is not obvious at all with our current presentation.
- For example: what are the terms that appear in the coproduct of

$$F_{\alpha_2+\alpha_3} = T_1 T_2 T_3 T_1(F_2)?$$

- **Rough Idea** "Use the braid relations to write a quantum root vector in the shortest possible form"

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Theorem (AC-C. Hohlhweg - P. Papi)

Let $w_k = s_{i_1} \dots s_{i_k}$, *and* $\beta = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$. *The set*

$$\{v \leq w_k \mid \beta \in N(v)\}$$

has a minimum $\bar{w}_k$. *Moreover, let* $\bar{w}_k = s_{j_1} \dots s_{j_l}$ *be a reduced decomposition, we have*

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- We call $s_{j_1} \dots s_{j_l}$ a **reduced expression** for F_{β_k}.
- We call the elements $F_{j_1}, T_{j_1}(F_{j_2}), \dots$ the **prefix root vectors** of F_{β_k}.

Theorem

Let F_{β_k} *be a quantum root vector with reduced expression* $\mathbf{j}$, *that is*

$$F_{\beta_k} = T_{j_1} \dots T_{j_l}(F_{j_{l+1}}).$$

The terms on the left-hand side of the coproduct of F_{β_k} *are products of its prefix root vectors.*

- We can read the terms that appear in the coproduct of F_{β_k} directly from its reduced expression!
- One just have to make the comparison between the prefix root vectors $\{\tilde{F}^k_{\beta_i}\}$ and quantum root vectors $\{\tilde{F}_{\beta_i}\}$.

- Let $\overline{w}_k$ be the minimum of the set $\{v \leq w_{i_1} \dots w_{i_k} \mid \beta_k \in N(v)\}$. Fix a reduced decomposition $\overline{w}_k = w^k_{j_1} \dots w^k_{j_r}$.

Theorem

The quantum root vectors $\{F_{\beta_k}\}$ associated to $\mathbf{i}$ *span a quantum tangent space iff for every k the following hold*

1. $\{\tilde{F}_{\beta_i,k}\} = \{F_{\beta_i}\}$.
2. *Suppose $\beta = \gamma + \delta$, $\gamma < \beta$ in the order induced $<$ by* $\mathbf{j}$*, $\delta \in Q^+$. Then, for every decomposition of δ into a sum of simple positive roots, all summands follow β in the order.*

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Proposition

Let $\mathfrak{g} = \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_8$. There exist no reduced decomposition of w_0, the longest element of $\mathcal{W}_\mathfrak{g}$ such that the corresponding negative quantum root vectors span a quantum tangent space in $U_q(\mathfrak{g})$.

- We can give explicitely all the reduced decomposition of w_0 that give a quantum tangent space for given $\mathfrak{g}$!
- Two reduced decomposition give the same set of quantum root vectors if and only if they are in the same commutation class. We consider reduced decomposition up to this equivalence.

- We can give explicitely all the reduced decomposition of w_0 that give a quantum tangent space for given $\mathfrak{g}$!
- Two reduced decomposition give the same set of quantum root vectors if and only if they are in the same commutation class. We consider reduced decomposition up to this equivalence.
- For a simple root $\alpha_i \in \Pi$, let $W_{(i)} := W_{S/\{s_i\}}$ with longest element denoted by $w_{0,(i)}$. Let $d^{\Pi}_{(i)} = w_0 w_{0,(i)}$. We can iterate this construction by removing one simple root at the time, considering for example $d^{\Pi/(i)}_{(i)}$.

Theorem

The negative quantum root vectors corresponding to a reduced decomposition of w_0 span a quantum tangent space iff the reduced decomposition is equivalent to

$$w_0 = d^{\Pi}_{(i_1)} d^{\Pi/\{i_1\}}_{(i_2)} \dots d^{\{s_{i_{n-1}}, s_{i_n}\}}_{s_{i_{n-1}}} s_{i_n}$$

where α_{i_k} is cominuscule in the root system spanned of the Dynkin diagram of $\Pi/\{i_1, \dots, i_{k-1}\}$.

- Let's take as first example A_4. Let's say we want $\alpha_2 < \alpha_1 < \alpha_3 < \alpha_4$.
- We obtain the reduced decomposition $w_0 = d^{\Pi}_{(2)} d^{\Pi/\{2\}}_{(1)} d^{s_3,s_4}_{(3)} s_4 = 124312 \cdot 1 \cdot 34 \cdot 3$.

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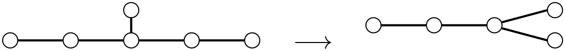
- Why not E_6? We follow a step-by-step approach. Here we have

$$theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$$

So we have to start with either $d^{E_6}_{(1)}$ or $d^{E_6}_{(5)}$. We then remove the corresponding node:



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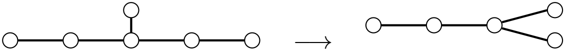

- We now have a D_5 root system, where

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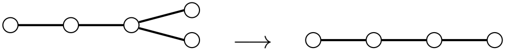
and we can take either $d^{D_5}_{(1)}$, $d^{D_5}_{(4)}$, or $d^{D_5}_{(6)}$. The next step is



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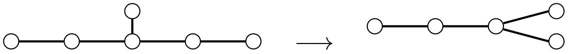

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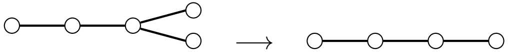
$$\theta' = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6$$

and we can take either $d_{(1)}^{D_5}$, $d_{(4)}^{D_5}$, or $d_{(6)}^{D_5}$. The next step is



- And finally we can repeat what we have already done with A_4! So a good reduced decomposition of w_0 for E_6 is given by

$$w_0 = d_{(5)}^{E_6} \cdot d_{(1)}^{D_5} \cdot d_{(6)}^{A_4} \cdot d_{(4)}^{A_3} \cdot 23 \cdot 2.$$

Thank you!