Torsion and Lorentz symmetry from twisted spectral triples

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in collaboration with

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Workshop on Noncommutative Geometry

Corfu, 18th September 2024

Introduction

Noncommutative geometry a la Connes provides a unified description of

- the lagrangian of the Standard Model of fundamental interactions;
- minimally coupled to the Einstein-Hilbert action of general relativity;
- including right handed neutrinos; (Chamseddine, Connes, Lott, Marcolli)
- where the Higgs boson comes out naturally on the same footing as the other bosons, that is the local expression of a connection 1-form.

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Works well on riemannian manifolds: in 4D metric with euclidean signature $(+,+,+,+)$. The generalisation to lorentzian manifolds, with signature $(+,-,-,-)$, is far from obvious. Some attempts to implement lorentzian signature from the beginning. (Barrett, Besnard, Eckstein, Franco, Wallet, Dungen, Bochniak, Sitarz etc)

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Alternative way: starting in riemannian signature, and generating the lorentzian structure by twisting the spectral triple.

- ▶ Unveils an unexpected interplay between torsion and change of signature.

1. **Standard model in noncommutative geometry**

2. **Twisted spectral triples & torsion**

3. **Twisted unitaries**

4. **Change of signature from the fermionic action**

1. Standard model in noncommutative geometry

Spectral triple

An algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ together with selfadjoint operator D with compact resolvent, such that

$$[D, a] \text{ is bounded} \quad \forall a \in \mathcal{A}.$$

Graded spectral triple: there exists $\Gamma = \Gamma^*$, $\Gamma^2 = \mathbb{I}$, such that

$$\{\Gamma, D\} = 0, \quad [\Gamma, a] = 0 \quad \forall a \in \mathcal{A}.$$

Real spectral triple: there exists antilinear operator J such that

$$J^2 = \epsilon\mathbb{I}, \; JD = \epsilon' DJ, \; J\Gamma = \epsilon'' \Gamma J$$

where $\epsilon, \epsilon', \epsilon'' = \pm 1$ define the KO-dimension $k \in [0, 7]$.

Connes' reconstruction theorem

Extra-conditions yield the following spectral characterization of manifolds:

- Closed Riemannian manifold $\mathcal{M} \Longrightarrow$ spectral triple $\left(C^\infty\left(\mathcal{M}\right), L^2(\mathcal{M}, S), \partial\!\!\!/\right)$

  with $C^\infty(\mathcal{M})$ the (commutative) algebra of smooth functions on $\mathcal{M}$, $L^2(\mathcal{M},S)$ the space of square integrable spinors on $\mathcal{M}$, and

  $$\partial\!\!\!/ = -i\gamma^\mu(\partial_\mu + \omega_\mu) \quad \text{with} \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I} \quad (\mu = 1, 2, 3, 4)$$

  the Dirac operator, with (ω_μ the lift of the Levi-Civita connection to the spinor bundle.

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- $\mathcal{M}$ such that $\mathcal{A} = C^\infty\left(\mathcal{M}\right) \Longleftarrow (\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}$ commutative, unital.

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commutative spectral triple	$\rightarrow$	**noncommutative spectral triple**
$\updownarrow$		$\downarrow$
Riemannian geometry		**non-commutative geometry**

Standard Model

Product of a $4D$ riemannian closed spin manifold $\mathcal{M}$ with a finite dimensional noncommutative spectral triple:

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F$$

in which

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \;\; \mathcal{H}_F = \mathbb{C}^{96}, D_F$$

where

$$\underbrace{\Big(\big((e^-, \nu_e) + (u, d) \times 3 \text{ colors}\big) \times 2 \text{ chiralites} \times 2\Big) \times 3 \text{ generations}}_{(2+6)\times 2\times 2\times 3 = 96}$$

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- sections of $\mathcal{H} \to$ fermions.

The bosons are obtained by fluctuation of the metric,

$$D \to D_A =: D + A + J\,A\,J^{-1}$$

with A a generalized 1-forms

$$\Omega^1_D(\mathcal{A}) := \left\{ a^i\,[D, b_i],\ a^i, b_i \in \mathcal{A} \right\}.$$

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For the Standard Model:

$$\left.\begin{array}{rcl} \mathcal{A} & = & C^\infty(\mathcal{M}) \otimes \mathcal{A}_F \\ \mathcal{H} & = & L_2(\mathcal{M}, S) \otimes \mathcal{H}_F \\ D & = & \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F \end{array}\right\} \Longrightarrow A = \gamma^5 \otimes H - i \sum_\mu \gamma^\mu \otimes A_\mu.$$

- H: scalar field on $\mathcal{M}$ with value in $\mathcal{A}_F$ $\to$ Higgs.
- A_μ: 1-form field with value in $Lie(U(\mathcal{A}_F))$ $\to$ gauge field.

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The asymptotic expansion $\Lambda \to \infty$ of the spectral action

$$\mathrm{Tr}\, f\left(\frac{D_A^2}{\Lambda^2}\right)$$

(f a smooth approximation of the characteristic function of $[0,1]$) yields the bosonic Lagrangian of the Standard Model coupled with Einstein-Hilbert action in euclidean signature.

Problem with Lorentzian signature

The space $L^2(\mathcal{M}, S)$ of square integrable spinors on a riemannian manifold $\mathcal{M}$ with spin structure S is an Hilbert space with inner product

$$(\psi, \varphi) = \int_{\mathcal{M}} \psi^\dagger \varphi \, \nu_g,$$

where

$$\nu_g = \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^n$$

is the volume form associated with the riemannian metric g on $\mathcal{M}$.
The Dirac operator

$$\partial\!\!\!/ = -i\gamma^\mu(\partial\mu + \omega_\mu)$$

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In lorentzian signature, the Dirac operator (built from lorentzian Dirac matrices) is no longer selfadjoint. It is so with respect to the Krein product

$$(\psi, \varphi) = \int_{\mathcal{M}} \psi^\dagger \gamma^0 \varphi \, \nu_g.$$

But then $L^2(\mathcal{M}, S)$ is no longer an Hilbert space.

Twisted spectral triples

Given a triple $(\mathcal{A}, \mathcal{H}, D)$, instead of asking the commutators $[D, a]$ to be bounded, one asks the boundedness of the twisted commutators Connes, Moscovici 2008

$$[D, a]_\rho := Da - \rho(a)D \qquad \text{for some fixed} \quad \rho \in \mathrm{Aut}(\mathcal{A}).$$

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Compatible with the real structure: twisted flucttutation

$$D \to D_{A_\rho} := D + A_\rho + J\, A_\rho\, J^{-1}$$

where A_ρ is an element of the set of twisted 1-forms

$$\Omega^1_D(\mathcal{A}, \rho) := \left\{ a^i [D, b_i]_\rho,\ a_i, b_i \in \mathcal{A} \right\}.$$

Devastato, Landi, PM 2016/17

Minimal twist of a spectral triple

Associate a twisted partner to any graded spectral triple $(\mathcal{A} \xrightarrow{\pi_0} \mathcal{H}, D)$, keeping $\mathcal{H}, \mathcal{D}$ untouched but doubling the algebra to $\mathcal{A} \otimes \mathbb{C}^2$ by making each copy of $\mathcal{A}$ act independently on the eigenspaces $\mathcal{H}_\pm$ of the grading Γ. Landi, PM 2016.

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Proposition Landi, PM 2016

The triple

$$(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D), \rho$$

with representation

$$\pi((a, a')) := \frac{1}{2}(\mathbb{I} + \Gamma)\,\pi_0(a) + \frac{1}{2}(\mathbb{I} - \Gamma)\,\pi_0(a)$$

and twisting automorphism

$$\rho((a, a')) = (a', a) \quad \forall (a, a') \in \mathcal{A} \otimes \mathbb{C}^2$$

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- fermionic content (i.e. D and $\mathcal{H}$) preserved, but new bosonic fields allowed:

$$[D, a] = 0 \;\forall a \in \mathcal{A} \quad \text{does not mean} \quad [D, (a, a')]_\rho = 0 \;\forall (a, a') \in \mathcal{A} \otimes \mathbb{C}^2.$$

Example: minimal twist of a manifold $\mathcal{M}$ (closed, spin, riemannian, dim. $2m$).

The eigenspaces of the grading γ_{2m+1} (which is γ^5 in dimension $2m = 4$) are the left /right handed spinors, thus one obtains

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, S), \quad D = \partial\!\!\!/; \quad \rho$$

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- In KO-dimension $0, 4$, there exist non-zero selfadjoint twisted fluctuations:

$$\partial\!\!\!/ \to D_{A_\rho} = \partial\!\!\!/ - i\, f_\mu \gamma^\mu \gamma_{2m+1} \quad \text{with} \;\; f_\mu \in C^\infty(\mathcal{M}, \mathbb{R})$$

Devastato, Lizzi, Farnsworth, PM 2017

- In the non twisted case, such fluctuations vanish.

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- In the non twisted case, such fluctuations vanish.

What is the meaning of the extra-term $if_\mu\gamma^\mu\gamma_{2m+1}$?

Torsion

The contorsion of an arbitrary connection ∇ on $T\mathcal{M}$ is the $(2,1)$ tensor field

$$K := \nabla - \bar{\nabla}$$

where $\bar{\nabla}$ is the Levi-Civita connection. ∇ has the same geodesics as $\bar{\nabla}$ iff $K(X,Y) = -K(Y,X)$. It is orthogonal (i.e. compatible with the metric) iff

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- ∇ is orthogonal and geodesic preserving iff $K^\flat$ is totally antisymmetric.

In that case $K^\flat$ is the torsion 3-form.

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Proposition Nieuviarts, Zeitoun, PM 2023

In dimension 4, the twisted covariant Dirac operator D_{A_ρ} is the lift to spinors of an orthogonal and geodesic preserving connection, with torsion 3-form $-\star\omega_f$. More generally

$$i\, f_\mu \gamma^\mu \gamma_{2m+1} = \frac{(-i)^{m+1}}{2m}\, c(\star\omega_f)$$

where

$$\omega_f := f_\mu dx^\mu.$$

Twisted product

The twisting automorphism

$$\rho(f,g) = (g,f) \quad \forall f,g \in C^\infty(\mathcal{M})$$

extends to an inner automorphism of $\mathcal{B}(L^2(\mathcal{M},S))$:

$$\rho(\mathcal{O}) = \gamma^0 \mathcal{O} \gamma^0 \qquad \forall \mathcal{O} \in \mathcal{B}(\mathcal{H}).$$

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This induces a new inner product on $L^2(\mathcal{M},S)$:

$$\langle \psi, \varphi \rangle_{\gamma^0} := \langle \psi, \gamma^0 \varphi \rangle,$$

with respect to whom the adjoint of an operator $\mathcal{O}$ is

$$\mathcal{O}^+ := \rho(\mathcal{O})^\dagger.$$

- This is the Krein product for spinors in Lorentzian signature.

Devastato, Farsworth, Lizzi, PM (2018)

Generating torsion by group action

The group of twisted unitaries is

$$\mathcal{U}_\rho := \left\{u_\rho \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2,\ u_\rho^+ u_\rho = u_\rho u_\rho^+ = \mathbf{1}\right\},$$
$$= \{h \in C^\infty(\mathcal{M}),\ h(x) \neq 0\ \forall x \in \mathcal{M}\}.$$

Adjoint action:

$$\mathrm{Ad}(u_\rho)\,\psi := u_\rho J u_\rho J^{-1}\,\psi \qquad \forall \psi \in L^2(\mathcal{M}, S).$$

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Proposition — Nieuviarts, PM (2024)

The conjugate action, with respect to the initial involution †, of $\mathrm{Ad}(u_\rho)$ generates all the torsion terms with co-exact torsion form: given

$$D_{\omega_f} := \partial\!\!\!/ - i f_\mu \gamma^\mu \gamma_{2m+1},$$

one has

$$\mathrm{Ad}(u_h)\, D_{\omega_f}\, \mathrm{Ad}(u_h)^\dagger = D_{\omega_{f'}} \text{ where } \omega_{f'} = \omega_f + d(\ln|h|^2).$$

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- In the non-twisted case, unitaries generate the fluctuations of the metric.
- Here, there is an intertwining of the two involutions $+$ and †.
- When $2m = 4$, the Lorentz group is a subgroup of the twisted unitaries of $\mathcal{B}(L^2(\mathcal{M}, S))$.

4. Change of signature from the fermionic action

One defines the twisted fermionic action

$$S(D_{\omega_f}) := \langle J\tilde{\xi}, \gamma^0 \, D_{\omega_f} \, \tilde{\xi} \rangle$$

for $\xi \in \mathcal{H}_0 := \{\xi \in L^2(\mathcal{M}, S), \gamma^0\xi = \xi\}$, and $\tilde{\cdot}$ the Grassmann variables.

Devastato, Lizzi, Farnsworth, PM 2018

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- The twisted product guarantees the invariance under twisted gauge transformation.
- Restricting to $\mathcal{H}_0$ is to make the bilinear form antisymmetric.

Does it make sense physically ?

Twisted riemannian manifold and Weyl lagrangian:

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, S), \quad D = \partial\!\!\!/; \quad \rho.$$

Twisted fluctuation:

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The twisted fermionic action (in dimension 4) is

$$S^f(D_{\omega_f}) = 2 \int_{\mathcal{M}} d\mu\, \bar{\tilde{\zeta}}^\dagger \sigma_2 \left(i f_0 \mathbb{I}_2 - \sum_{j=1}^{3} \sigma_j \partial_j \right) \tilde{\zeta} \quad \text{where} \quad \xi = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \in \mathcal{H}_0.$$

- The ∂_0 derivative is substituted with the component f_0 of the fluctuation.

Twisted riemannian manifold and Weyl lagrangian:

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, S), \quad D = \partial\!\!\!/; \quad \rho.$$

Twisted fluctuation:

$$D_{A_\rho} = \partial\!\!\!/ - i\, f_\mu \gamma^\mu \gamma_{2m+1} \quad \text{with} \;\; f_\mu \in C^\infty(\mathcal{M}, \mathbb{R}).$$

The twisted fermionic action (in dimension 4) is

$$S^f(D_{\omega_f}) = 2 \int_{\mathcal{M}} d\mu \, \bar{\tilde{\zeta}}^\dagger \sigma_2 \left(i f_0 \mathbb{I}_2 - \sum_{j=1}^{3} \sigma_j \partial_j \right) \tilde{\zeta} \quad \text{where} \quad \xi = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \in \mathcal{H}_0.$$

- The ∂_0 derivative is substituted with the component f_0 of the fluctuation.

It reminds the Weyl lagrangian in lorentzian signature

$$\psi_l^\dagger \, \tilde{\sigma}_M^\mu \, \partial_\mu \psi_l \qquad \text{where} \quad \tilde{\sigma}_M^\mu := \{\mathbb{I}_2, -\sigma_j\}.$$

Tempting to identify $\partial_0 \psi_l = i f_0 \tilde{\zeta}$, that is

$$\tilde{\zeta}(t, \mathbf{x}) = \psi_l(t, \mathbf{x}) = e^{itf_0} \psi_l(\mathbf{x}).$$

But then it is not true that $\bar{\tilde{\zeta}}^\dagger \sigma^2 \neq \psi_l^\dagger$.

The twist of a doubled manifold

$$\mathcal{A} = \left(C^\infty(\mathcal{M}) \otimes \mathbb{C}^2\right) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^2, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_2$$

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- $\mathcal{H}_0$ spanned by $\{\xi \otimes e, \phi \otimes \bar{e}\}$ with $\xi = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}$, $\phi = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$, $\{e, \bar{e}\}$ basis of $\mathbb{C}^2$.

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The fermionic action is the integral of

$$\mathcal{L}^f_\rho := \bar{\tilde{\varphi}}^\dagger \sigma_2 \left(if_0 - \sum_{j=1}^3 \sigma_j \partial_j\right) \tilde{\zeta}, \qquad f_0 \in C^\infty(\mathcal{M}, \mathbb{R}).$$

This yields the Weyl lagrangian identifying $\Psi_l := \tilde{\zeta}$, $\Psi_l^\dagger := -i\bar{\tilde{\varphi}}^\dagger \sigma_2$ and assuming $\partial_0 \Psi_l = if_0 \Psi_l$, that is

$$\Psi_l(x_0, x_j) = \Psi_l(x_j) e^{if_0 x_0}.$$

- The twisted fermionic action for a twisted doubled riemannian manifold describes a plane wave solution of Weyl equation (in lorentzian signature).

Twist of electrodynamics

$$\mathcal{A}_{\text{ED}} = \left(C^\infty(\mathcal{M}) \otimes \mathbb{C}^2\right) \otimes \mathbb{C}^2, \; \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^4, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_4 + \gamma_{\mathcal{M}} \otimes D_{\mathcal{F}}$$

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- The twisted fermionic action coincides with the Dirac action in lorentzian signature.

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- Link with thermal time hypothesis: in Connes-Moscovici, $\rho = \sigma_i$ for a 1-parameter group of automorphism σ_s related to Tomita-Takesaki.
  - Which (modular) group is behind the flip $(f, g) \mapsto (g, f)$? Hints: for γ matrices, Wick rotation

    $$W(\gamma^0) = \gamma^0, \quad W(\gamma^i) = i\gamma^i$$

    is the square root of the flip

    $$\rho(\gamma^0) = \gamma^0, \quad \rho(\gamma^i) = -\gamma^i.$$

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