Workshop on Noncommutative and Generalized Geometry in String theory, Gauge theory and Related Physical Models

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Homotopy Algebra Techniques for Noncommutative Quantum Field Theories

Djordje Bogdanović

University of Belgrade, Faculty of Physics,
Belgrade, Serbia

based on:

DjB, M. D. Ćirić, V. Radovanović, R. J. Szabo, *BV quantization of braided scalar field theory*, **arXiv:2304.14073**, DjB, M. D. Ćirić, V. Radovanović, R. J. Szabo, G. Trojani, *Braided scalar quantum field theory*, **arXiv:2406.0237** and with F. Lizzi, P. Vitale, R. J. Szabo, M. D. Ćirić, *Work in progress*

Talk overview

Brief motivation

L_∞-algebras of classical field theories

Braided L_∞-algebra of ϕ^3 theory

Homological perturbation theory

Results:

- Correlation functios
- Schwinger-Dayson equations
- Braided Wick theorem
- ρ-Minkowski noncommutativity

Outlook

Brief motivation

- ⋆ BV formalism is developed for gauge quantum field theories [Weinberg '96; Gomis et al '94]
- ⋆ BV formalism has natural structure encoded in L_∞-algebra [Hohm, Zwiebach '17; Jurco et al. '18; Costello, Gwilliam '16, '21]

- ⋆ Amplitude program in quantum field theories (recursion relations) [Elvang, Huang '15]
- ⋆ Double copy method connects gauge theories to quantum gravity [Berm et all '10; Borsten et al '21].

- ⋆ Consistent quantization of nonperturbative noncommutative field theories and resolving the issues of UV/IR mixing and existence of non-planar diagrams [Minwalla et al. '99; Balachandran et al. '06; Bu et al. '06 Fioere, Wess '07; Aschieri et al. '08]

L_∞-algebras of classical field theories - mini dictionary

Classical field theory	$\rightarrow$	L_∞-algebra $(V, \ell_n, \langle \_, \_ \rangle)$
Fields, ghosts and antifields	$\rightarrow$	$V = \cdots \oplus V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus \ldots$ $\cdots \oplus$ **ghosts** $\oplus$ **fields** $\oplus$ **EoM** $\oplus$ **Noether id** $\oplus \ldots$
Classical action S	$\rightarrow$	$\mathcal{S}(A) = \sum_{n=1}^{\infty} (-1)^{\frac{n(n-1)}{2}} \langle A, \ell_n(A, \ldots, A) \rangle$
Spacetime noncommutativity via Drinfel'd twist formalism	$\rightarrow$	Braided L_∞-algebra $(V, \ell_n^\star, \langle \_, \_ \rangle_\star)$
Polynomials in fields φ^2 e.g. φ^3, φ^4	$\rightarrow$	Simmetised tensor algebra $\mathrm{Sym}_{\mathcal{R}}(V[2])$ $v_1 \odot_\star v_2 = (-1)^{\lvert v_1 \rvert \lvert v_2 \rvert} R_\alpha(v_2) \odot_\star R^\alpha(v_1)$
Tensor product algebra	$\rightarrow$	Extending algebraic structure to new L_∞-algebra $\mathrm{Sym}_{\mathcal{R}}(V[2]) \otimes V$ where brackets and pairing respect: $\ell_2^{\star ext}(a_1 \otimes v_1, a_2 \otimes v_2) = \pm(a_1 \odot_\star \mathrm{R}_\alpha(a_2)) \otimes \ell_2^\star(\mathrm{R}^\alpha(v_1), v_2)$ $\langle\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle\rangle_\star = \pm(a_1 \odot_\star \mathrm{R}_\alpha(a_2)) \cdot \langle \mathrm{R}^\alpha(v_1), v_2 \rangle_\star$
Poisson structure	$\rightarrow$	$\{\_, \_\}_\star : \mathrm{Sym}_{\mathcal{R}}(V[2]) \otimes \mathrm{Sym}_{\mathcal{R}}(V[2]) \rightarrow (\mathrm{Sym}_{\mathcal{R}}(V[2]))[1]$

L_∞-algebras of classical field theories - mini dictionary

Classical BV action S_{BV}	$\rightarrow$	Braided BV action $\mathcal{S}^\star_{BV} \in \mathrm{Sym}_{\mathcal{R}}(V[2])$
Classical master equation $\{S_{BV}, S_{BV}\}_{PB} = 0$	$\rightarrow$	$\{\mathcal{S}^\star_{BV}, \mathcal{S}^\star_{BV}\}_\star = 0$
Solution as expansion in antifields: $S_{BV} = S + (antifield) \cdot (\dots) + \dots$	$\rightarrow$	Solution as expansion in brackets via contracted coordinate functions $\xi = \tau_k \otimes \tau^k \in \mathrm{Sym}_{\mathcal{R}}(V[2]) \otimes V$: $\mathcal{S}^\star_{\mathrm{BV}} = \frac{1}{2} \langle\langle \boldsymbol{\xi}, \ell_1^{\star ext}(\boldsymbol{\xi}) \rangle\rangle_\star - \frac{1}{3!} \langle\langle \boldsymbol{\xi}, \ell_2^{\star ext}(\boldsymbol{\xi}, \boldsymbol{\xi}) \rangle\rangle_\star + \dots$ where we identify $\mathcal{S}^\star_{\mathrm{BV}} = \mathcal{S}^\star_{(0)} + \mathcal{S}^\star_{\mathrm{int}}$
BV Laplacian Δ_{BV} appears in quantum master equation $\frac{1}{2}\{S_{BV}, S_{BV}\} = \mathrm{i}\hbar \Delta_{BV} S_{BV}$	$\rightarrow$	Braided BV Laplacian nontrivialy defined via pairing of field φ and corresponding antifield φ^+: $\Delta_{\mathrm{BV}}(\varphi \odot_\star \varphi^+) = \pm \langle \varphi, \varphi^+ \rangle_\star$

Braided L_∞-algebra of ϕ^3 theory

- Massive real scalar field in 4D Minkowski spacetime and qubic interaction ϕ^3
- The underlying graded vector space is $V = V_1 \oplus V_2$, where

  $V_1 = V_2 = \Omega^0(\mathbb{R}^{1,3})$
- V_1 is the space of fields ϕ, V_2 is the space of antifields/EoM ϕ^+
- There are just the first two brackets

$$\ell_1^\star(\phi) = \ell_1(\phi) = -(\Box + m^2)\phi \ \ \& \ \ \ell_2^\star(\phi_1, \phi_2) = -\lambda \phi_1 \star \phi_2$$

- Equipping the structure with the cyclic pairing

$$\langle \phi, \phi^+ \rangle_\star = \int \mathrm{d}^4 x \ \phi \star \phi^+$$

- Braided MC action is:

$$\begin{aligned} S_\star(\phi) &= \frac{1}{2}\langle \phi, \ell_1(\phi)\rangle_\star - \frac{1}{3!}\langle \phi, \ell_2(\phi,\phi)\rangle_\star \\ &= \int \mathrm{d}^4 x \ \Big(\frac{1}{2}\,\phi\,\big(-\Box - m^2\big)\,\phi - \frac{\lambda}{3!}\phi \star \phi \star \phi\Big). \end{aligned}$$

- At the classical level, this action is the same as in the usual $\phi^3_\star$ theory!

Braided L_∞-algebra of ϕ^3 theory

- Define contracted coordinate functions $\xi \in \mathrm{Sym}_{\mathcal{R}}(V[2]) \otimes V$

$$\boldsymbol{\xi} = \int_k \big(\mathsf{e}_k \otimes \mathsf{e}^k + \mathsf{e}^k \otimes \mathsf{e}_k\big),$$

- Define and calculate the interaction action

$$\mathcal{S}^{\star}_{\rm int} = -\frac{1}{6}\langle\!\langle \xi, \ell_2^{\star\,ext}(\xi,\xi)\rangle\!\rangle_\star = \int_k V(k_1,k_2,k_3) e_1^k \odot_\star e_2^k \odot_\star e_3^k,$$

- We naturally chose plane waves as basis vectors in momentum space, $\mathsf{e}^k(x) = e^{\mathrm{i}k\cdot x}$ and $\mathsf{e}_k(x) = e^{-\mathrm{i}k\cdot x}$

- The twist we use is the Moyal-Weyl twist $\mathcal{F} = e^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}$

- Vertex has a simple form implying regular momentum conservation law:

$$V(k_1,k_2,k_3) = -\frac{\lambda}{3!} e^{\frac{\mathrm{i}}{2}\sum_{a<b} k_i\cdot\theta k_j} (2\pi)^4 \delta(k_1+k_2+k_3)$$

- The twist we can also use is the ρ-Minkowski twist $\mathcal{F} = e^{-\frac{\mathrm{i}\theta}{2}(\partial_z\otimes\partial_\varphi - \partial_\varphi\otimes\partial_z)}$

Homological perturbation theory

- ⋆ ℓ_1 acts as differential creating cochain complex (V, ℓ_1) that can be related to cochain complex of its cohomology $(H^\bullet(V), 0)$ via maps: contracting homotopy (in our case it it the propagator) $\mathrm{h} : V \to V$ of degree -1, inclusion $i : H^\bullet(V) \to V$ and projection $p : V \to H^\bullet(V)$
- ⋆ Strong deformation retract is defined when aforementioned maps fulfill certain conditions
- ⋆ Homological perturbation lemma states that strong deformation retract is stable i.e. deformation $\ell_1 \to \ell_1 + \delta$ deforms other maps $(\tilde{i}, \tilde{p}, \tilde{h})$ such that strong deformation retracts conditions hold
- ⋆ Correlation functions in momentum space are then:

$$\tilde{G}_n(p_1, \dots, p_n) = \sum_{m=1}^{\infty} \mathrm{P}\big((\delta \mathrm{H})^m (e_1^p \odot_\star \cdots \odot_\star e_n^p\big)$$

- ⋆ Deformation can be chosen to be $\delta = \mathrm{i}\hbar\Delta_{\mathrm{BV}}$ for free theory or $\delta = \{\mathcal{S}^\star_{int}, \_\} + \mathrm{i}\hbar\Delta_{\mathrm{BV}}$ for interacting theory

Results: Correlation functions in ϕ^3 theory

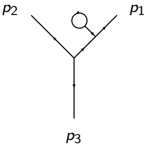
- ⋆ Propagator in free theory is the same as in regular theory

$$G_2^\star(p_1, p_2)^{(0)} = \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\mathsf{H}\big(\mathrm{e}^{p_1} \odot_\star \mathrm{e}^{p_2}\big) = (\mathrm{i}\,\hbar)\,\frac{(2\pi)^4\,\delta(p_1+p_2)}{p_1^2 - m^2}$$

- ⋆ In MW case, two point function at 1-loop have no NC contributions, no nonplanar diagrams and no UV/IR mixing and is the same as in regular theory Consistent with [Oeckel '00]
- ⋆ In MW case, the final result for the connected 3-point function is:

$$G_3^\star(p_1, p_2, p_3)^{(1)}|_{connected} = \sum_{cyclic} \mathrm{e}^{\frac{\mathrm{i}}{2}\,p_3\cdot\theta\,p_2} \times$$

$\Big($ 108 ×



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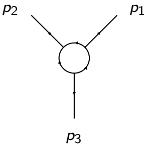

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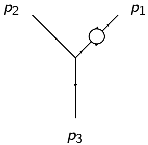

$\Big)$

- ⋆ NC contribution appears as a phase factor in external momenta. No UV/IR mixing! Consistent with [Oeckel '00]

Results: Schwinger-Dayson equations

- Schwinger-Dyson equations are EoM corresponding to Green's functions
- SD equations were analyzed in the commutative QFT and from the perspective of homotopy algebras [K. Konosu '23; K. Konosu and J. Totsuka-Yoshinaka '24; K. Konosu and Y. Okawa '24]
- In this approach, SD equations are coming from the Homological perturbation lemma in the recursion of the form:

$$\tilde{\mathsf{P}} = \tilde{\mathsf{P}}\delta\mathsf{H}$$

- $\tilde{\mathsf{P}}$ is deformed projection map P, an extension of map p. When acting on $\mathrm{Sym}_{\mathcal{R}}(V[2])$ it generates all n-point functions G_n
- Acting on the symmetrized product of basis elements, it recursively relates different correlation functions $G_k^{\star}$:

$$\tilde{G}_n(p_1, \ldots, p_n) = \tilde{\mathsf{P}}\delta\mathsf{H}(e_1^p \odot_\star \cdots \odot_\star e_n^p)$$

Results: Braided Wick theorem

- In free theory and using MW twist, where $\delta = \mathrm{i}\hbar\Delta_{\mathrm{BV}}$, SD equation is:

$$\tilde{G}^{\star 0}_{2n}(p_1, \ldots, p_{2n}) = \frac{1}{2n} \sum_{\alpha \neq \beta}^{n} \mathrm{e}^{\mathrm{i}\, p_\beta \cdot \theta (p_{\alpha+1} + \cdots + p_{\beta-1})} \, \underbrace{\phi_\alpha \, \phi_\beta} \cdot$$

$$\cdot \, \tilde{G}^{\star 0}_{2n-2}(p_1, \ldots, \hat{p}_\alpha, \ldots, \hat{p}_\beta, \ldots, p_{2n})$$

- The solution of SD equation in free theory is the general expression for the braided Wick there:

$$\tilde{G}^{\star 0}_{2n}(p_1, \ldots, p_{2n}) = \frac{1}{n!\, 2^n} \sum_{\sigma \in S_{2n}} \mathrm{e}^{-\frac{\mathrm{i}}{2} \sum\limits_{i<j} p_i \cdot \theta \, p_j} \prod_{k=1}^{n} \underbrace{\phi_{\sigma(2k-1)} \, \phi_{\sigma(2k)}} \, ,$$

- Braided Wick theorem for 4-point function reads:

$$\tilde{G}^{\star}_4(k_1, k_2, k_3, k_4)^{(0)} = \underbrace{\phi_1 \, \phi_2} \, \underbrace{\phi_3 \, \phi_4} + \underbrace{\phi_1 \, \phi_4} \, \underbrace{\phi_2 \, \phi_3} + \mathrm{e}^{\mathrm{i}\, k_3 \cdot \theta \, k_2} \, \underbrace{\phi_1 \, \phi_3} \, \underbrace{\phi_2 \, \phi_4}$$

General: ρ-Minkowski noncommutativity

- $\star$ In Cartesian and polar coordinates ρ-Minkowski twist is:

$$\mathcal{F} = e^{-\frac{i\theta}{2}(\partial_z \otimes (x\partial_y - y\partial_x) - (x\partial_y - y\partial_x)\otimes \partial_z)} = e^{-\frac{i\theta}{2}(\partial_z \otimes \partial_\varphi - \partial_\varphi \otimes \partial_z)}$$

- $\star$ In Cartesian coordinates it describes space-time noncommutativity of Lie algebra type, in polar coordinate of MW type:

$$[\hat{z}, \hat{x}] = -i\theta\hat{y}, \quad [\hat{z}, \hat{y}] = +i\theta\hat{x}; \qquad \left[\hat{z}, e^{i\hat{\varphi}}\right] = i\theta e^{i\hat{\varphi}}$$

- $\star$ Standard noncommutative quantization, based on $\star$-product approach, was done in ϕ^4 case [M. D. Ćirić et al '18]

- $\star$ The phenomenon of UV/IR mixing appears and the model contains nonplanar diagrams. Conservation of momenta is deformed.

Preliminar results: ρ-Minkowski braiding of ϕ^3 theory

- Instead of MW twist, we applied ρ-Minkowski twist to our plane waves e^k and produced the fallowing vertex:

$$V(k_1, k_2, k_3) = -\frac{\lambda}{3!} e^{\theta \sum_{a<b}(k_{bz}(k_{ay}\partial_{k_{ax}} - k_{ax}\partial_{k_{ay}}) - k_{az}(k_{by}\partial_{k_{bx}} - k_{bx}\partial_{k_{by}}))} \cdot$$
$$\cdot (2\pi)^4 \delta^\star(k_1 +_\star k_2 +_\star k_3)$$

- Deformed momentum conservation law apeares!

$$\delta^\star(k_1 +_\star k_2 +_\star k_3) = \int_x e^{-\mathrm{i}k_1 x} \star e^{-\mathrm{i}k_2 x} \star e^{-\mathrm{i}k_3 x}$$

- Two point function at one loop level contains a nonplanar diagram leading to UV/IR mixing!

Preliminar results: ρ-Minkowski braiding of ϕ^3 theory

- Since Cartesian coordinates don't respect the symmetry of our twist, we can change the basis: $(x, y) \to (\rho, \varphi)$
- Functions that solve EoM and can be used as basis vectors for the space of (anti)fields are of the form:

$$e_{E,k_z,l,\alpha}(t, r, z, \varphi) = \sqrt{\alpha} J_l(\alpha r) \cdot e^{il\varphi} \cdot e^{-iEt} \cdot e^{ik_z z}, \quad EoM : \alpha^2 = k_x^2 + k_y^2$$

- Calculations so far suggest that there are no traces of nonplanar diagrams and UV/IR mixing in this basis!
- Since noncommutativity in this basis is of MW form, it can be expected that results are analogous to MW case.
- How can this be?! We have to understand the results better. *Work in progress...*

Outlook

- ⋆ Well established algebraic techniques were applied in details in ϕ^3 theory using Moyal-Wayle twist
- ⋆ Some further algebraic techniques were developed
- ⋆ ρ-Minkowski twist in ϕ^3 theory is currently under investigation with very interesting preliminary results that should be clarified
- ⋆ Future work will be dedicated to the analysis of non-Abelian gauge theories
- ⋆ Aiming for construction of amplitudes needed for double copy approach

Results: Schwinger-Dyson equations, an example

- ⋆ In the interacting ϕ^3 theory, using MW twist and deformation of form $\delta = \{\mathcal{S}^{\star}_{int}, \_\} + \mathrm{i}\hbar\Delta_{\mathrm{BV}}$, SD equation in case of $n = 2$ yields: