BASIC CURVATURE TENSORS

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Based on collaborations & WiP with:

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A MOTIVATING EXERCISE

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The curvature of a connection ∇ on a manifold M is

$$R_\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \,.$$

Suppose that it is a torsion-free connection, $T_\nabla = 0$, in which case

$$\nabla_X Y - \nabla_Y X = [X, Y] \,.$$

A little algebra and the algebraic Bianchi identity for the curvature reveal that

$$R_\nabla(X, Y)Z = -\Big(\nabla_Z[X, Y] - [\nabla_Z X, Y] - [X, \nabla_Z Y] + \nabla_{\nabla_X Z} Y - \nabla_{\nabla_X Y} Z\Big) =: -S_\nabla(X, Y)Z \,.$$

The right hand side measures whether the connection is a derivation of the Lie bracket...

The ("basic curvature") tensor S_∇ has a separate life, in general.

In the present case of the Lie algebroid $(TM, \mathrm{id}, [-, -])$ it is the (opposite of) curvature.

MOTIVATIONS

- General Lie algebroids are important in various contexts.

  constrained Hamiltonian systems, gauge theory, Poisson geometry, foliation theory &c.

  mergers of Lie algebras and tangent bundles

- Not only Lie, but also higher algebroid structures, e.g. Courant & Lie n-algebroids.

  strings & generalised geometry, flux backgrounds, topological field theories in higher dimensions, &c.

- Connections & tensors are also important ingredients.

  Geometric structure of variational problems, representation theory for algebroids, covariant formulations, &c.

CONTENT

Focus on the “basic curvature” (compatibility of connections & brackets), & on its relation to other notions & constructions.

In this talk:

- LAoid representations up to homotopy (ruths), Lie 2-algebroids.
- (Open) Gauge algebras in topological sigma models.

Not in this talk:

- Supersymmetric Poisson sigma models. see Thomas Basile’s talk.
- Atiyah class of dg-manifolds. WiP w/ Lara Jonke & Dima Roytenberg; next year’s talk?

BASIC CONNECTIONS & BASIC CURVATURE

Consider the pair (E, ∇) of a LAoid $(E \to M, \rho : E \to TM, [-,-] : E \wedge E \to E)$
and a "TM-on-E" linear connection $\nabla : \Gamma(TM \otimes E) \to \Gamma(E)$. R_∇ defined as usual, T_∇ in general not.

A canonical induced E-on-E connection is always defined: $\dot{\nabla}^E_e := \nabla_{\rho(e)}$.
Does a torsion-free E-on-E connection exist?

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Does a torsion-free E-on-E connection exist? Yes, by "averaging with the opposite":

$$\nabla^E_{\text{avg}} = \frac{1}{2}(\dot{\nabla}^E + \overline{\nabla}^E), \qquad \overline{\nabla}^E_e e' = \nabla_{\rho(e')} e + [e, e'] \quad \rightsquigarrow \quad T_{\nabla^E_{\text{avg}}} = 0\,.$$

BASIC CONNECTIONS & BASIC CURVATURE

Consider the pair (E, ∇) of a LAoid $(E \to M, \rho : E \to TM, [-,-] : E \wedge E \to E)$
and a "*TM*-on-*E*" linear connection $\nabla : \Gamma(TM \otimes E) \to \Gamma(E)$. R_∇ defined as usual, T_∇ in general not.

A canonical induced *E*-on-*E* connection is always defined: $\dot{\nabla}^E_e := \nabla_{\rho(e)}$.
Does a torsion-free *E*-on-*E* connection exist? Yes, by "averaging with the opposite":

$$\nabla^E_{\rm avg} = \frac{1}{2}(\dot{\nabla}^E + \overline{\nabla}^E)\,, \qquad \overline{\nabla}^E_e e' = \nabla_{\rho(e')} e + [e, e'] \quad \rightsquigarrow \quad T_{\nabla^E_{\rm avg}} = 0\,.$$

What is the curvature of these two induced connections?

$$R_{\dot{\nabla}^E}(e, e')e'' = R_\nabla(\rho(e), \rho(e'))e''\,, \qquad R_{\overline{\nabla}^E}(e, e')e'' = -S_\nabla(e, e')\rho(e'')\,,$$

where the basic curvature tensor of the LAoid connection ∇ is defined as $X \in \Gamma(TM)$

$$S_\nabla(e, e')X = \nabla_X[e, e'] - [\nabla_X e, e'] - [e, \nabla_X e'] - \nabla_{\overline{\nabla}^E_{e'} X} e + \nabla_{\overline{\nabla}^E_e X} e'\,,$$

which measures the compatibility of ∇ and $[-,-]$ and uses an *E*-on-*TM* connection:

$$\overline{\nabla}^E_e X = \rho(\nabla_X e) + [\rho(e), X]\,, \qquad R_{\overline{\nabla}^E}(e, e')X = -\rho\big(S_\nabla(e, e')X\big)\,.$$

INTERMEZZO: CARTAN–LIE ALGEBROIDS

The concept of basic curvature was implicitly introduced in Blaom '04.

A Cartan connection on a LAoid is one whose basic curvature vanishes.

A Cartan-LAoid is a LAoid equipped with a Cartan connection.

A LAoid is associated to a LAa action iff it admits a flat ∇ s.t. $S_\nabla = 0$. Blaom '04; Abad, crainic '09

A bundle of LAas if a LAa bundle iff it admits a ∇ s.t. $S_\nabla = 0$. Abad, Crainic '09

RUTHS

As for Lie algebras, representations of Lie algebroids are flat E-on-V connections ∇^E.

Too restrictive. More generally, representations up to homotopy. Abad, Crainic '09

A LAoid E and a graded VB $\mathcal{V}$ (chain complex) with a flat-uth E-on-$\mathcal{V}$ connection.

Ingredients:

- deg 1 operator $\partial : \mathcal{V} \to \mathcal{V}$ s.t. $(\mathcal{V}, \partial)$ is a complex.
- E-on-$\mathcal{V}$ connection: ∇^E.
- deg 1, End-valued 2-form $\omega_2 \in \Omega^2(E; \mathrm{End}^{-1}(\mathcal{V}))$ s.t. $[\partial, \omega_2] + R_{\nabla^E} = 0$.
- deg 1, End-valued n-forms, $\omega_{n>2} \in \Omega^n(E; \mathrm{End}^{1-n}(\mathcal{V}))$ s.t. recursive relations ...

ADJOINT & CO.

The adjoint representation of a LAoid is a ruth with $\mathcal{V}$ the adjoint complex $E \xrightarrow{\rho} TM$.

Abad, Crainic '09

Ingredients:

- $\partial = \rho$.
- $\nabla^E = \overline{\nabla}^E$, the basic connection:

$$\overline{\nabla}^E_e \begin{pmatrix} e' \\ X \end{pmatrix} = \begin{pmatrix} \nabla_{\rho(e')} e + [e, e'] \\ \rho(\nabla_X e) + [\rho(e), X] \end{pmatrix}, \quad \text{satisfying} \quad [\overline{\nabla}^E, \rho] = 0\,.$$

- $\omega_2 = S_\nabla$, the basic curvature, satisfying: $R_{\overline{\nabla}^E} + [\rho, S_\nabla] = 0\,.$
- $\omega_{n>2} = 0$, since the basic curvature is $\mathrm{d}_{\overline{\nabla}^E}$-closed.

The coadjoint representation is a ruth with $\mathcal{V}$ the dual complex $T^*M \xrightarrow{\rho^*} E^*$ and dual data.

GAUGE ALGEBRA OF (TWISTED) PSM

In physics, the basic curvature appears in the Poisson sigma model. Ikeda, Strobl '19

2D dilaton gravity, deformation quantization of Poisson manifolds, AKSZ construction, ...

The PSM is a 2D gauge theory of scalars X^μ and 1-forms A_μ, with open gauge algebra

$$[\delta_1, \delta_2] A_\mu = \delta_{12} A_\mu + S_{\mu\nu}{}^{\rho\sigma}(X) \epsilon_{1\rho} \epsilon_{2\sigma} F^\nu \,, \quad \epsilon_{12\mu} = \nabla_\mu \Pi^{\nu\rho}(X) \epsilon_{1\nu} \epsilon_{2\rho} \,.$$

Bonus: this remains true for the 3-form twisted PSM for a connection with torsion.

when the target space has a twisted Poisson structure Ševera, Weinstein '01

Another bonus: still true for other LAoid-based models, like Dirac sigma models.

Th. Ch., Jonke '23

Remark: ∇ plays an auxiliary role here, manifesting target space covariance.

Baulieu, Losev, Nekrasov '01

FOR COURANT ALGEBROIDS . . .

Courant algebroid: $(E, [-,-], \rho : E \to TM, \langle\cdot,\cdot\rangle \equiv \eta)$, $(\Gamma(E), [-,-])$ is a Leibniz algebra,

$$[e,[e',e'']] = [[e,e'],e''] + [e',[e,e'']]\,,$$

$$\eta(e,[e',e']) = \tfrac{1}{2}\rho(e)\eta(e',e') = \eta([e,e'],e')\,.$$

Consider a *TM*-on-*E* connection ∇. Is it a derivation of the Dorfman bracket $[-,-]$?

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Consider a TM-on-E connection ∇. Is it a derivation of the Dorfman bracket $[-,-]$?

Basic curvature of ∇ w.r.t $[-,-]$ is defined, using $\overline{\nabla}^E_e X = \rho(\nabla_X e) + [\rho(e), X]$, as:

Th. Ch., Jonke '23

$$S_\nabla(e,e')X = \nabla_X[e,e'] - [\nabla_X e, e'] - [e, \nabla_X e'] - \nabla_{\overline{\nabla}^E_{e'}X} e + \nabla_{\overline{\nabla}^E_e X} e' + \eta(\nabla_{\overline{\nabla}^E X} e, e')\,.$$

FOR COURANT ALGEBROIDS ...

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Remarks:

- The last term is there to tensorize it, cf. torsion of CAoid E-on-E connections

  Gualtieri '07

$$T_{\nabla^E}(e,e') = \nabla^E_e e' - \nabla^E_{e'} e - [e,e'] + \eta(\nabla^E e, e')\,.$$

- The naive formula for E-on-E basic connection is a Dorfman connection Jotz-Lean '12

$$D_e e' = \nabla_{\rho(e')} e + [e,e'] \qquad vs. \qquad \overline{\nabla}^E_e e' = \nabla_{\rho(e')} e + [e,e'] - \eta(\nabla_{\rho(-)} e, e')\,.$$

  Dorfman connections are not homogeneous: $D_{f\,e} e' \neq f\, D_e e'$. "Fork in the road".

GAUGE ALGEBRA OF (TWISTED) CSM

The CAoid basic curvature appears in the gauge algebra of Courant sigma models.

Th. Ch., Ikeda, Jonke '24

The CSM is a 3D gauge theory of scalars X^μ, 1-forms $A^a = (A_\mu, A^\mu)$ and 2-forms B_μ.

in fact, even to define it in terms of differential forms, you need a connection

2 BF theories in 3D: scalar/2-form and 1-form/1-form (Chern-Simons): couple them.

Ikeda; Hofman, Park; Roytenberg

$$S_{\text{CSM}} = \int -B_\mu \wedge \mathrm{d}X^\mu + \frac{1}{2}\eta_{ab}A^a \wedge \mathrm{d}A^b + \rho_a{}^\mu(X)B_\mu \wedge A^a + \frac{1}{3!}C_{abc}(X)A^a \wedge A^b \wedge A^c\,.$$

The gauge algebra is once again open and in the covariant formulation it features S_∇:

$$[\delta_1, \delta_2]B_\mu = \delta_{12}B_\mu + (\nabla_\mu S_{\nu abc}\,\epsilon_1^a\,\epsilon_2^b\,A^c + \dots) \wedge F^\nu + (\dots)F^\nu \wedge F^\rho + S_{\mu abc}\,\epsilon_1^a\,\epsilon_2^b\,G^c\,.$$

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Bonus: it allows for straightforwardly introducing the effect of a WZ term.

models based on 4-form twisted CAois / twist being responsible for the nonlinear openness of the gauge algebra

Remark: analogous notions for examples of Ševera's higher-dim Hamiltonian mechanics.

Th. Ch. '21; Th. Ch., Kodzoman, Škoda '24

COURANT ALGEBROIDS & QP2 MANIFOLDS

- Graded-geometric description of CAoid as symplectic subman'd of $M_2 = T^*[2]E[1]$.
  Roytenberg
- A canonical degree 2 symplectic structure Ω and a homological vector field Q.

  $$|Q| = 1 \quad \text{and} \quad \{Q, Q\} = 0\,.$$

  Compatibility of the graded symplectic and Q structures.

  $$L_Q \Omega = 0\,.$$

  A degree 3 Hamiltonian function.

  $$Q = \{\Theta, -\}\,, \quad \{\Theta, \Theta\} = 0\,.$$

- The Dorfman bracket is a derived bracket, together with ρ and η are given as

  $$[e, e'] = \{\{\Theta, e\}, e'\}$$
  $$\rho(e)f = \{\{\Theta, e\}, f\}\,,$$
  $$\eta(e, e') = \{e, e'\}\,.$$

COURANT ALGEBROIDS & LIE 2–ALGEBROIDS

- In general, split Qn manifolds correspond to Lie n-algebroids.
  Voronov; Sheng, Zhu; Bonavolonta, Poncin; ...
- A Courant algebroid is not a split QP2 manifold, in general.
- Price to pay: an η-compatible TM-on-E connection $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$.
- Then the split graded vector bundle $T^*[1]M \oplus E$ admits a Lie 2-algebroid structure.
- The brackets are given as higher derived brackets ($C^\infty(M)$-linear for $k \geq 3$)

$$\ell_k(e_1, \ldots, e_k) = \{\ldots\{\{Q^{(k-1)}, e_1\}, e_2\}\ldots e_k\},$$

  where the arity k $Q^{(k)}$ w.r.t. the (unweighted) Euler vector field is

$$\{Q^{(k)}, \varepsilon\} = k\, Q^{(k)}.$$

- For Q1 manifolds, only $k = 1$, giving the Lie bracket of the Lie (1-)algebroid.
  no fork in the road ...

COURANT ALGEBROIDS & LIE 2-ALGEBROIDS

- For Courant algebroids, there are $Q^{(k)}$ for $k = 0, 1, 2$.
- The Lie 2-algebroid on $T^*[1]M \oplus E$ is given by the anchor ρ and the brackets

$$\ell_1 = -\rho^\sharp$$

$$\ell_2(e, e') = [e, e'] - \eta^\sharp(\dot{\nabla}^E_- e, e'), \quad \ell_2(e, \sigma) = L_{\rho(e)}(\sigma) - (\nabla_- e, \rho^*(\sigma)),$$

$$\ell_3(e, e', e'') = -\eta(S_\nabla(e, e')(-), e'') - \eta(R_\nabla(-, \rho(e''))(e), e')$$

- Here $S^\nabla \in \Gamma(E \otimes E \otimes E \otimes TM)$ is the basic curvature for the TM-on-E wrt ℓ_2:

$$\mathcal{S}^\nabla(e, e')X = \nabla_X(\ell_2(e, e')) - \ell_2(\nabla_X e, e') - \ell_2(e, \nabla_X e') - \nabla_{\overline{\nabla}^E_{e'} X} e + \nabla_{\overline{\nabla}^E_e X} e.$$

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- Here $S^\nabla \in \Gamma(E \otimes E \otimes E \otimes TM)$ is the basic curvature for the TM-on-E wrt ℓ_2:

$$\mathcal{S}^\nabla(e,e')X = \nabla_X(\ell_2(e,e')) - \ell_2(\nabla_X e, e') - \ell_2(e, \nabla_X e') - \nabla_{\overline{\nabla}^E_{e'}X} e + \nabla_{\overline{\nabla}^E_e X} e\,.$$

Remarks:

- Canonical definition of S_∇ wrt ℓ_2 vs wrt $[-,-]$. (n.b.: $\overline{\nabla}^E_e e' = \nabla_{\rho(e')} e + \ell_2(e,e')$.) This is due to ℓ_2 being skew and $C^\infty(M)$-Leibniz (in both entries).
- As such, ℓ_2 defines canonical tensors for E-n-E connections too:

$$T_{\nabla^E}(e,e') = \nabla^E_e e' - \nabla^E_{e'} e - \ell_2(e,e')\,, \qquad R_{\nabla^E}(e,e') = [\nabla^E_e, \nabla^E_{e'}] - \nabla^E_{\ell_2(e,e')}\,.$$

Another choice of ∇ gives L_∞ quasi-isomorphic Lie 2-algebroid. Different construction than Roytenberg-Weinstein Lie 2-algebra

OUTLOOK

- The basic curvature tensor appears naturally in the study of connections & representations of LAois, CAois & beyond.
- Also in physical settings where these structures appear, e.g. gauge theory & CHSs.
- For Lie $(n > 1)$-algebroids, it ties up the structure $\rightsquigarrow$ canonical definitions of tensors.

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- General study of connections on dg-manifolds & Atiyah class.
- Physical applications (e.g. nontopological / susy “AKSZ”, quantization, ...)
- Utility of Lie 2-algebroid construction (e.g. in gravity models).

Thank You