



# Exceptional symmetry as a symmetry principle for sigma models

David Osten (University of Wrocław)

Workshop on Noncommutative and Generalized Geometry  
in String theory, Gauge theory and Related Physical Models,

Corfu, September 22<sup>th</sup>

based on: 2402.10269, 2306.11093, 2103.03267



Uniwersytet  
Wrocławski



POLONEZ BIS

# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

---

# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

---



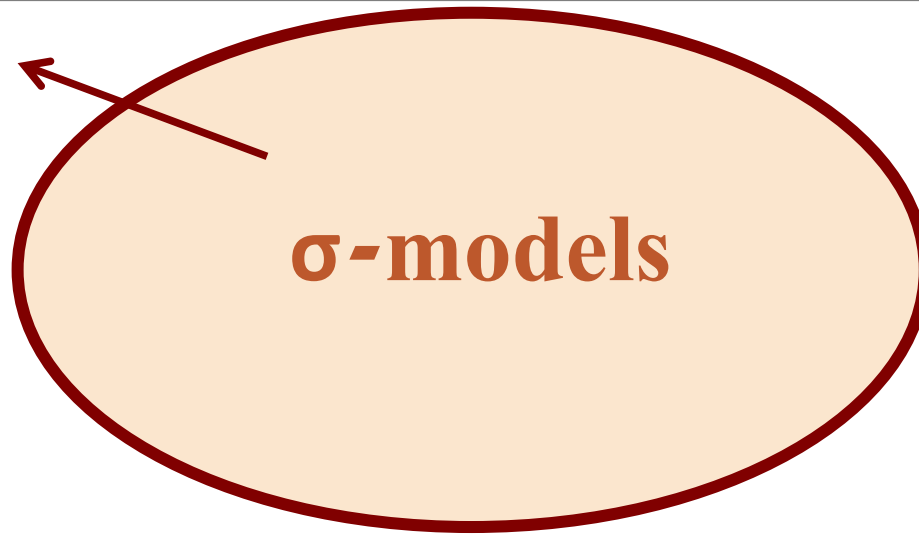
**$\sigma$ -models**

# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

---

**Particle physics**  
effective theories of  
mesons, skyrmions

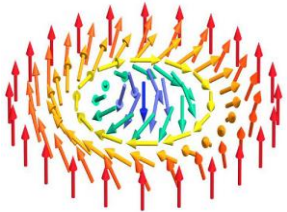


# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

## Particle physics

effective theories of mesons, skyrmions



$\sigma$ -models

## Condensed matter

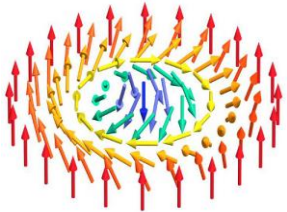
ferromagnetism, magnetic skyrmions, superfluidity, quantum Hall effect, ...

# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

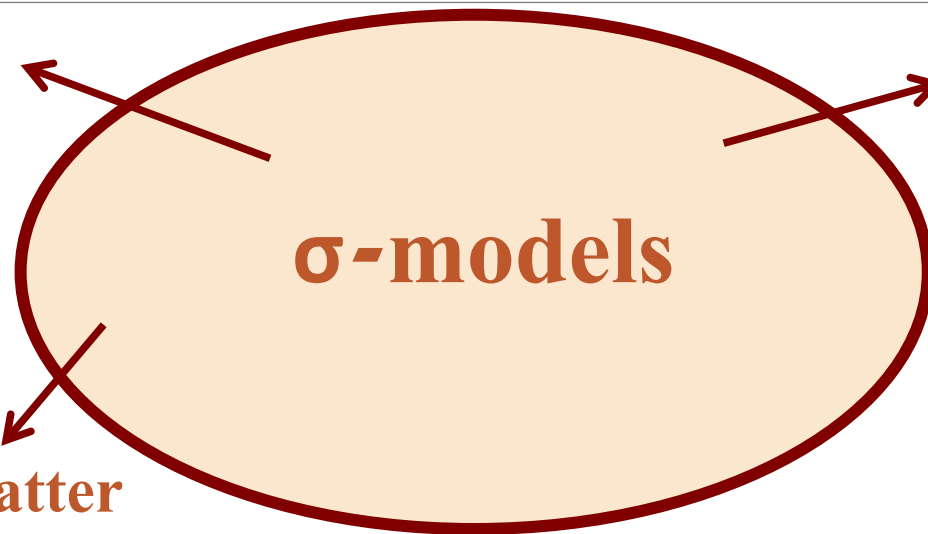
## Particle physics

effective theories of mesons, skyrmions



## Condensed matter

ferromagnetism, magnetic skyrmions, superfluidity, quantum Hall effect, ...



## Gravity

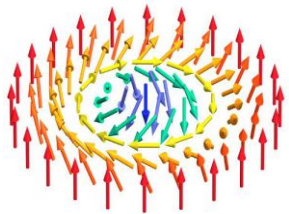
objects interacting with gravity, spacetimes with symmetries

# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

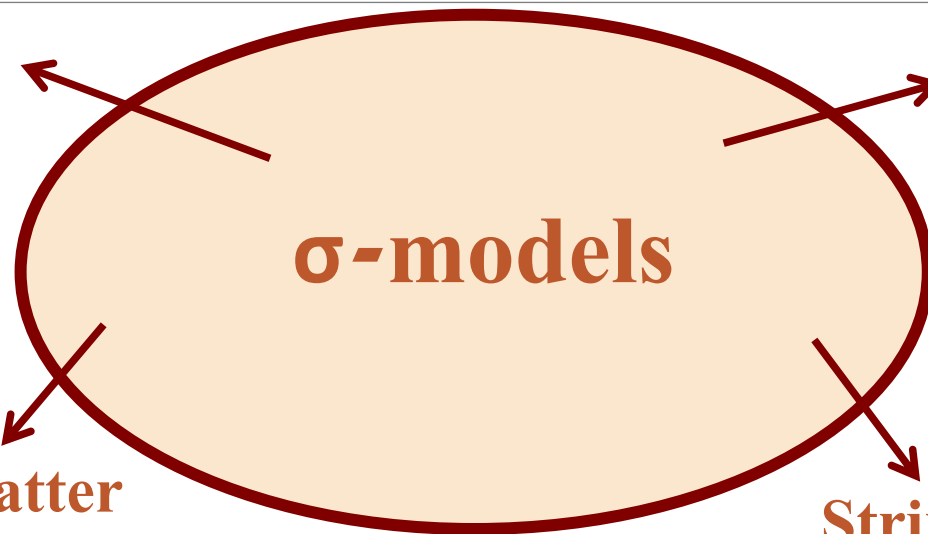
## Particle physics

effective theories of mesons, skyrmions



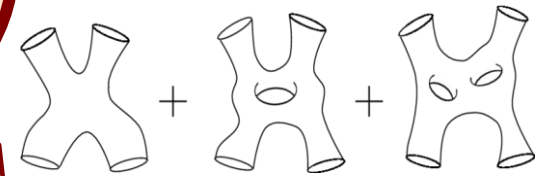
## Condensed matter

ferromagnetism, magnetic skyrmions, superfluidity, quantum Hall effect, ...



## Gravity

objects interacting with gravity, spacetimes with symmetries



## String and M-theory

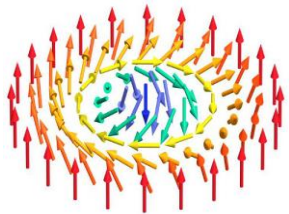
description of fundamental objects

# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

## Particle physics

effective theories of mesons, skyrmions



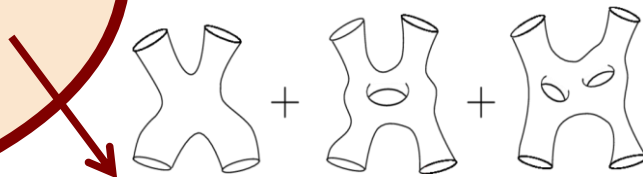
## Gravity

objects interacting with gravity, spacetimes with symmetries

$\sigma$ -models

## Condensed matter

ferromagnetism, magnetic skyrmions, superfluidity, quantum Hall effect, ...



## String and M-theory

description of fundamental objects

**methods (well-understood corners)**  
**integrability** (*exactly solvable models*),  
**topology** (*properties without quantum corrections*),  
**symmetry principles** (conformal symmetry, supersymmetry, ...)

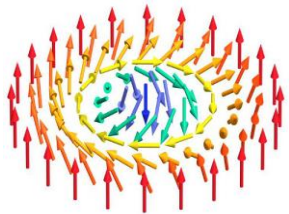


# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

## Particle physics

effective theories of mesons, skyrmions



## Gravity

objects interacting with gravity, spacetimes with symmetries

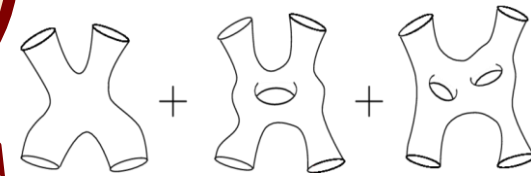
$\sigma$ -models

## Condensed matter

ferromagnetism, magnetic skyrmions, superfluidity, quantum Hall effect, ...

## String and M-theory

description of fundamental objects



### methods (well-understood corners)

**integrability** (*exactly solvable models*),

**topology** (*properties without quantum corrections*),

**symmetry principles** (conformal symmetry, supersymmetry, ...)

### open issues

pert. quantisation (**non-linearity**),  
less symmetry

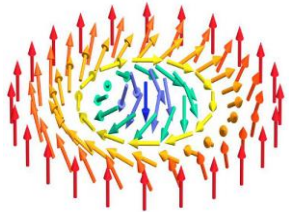
more than two dimensions

# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

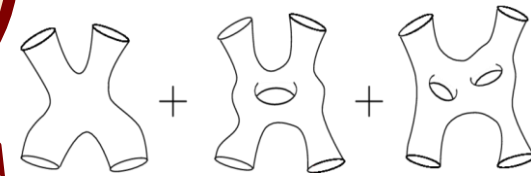
## Particle physics

effective theories of mesons, skyrmions



## Gravity

objects interacting with gravity, spacetimes with symmetries



**$\sigma$ -models**

## Condensed matter

ferromagnetism, magnetic skyrmions, superfluidity, quantum Hall effect, ...

## String and M-theory

description of fundamental objects

### methods (well-understood corners)

**integrability** (*exactly solvable models*),

**topology** (*properties without quantum corrections*),

**symmetry principles** (conformal symmetry, supersymmetry, ...)

### open issues

pert. quantisation (**non-linearity**),

less symmetry

more than two dimensions

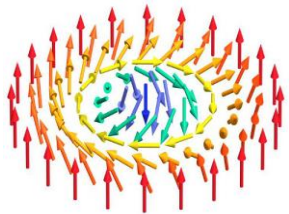
**this talk**

# Motivation

**(Non-linear)  $\sigma$ -models?** big class of theories for model building & fundamental physics

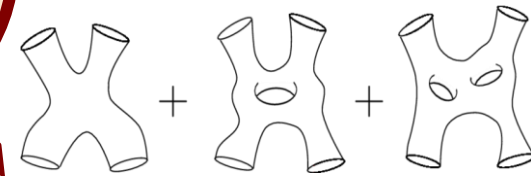
## Particle physics

effective theories of mesons, skyrmions



## Gravity

objects interacting with gravity, spacetimes with symmetries



$\sigma$ -models

## Condensed matter

ferromagnetism, magnetic skyrmions, superfluidity, quantum Hall effect, ...

## String and M-theory

description of fundamental objects

### methods (well-understood corners)

**integrability** (*exactly solvable models*),

**topology** (*properties without quantum corrections*),

**symmetry principles** (conformal symmetry, supersymmetry, ...)

### open issues

pert. quantisation (**non-linearity**),

less symmetry

more than two dimensions

**this talk**

(new symmetry principle (exceptional covariance) for special class of  $\sigma$ -models (sigma models with diff-invariance))

# Programme

---

1. Sigma models
2. (Exceptional) generalised geometry
3. exceptional covariance as a symmetry property

# (Non-linear) $\sigma$ -models

---

$$L \sim g_{mn}(x) \partial_\alpha x^m \partial^\alpha x^n (+C_{m_1 \dots m_p}) \partial_{\alpha_1} x^{m_1} \dots \partial_{\alpha_p} x^{m_p} \epsilon^{\alpha_1 \dots \alpha_p}$$

- fields  $x : \Sigma \rightarrow M$   
 $\sigma^\alpha$ : coordinates of  $\Sigma$ ,  $p = \dim \Sigma - 1$   
 $x^m$ : coordinates on field (target) space  $M$ ,
- 'coupling constants':  $g$  is a metric tensor on  $M$ ,  $C$  a  $(p+1)$ -form potential

# (Non-linear) $\sigma$ -models

$$L \sim g_{mn}(x) \partial_\alpha x^m \partial^\alpha x^n (+C_{m_1 \dots m_p}) \partial_{\alpha_1} x^{m_1} \dots \partial_{\alpha_p} x^{m_p} \epsilon^{\alpha_1 \dots \alpha_p}$$

- fields  $x : \Sigma \rightarrow M$   
 $\sigma^\alpha$ : coordinates of  $\Sigma$ ,  $p = \dim \Sigma - 1$   
 $x^m$ : coordinates on field (target) space  $M$ ,
- 'coupling constants':  $g$  is a metric tensor on  $M$ ,  $C$  a  $(p+1)$ -form potential

$p$	physical model	$\Sigma$	$M$	$C?$
1	point particle coupling to GR and EM	world-line	space-time	$A_\mu$
2	2d materials (continuum limit of spin chains, ...)	material	$O(3), O(N), \dots$	-
	(first quantised) string	world-sheet	space-time	yes
	Geroch model (GR with 2 Killing vectors)	eff. space-time	$SL(2, R)$	-
3	relativistic membrane (e.g. in M-theory)	world-volume	space-time	yes
4	effective theory for pions	space-time	$SU(N)$	-
	original Skyrme model	space-time	$\frac{SU(N) \times SU(N)}{SU(N)}$	Skyrme term

# Geometry and non-linear $\sigma$ -models

---

# Geometry and non-linear $\sigma$ -models

---

- **geometric paradigm:**  
**physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$**



# Geometry and non-linear $\sigma$ -models

---

- **geometric paradigm:**  
**physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$**

classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

# Geometry and non-linear $\sigma$ -models

---

- **geometric paradigm:**  
**physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$**

classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- **here:  $\sigma$ -models with diff. invariance on  $\Sigma$**   
**(,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )**

# Geometry and non-linear $\sigma$ -models

---

- **geometric paradigm:**  
**physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$**

classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- **here:  $\sigma$ -models with diff. invariance on  $\Sigma$**   
**(,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )**

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
after gauge fixing metric on  $\Sigma$ )

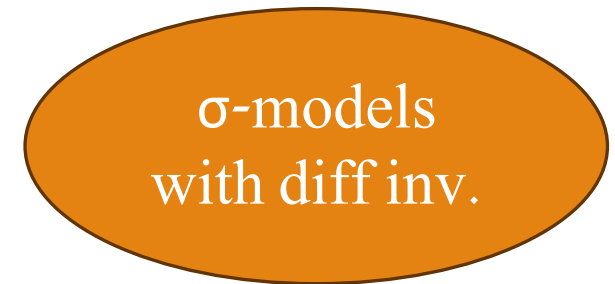
# Geometry and non-linear $\sigma$ -models

- **geometric paradigm:**  
**physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$**

classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- **here:  $\sigma$ -models with diff. invariance on  $\Sigma$**   
**(,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )**

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
after gauge fixing metric on  $\Sigma$ )



# Geometry and non-linear $\sigma$ -models

- **geometric paradigm:**

**physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$**

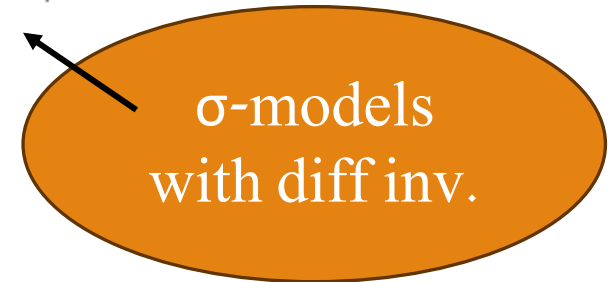
classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- **here:  $\sigma$ -models with diff. invariance on  $\Sigma$**   
(,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
after gauge fixing metric on  $\Sigma$ )

*Polyakov-type actions*

$$L \sim \sqrt{-\gamma} \gamma^{\alpha\beta} g_{mn}(x) \partial_\alpha x^m \partial_\beta x^n + (p-1) + \dots$$



# Geometry and non-linear $\sigma$ -models

- **geometric paradigm:**

**physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$**

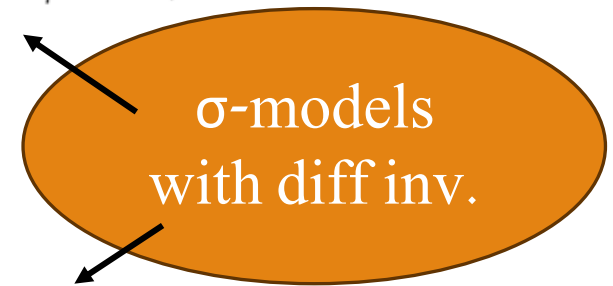
classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
 symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
 for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- **here:  $\sigma$ -models with diff. invariance on  $\Sigma$**   
 (,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
 after gauge fixing metric on  $\Sigma$ )

*Polyakov-type actions*

$$L \sim \sqrt{-\gamma} \gamma^{\alpha\beta} g_{mn}(x) \partial_\alpha x^m \partial_\beta x^n + (p-1) + \dots$$



*Nambu-Goto type actions*

$$L \sim \sqrt{-\det(\gamma_{\alpha\beta} + \alpha' F_{\alpha\beta})} \quad \gamma_{\alpha\beta} = g_{mn} \partial_\alpha x^m \partial_\beta x^n$$

# Geometry and non-linear $\sigma$ -models

- **geometric paradigm:**

**physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$**

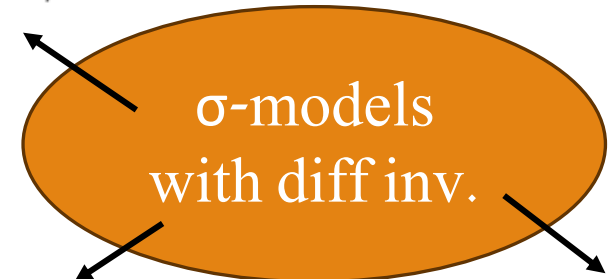
classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
 symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
 for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- **here:  $\sigma$ -models with diff. invariance on  $\Sigma$**   
 (,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
 after gauge fixing metric on  $\Sigma$ )

*Polyakov-type actions*

$$L \sim \sqrt{-\gamma} \gamma^{\alpha\beta} g_{mn}(x) \partial_\alpha x^m \partial_\beta x^n + (p-1) + \dots$$



*Nambu-Goto type actions*

$$L \sim \sqrt{-\det(\gamma_{\alpha\beta} + \alpha' F_{\alpha\beta})} \quad \gamma_{\alpha\beta} = g_{mn} \partial_\alpha x^m \partial_\beta x^n$$

*DBI type actions*

# Geometry and non-linear $\sigma$ -models

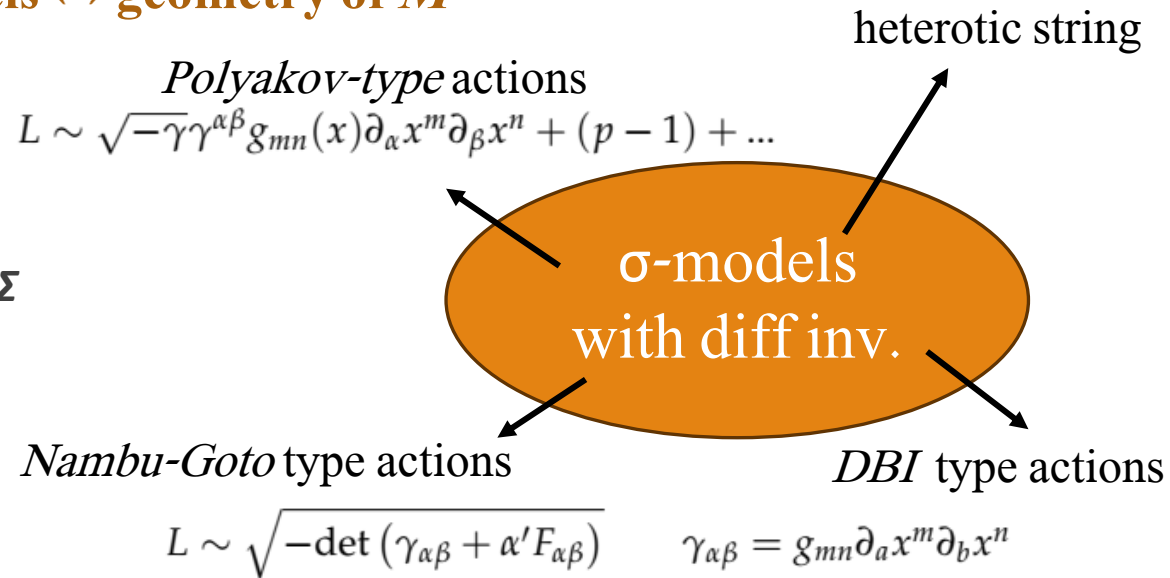
- geometric paradigm:

physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$

classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
 symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
 for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- here:  $\sigma$ -models with diff. invariance on  $\Sigma$   
 (,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
 after gauge fixing metric on  $\Sigma$ )





# Geometry and non-linear $\sigma$ -models

- geometric paradigm:

physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$

**which geometry?**

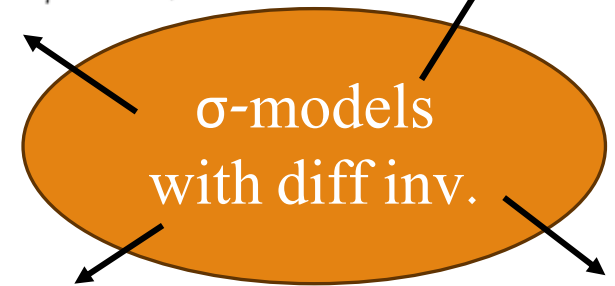
classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
 symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
 for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- here:  $\sigma$ -models with diff. invariance on  $\Sigma$   
 (,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
 after gauge fixing metric on  $\Sigma$ )

*Polyakov-type actions*

$$L \sim \sqrt{-\gamma} \gamma^{\alpha\beta} g_{mn}(x) \partial_\alpha x^m \partial_\beta x^n + (p-1) + \dots$$



heterotic string

*Nambu-Goto type actions*

$$L \sim \sqrt{-\det(\gamma_{\alpha\beta} + \alpha' F_{\alpha\beta})} \quad \gamma_{\alpha\beta} = g_{mn} \partial_\alpha x^m \partial_\beta x^n$$

*DBI type actions*

# Geometry and non-linear $\sigma$ -models

- geometric paradigm:

physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$

**which geometry?**

classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
 symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
 for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

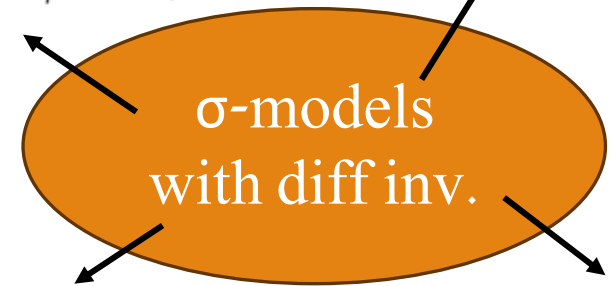
- here:  $\sigma$ -models with diff. invariance on  $\Sigma$   
 (,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
 after gauge fixing metric on  $\Sigma$ )

## Claims:

*Polyakov-type actions*

$$L \sim \sqrt{-\gamma} \gamma^{\alpha\beta} g_{mn}(x) \partial_\alpha x^m \partial_\beta x^n + (p-1) + \dots$$



heterotic string

*Nambu-Goto type actions*

$$L \sim \sqrt{-\det(\gamma_{\alpha\beta} + \alpha' F_{\alpha\beta})} \quad \gamma_{\alpha\beta} = g_{mn} \partial_a x^m \partial_b x^n$$

*DBI type actions*

# Geometry and non-linear $\sigma$ -models

- geometric paradigm:

physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$

**which geometry?**

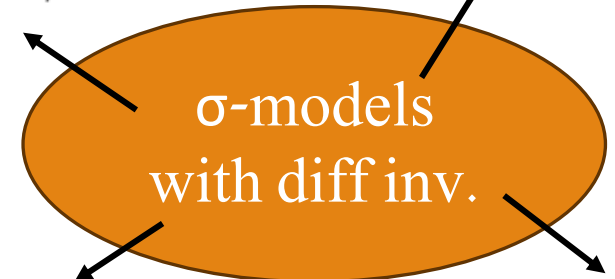
classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
 symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
 for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- here:  $\sigma$ -models with diff. invariance on  $\Sigma$   
 (,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
 after gauge fixing metric on  $\Sigma$ )

*Polyakov-type actions*  

$$L \sim \sqrt{-\gamma} \gamma^{\alpha\beta} g_{mn}(x) \partial_\alpha x^m \partial_\beta x^n + (p-1) + \dots$$



*Nambu-Goto type actions*

*DBI type actions*

$$L \sim \sqrt{-\det(\gamma_{\alpha\beta} + \alpha' F_{\alpha\beta})} \quad \gamma_{\alpha\beta} = g_{mn} \partial_\alpha x^m \partial_\beta x^n$$

## Claims:

- Generalised geometry unifying language of brane  $\sigma$ -models [Duff et al 90-, Siegel et al 92-, Tseytlin 92, Hull 06-, Alekseev/Strobl 04, Hatsuda et al 12-, Sakatani/Uehara 16-, Arvanitakis/Blair 17-, DO 19-]

# Geometry and non-linear $\sigma$ -models

- geometric paradigm:

physical properties of  $\sigma$ -models  $\leftrightarrow$  geometry of  $M$

→ which geometry?

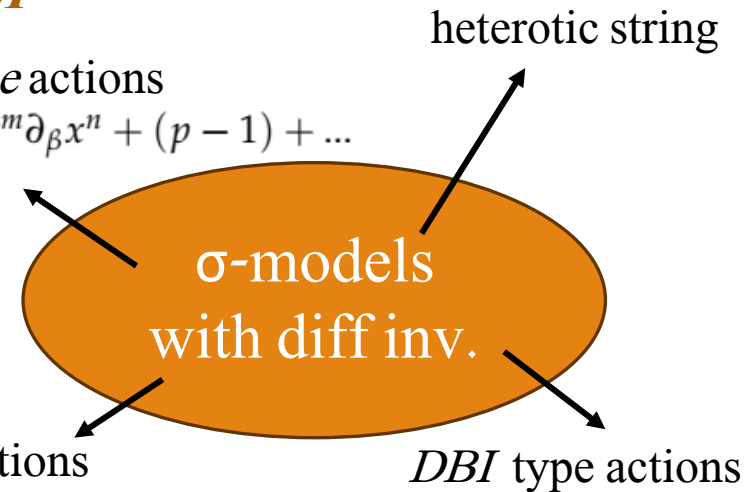
classical solutions  $\leftrightarrow$  minimal volumes in  $M$ ,  
 symmetries of  $\sigma$ -model  $\leftrightarrow$  symmetries of  $M$ ,  
 for simple models  $\beta$ -functions  $\leftrightarrow$  Ricci tensor to  $g_{mn}$  ...

- here:  $\sigma$ -models with diff. invariance on  $\Sigma$   
 (,brane  $\sigma$ -models', non-dyn gravity on  $\Sigma$ )

(In 2d, classically, every  $\sigma$ -model is of ,brane'-type  
 after gauge fixing metric on  $\Sigma$ )

Polyakov-type actions

$$L \sim \sqrt{-\gamma} \gamma^{\alpha\beta} g_{mn}(x) \partial_\alpha x^m \partial_\beta x^n + (p-1) + \dots$$



Nambu-Goto type actions

DBI type actions

$$L \sim \sqrt{-\det(\gamma_{\alpha\beta} + \alpha' F_{\alpha\beta})} \quad \gamma_{\alpha\beta} = g_{mn} \partial_\alpha x^m \partial_\beta x^n$$

## Claims:

- Generalised geometry unifying language of brane  $\sigma$ -models [Duff et al 90-, Siegel et al 92-, Tseytlin 92, Hull 06-, Alekseev/Strobl 04, Hatsuda et al 12-, Sakatani/Uehara 16-, Arvanitakis/Blair 17-, DO 19-]
- Covariance under exceptional generalised geometry restricts to  $\frac{1}{2}$ -BPS branes (string, M-branes, D-branes, non-perturbative/exotic branes, ...) in supergravity [DO 21,23,24, building on Arvanitakis/Blair 17-22]  
 - only considering bosonic part (i.e. no supersymmetry necessary)

# Examples: Hamiltonian formulation of brane $\sigma$ -models

---

# Examples: Hamiltonian formulation of brane $\sigma$ -models

- ex. 1: 2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$

*[Siegel 92, Tseytlin 92]*

# Examples: Hamiltonian formulation of brane $\sigma$ -models

- ex. 1: **2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
  - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$

*[Siegel 92, Tseytlin 92]*

# Examples:

## Hamiltonian formulation of brane $\sigma$ -models

- ex. 1: **2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
  - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$ 
    - $d$  canonical momenta
    - $d$  coordinate fields

[Siegel 92, Tseytlin 92]



# Examples:

## Hamiltonian formulation of brane $\sigma$ -models

- ex. 1: **2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
  - d canonical momenta*  $\swarrow$  *d coordinate fields*  $\searrow$
  - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$
  - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$

*[Siegel 92, Tseytlin 92]*

# Examples:

## Hamiltonian formulation of brane $\sigma$ -models

- ex. 1: **2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
  - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$ 
    - $d$  canonical momenta
    - $d$  coordinate fields
  - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$ 
    - $O(d,d)$ -inv. metric:  $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

[Siegel 92, Tseytlin 92]

# Examples:

## Hamiltonian formulation of brane $\sigma$ -models

- ex. 1: **2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (\overset{\text{d canonical momenta}}{p_m(\sigma)}, \overset{\text{d coordinate fields}}{\partial x^m(\sigma)})$
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
- O(d,d)-inv. metric:  $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

*[Siegel 92, Tseytlin 92]*

# Examples:

## Hamiltonian formulation of brane $\sigma$ -models

- ex. 1: **2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (\overset{\text{d canonical momenta}}{p_m(\sigma)}, \overset{\text{d coordinate fields}}{\partial x^m(\sigma)})$
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
- $O(d,d)$ -inv. metric:  $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$   
 generalised metric:  $\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}$

[Siegel 92, Tseytlin 92]

# Examples:

## Hamiltonian formulation of brane $\sigma$ -models

- ex. 1: **2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (\overset{\text{d canonical momenta}}{p_m(\sigma)}, \overset{\text{d coordinate fields}}{\partial x^m(\sigma)})$
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
    - Diff. (Virasoro) constraint  $0 = \eta^{MN} t_M(\sigma) t_N(\sigma)$
- $O(d,d)$ -inv. metric:  $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$   
 generalised metric:  $\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}$
- [Siegel 92, Tseytlin 92]*

# Examples:

## Hamiltonian formulation of brane $\sigma$ -models

- ex. 1: **2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$ 

*$d$  canonical momenta*       *$d$  coordinate fields*
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
    - Diff. (Virasoro) constraint  $0 = \eta^{MN} t_M(\sigma) t_N(\sigma)$
- [Siegel 92, Tseytlin 92]*

- ex. 2: **membrane  $\sigma$ -model ( $p=2$ ) in  $d$  dim.**  $S \sim \int \left( g_{mn} dx^m \wedge \star dx^n + C_{mnp} dx^m \wedge dx^n \wedge dx^p + \frac{1}{2} \star 1 \right)$

*[Hatsuda/Kamimura 12, DO 21]*

# Examples:

## Hamiltonian formulation of brane $\sigma$ -models

- **ex. 1: 2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$ 

*$d$  canonical momenta*       *$d$  coordinate fields*
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
    - Diff. (Virasoro) constraint  $0 = \eta^{MN} t_M(\sigma) t_N(\sigma)$
- [Siegel 92, Tseytlin 92]*

- **ex. 2: membrane  $\sigma$ -model ( $p=2$ ) in  $d$  dim.**  $S \sim \int \left( g_{mn} dx^m \wedge \star dx^n + C_{mnp} dx^m \wedge dx^n \wedge dx^p + \frac{1}{2} \star 1 \right)$ 
  - Currents  $t_M(\sigma) = (p_m(\sigma), dx^m(\sigma) \wedge dx^n(\sigma), 0, \dots)$

*[Hatsuda/Kamimura 12, DO 21]*

# Examples: Hamiltonian formulation of brane $\sigma$ -models

- **ex. 1: 2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$ 

*$d$  canonical momenta*       *$d$  coordinate fields*
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
    - Diff. (Virasoro) constraint  $0 = \eta^{MN} t_M(\sigma) t_N(\sigma)$
- [Siegel 92, Tseytlin 92]*

- **ex. 2: membrane  $\sigma$ -model ( $p=2$ ) in  $d$  dim.**  $S \sim \int \left( g_{mn} dx^m \wedge \star dx^n + C_{mnp} dx^m \wedge dx^n \wedge dx^p + \frac{1}{2} \star 1 \right)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), dx^m(\sigma) \wedge dx^n(\sigma), 0, \dots)$
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{p,MN} dx^p \wedge d\delta(\sigma - \sigma')$
- [Hatsuda/Kamimura 12, DO 21]*



# Examples: Hamiltonian formulation of brane $\sigma$ -models

- **ex. 1: 2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$ 

*$d$  canonical momenta*  $\swarrow$   $\nwarrow$   *$d$  coordinate fields*
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
    - Diff. (Virasoro) constraint  $0 = \eta^{MN} t_M(\sigma) t_N(\sigma)$
- [Siegel 92, Tseytlin 92]*

- **ex. 2: membrane  $\sigma$ -model ( $p=2$ ) in  $d$  dim.**  $S \sim \int \left( g_{mn} dx^m \wedge \star dx^n + C_{mnp} dx^m \wedge dx^n \wedge dx^p + \frac{1}{2} \star 1 \right)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), dx^m(\sigma) \wedge dx^n(\sigma), 0, \dots)$
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{p,MN} dx^p \wedge d\delta(\sigma - \sigma')$
- [Hatsuda/Kamimura 12, DO 21]*

# Examples:

## Hamiltonian formulation of brane $\sigma$ -models

- **ex. 1: 2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$ 

*$d$  canonical momenta*  $\swarrow$   $\nwarrow$   *$d$  coordinate fields*
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
    - Diff. (Virasoro) constraint  $0 = \eta^{MN} t_M(\sigma) t_N(\sigma)$
- [Siegel 92, Tseytlin 92]*

- **ex. 2: membrane  $\sigma$ -model ( $p=2$ ) in  $d$  dim.**  $S \sim \int \left( g_{mn} dx^m \wedge \star dx^n + C_{mnp} dx^m \wedge dx^n \wedge dx^p + \frac{1}{2} \star 1 \right)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), dx^m(\sigma) \wedge dx^n(\sigma), 0, \dots)$
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{p,MN} dx^p \wedge d\delta(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, C) t_M(\sigma) \wedge \star t_N(\sigma)$
- [Hatsuda/Kamimura 12, DO 21]*

# Examples: Hamiltonian formulation of brane $\sigma$ -models

- **ex. 1: 2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$ 

*$d$  canonical momenta*  $\swarrow$   $\nwarrow$   *$d$  coordinate fields*
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
    - Diff. (Virasoro) constraint  $0 = \eta^{MN} t_M(\sigma) t_N(\sigma)$
- [Siegel 92, Tseytlin 92]*

- **ex. 2: membrane  $\sigma$ -model ( $p=2$ ) in  $d$  dim.**  $S \sim \int \left( g_{mn} dx^m \wedge \star dx^n + C_{mnp} dx^m \wedge dx^n \wedge dx^p + \frac{1}{2} \star 1 \right)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), dx^m(\sigma) \wedge dx^n(\sigma), 0, \dots)$
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{p,MN} dx^p \wedge d\delta(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, C) t_M(\sigma) \wedge \star t_N(\sigma)$
- [Hatsuda/Kamimura 12, DO 21]*

# Examples: Hamiltonian formulation of brane $\sigma$ -models

- **ex. 1: 2dim  $\sigma$ -models and  $O(d,d)$  generalised geometry**  $S \sim \int (g_{mn} dx^m \wedge \star dx^n + B_{mn} dx^m \wedge dx^n)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), \partial x^m(\sigma))$ 



*$d$  canonical momenta*  $\swarrow$   *$d$  coordinate fields*  $\searrow$
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, B) t_M(\sigma) t_N(\sigma)$
    - Diff. (Virasoro) constraint  $0 = \eta^{MN} t_M(\sigma) t_N(\sigma)$
- O(d,d)-inv. metric:  $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$   
 generalised metric:  $\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}$
- [Siegel 92, Tseytlin 92]*

- **ex. 2: membrane  $\sigma$ -model ( $p=2$ ) in  $d$  dim.**  $S \sim \int \left( g_{mn} dx^m \wedge \star dx^n + C_{mnp} dx^m \wedge dx^n \wedge dx^p + \frac{1}{2} \star 1 \right)$ 
    - Currents  $t_M(\sigma) = (p_m(\sigma), dx^m(\sigma) \wedge dx^n(\sigma), 0, \dots)$
    - Poisson brackets  $\{t_M(\sigma), t_N(\sigma')\} = \eta_{p,MN} dx^p \wedge d\delta(\sigma - \sigma')$
    - Hamiltonian  $H = \frac{1}{2} \mathcal{H}^{MN}(g, C) t_M(\sigma) \wedge \star t_N(\sigma)$
    - Spatial diff. constraints  $0 = \eta^{p,MN} t_M(\sigma) \wedge \star t_N(\sigma)$
- invariant of  $E_{d(d)}$   
 generalised metric of  $E_{d(d)}$
- [Hatsuda/Kamimura 12, DO 21]*

# Generalised geometry (overview)



	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<p><i>‘tensor hierarchy of bundles’</i></p> <ul style="list-style-type: none"> <li>• generalised tangent bundle <i>‘<math>R_1</math>-bundle’</i>: <math>TM \oplus \bigwedge^p T^*M \oplus \dots</math></li> <li>• <i>‘<math>R_2</math>-bundle’</i>: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>• ...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique)
curvature	Riemann tensor	generalised Riemann tensor (not unique)

# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<p><i>,tensor‘ hierarchy of bundles</i></p> <ul style="list-style-type: none"> <li>• generalised tangent bundle <i>,<math>R_1</math>-bundle‘</i>: <math>TM \oplus \bigwedge^p T^*M \oplus \dots</math></li> <li>• <i>,<math>R_2</math>-bundle‘</i>: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>• ...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique) 
curvature	Riemann tensor	generalised Riemann tensor (not unique) 

See Larisa's & Falk's talks



# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<p><i>,tensor‘ hierarchy of bundles</i></p> <ul style="list-style-type: none"> <li>• generalised tangent bundle <i>,<math>R_1</math>-bundle‘</i>: <math>TM \oplus \bigwedge^p T^*M \oplus \dots</math></li> <li>• <i>,<math>R_2</math>-bundle‘</i>: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>• ...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique) 
curvature	Riemann tensor	generalised Riemann tensor (not unique) 

See Larisa's & Falk's talks

- two aims:

# Generalised geometry (overview)



	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<p><i>,tensor‘ hierarchy of bundles</i></p> <ul style="list-style-type: none"> <li>• generalised tangent bundle <i>,<math>R_1</math>-bundle‘</i>: <math>TM \oplus \bigwedge^p T^*M \oplus \dots</math></li> <li>• <i>,<math>R_2</math>-bundle‘</i>: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>• ...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique) 
curvature	Riemann tensor	generalised Riemann tensor (not unique) 

See Larisa's & Falk's talks

- two aims:
  - **unified treatment of metric and  $p$ -form gauge fields**  
e.g. *,gauge transformations‘* for metric and  $p$ -form gauge field



# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<p><i>,tensor‘ hierarchy of bundles</i></p> <ul style="list-style-type: none"> <li>• generalised tangent bundle <i>,<math>R_1</math>-bundle‘</i>: <math>TM \oplus \bigwedge^1 T^*M \oplus \dots</math></li> <li>• <i>,<math>R_2</math>-bundle‘</i>: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>• ...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique) 
curvature	Riemann tensor	generalised Riemann tensor (not unique) 

See Larisa's & Falk's talks

- two aims:

- **unified treatment of metric and  $p$ -form gauge fields**

e.g. *,gauge transformations‘* for metric and  $p$ -form gauge field  $\phi = v + \zeta \in \Gamma(TM \oplus \bigwedge^p T^*M)$

# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<i>tensor hierarchy of bundles</i> • generalised tangent bundle ‘ $R_1$ -bundle’: $TM \oplus \bigwedge^p T^*M \oplus \dots$ • ‘ $R_2$ -bundle’: $\bigwedge^{p-1} T^*M \oplus \dots$ • ...
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique)
curvature	Riemann tensor	generalised Riemann tensor (not unique)

• two aims:

- **unified treatment of metric and  $p$ -form gauge fields**

e.g. ‘gauge transformations’ for metric and  $p$ -form gauge field

vector (diff.)       $p$ -form ( $p+1$ -form gauge transformation)

$$\phi = v + \zeta \in \Gamma(TM \oplus \bigwedge^p T^*M)$$

generalised vector (diffeomorphism)

# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<i>tensor hierarchy of bundles</i> <ul style="list-style-type: none"> <li>generalised tangent bundle '<math>\mathcal{R}_1</math>-bundle': <math>TM \oplus \bigwedge^p T^*M \oplus \dots</math></li> <li>'<math>\mathcal{R}_2</math>-bundle': <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique)
curvature	Riemann tensor	generalised Riemann tensor (not unique)

- two aims:

- unified treatment of metric and  $p$ -form gauge fields**

e.g. ' $\mathcal{R}_1$ -gauge transformations' for metric and  $p$ -form gauge field

generalised Lie derivative (algebra of gauge transformations)

vector (diff.)       $p$ -form ( $p+1$ -form gauge transformation)

$$\phi = v + \zeta \in \Gamma(TM \oplus \bigwedge^p T^*M)$$

generalised vector (diffeomorphism)

# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<i>tensor hierarchy of bundles</i> <ul style="list-style-type: none"> <li>generalised tangent bundle <i>,<math>R_1</math>-bundle'</i>: <math>TM \oplus \bigwedge^p T^*M \oplus \dots</math></li> <li><i>,<math>R_2</math>-bundle'</i>: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique)
curvature	Riemann tensor	generalised Riemann tensor (not unique)

- two aims:

- unified treatment of metric and  $p$ -form gauge fields**

e.g. *'gauge transformations'* for metric and  $p$ -form gauge field

generalised Lie derivative (algebra of gauge transformations)

$$\mathcal{L}_{\phi_1} \phi_2^N = 2\phi_{[1}^M \partial_M \phi_2^N] + Y^{MN}{}_{KL} \partial_M \phi_1^K \phi_2^L$$

vector (diff.)       $p$ -form ( $p+1$ -form gauge transformation)

$$\phi = v + \xi \in \Gamma(TM \oplus \bigwedge^p T^*M)$$

generalised vector (diffeomorphism)

# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<i>tensor hierarchy of bundles</i> <ul style="list-style-type: none"> <li>generalised tangent bundle <i>,<math>R_1</math>-bundle'</i>: <math>TM \oplus \bigwedge^p T^*M \oplus \dots</math></li> <li><i>,<math>R_2</math>-bundle'</i>: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique) <span style="color: red;">↗ See Larisa's &amp; Falk's talks</span>
curvature	Riemann tensor	generalised Riemann tensor (not unique)

- two aims:

- unified treatment of metric and  $p$ -form gauge fields**

e.g. *'gauge transformations'* for metric and  $p$ -form gauge field

generalised Lie derivative (algebra of gauge transformations)

$$\mathcal{L}_{\phi_1} \phi_2^N = 2\phi_{[1}^M \partial_M \phi_2^N] + \underbrace{Y^{MN}}_{\text{G-invariant}} \partial_M \phi_1^K \phi_2^L$$

vector (diff.)       $p$ -form ( $p+1$ -form gauge transformation)

$$\phi = v + \zeta \in \Gamma(TM \oplus \bigwedge^p T^*M)$$

generalised vector (diffeomorphism)

G-invariant

# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<i>tensor hierarchy of bundles</i> <ul style="list-style-type: none"> <li>generalised tangent bundle ‘<math>R_1</math>-bundle’: <math>TM \oplus \bigwedge^p T^*M \oplus \dots</math></li> <li>‘<math>R_2</math>-bundle’: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique) <span style="color: red;">↗ See Larisa’s &amp; Falk’s talks</span>
curvature	Riemann tensor	generalised Riemann tensor (not unique)

- two aims:

- unified treatment of metric and  $p$ -form gauge fields**

e.g. ‘gauge transformations’ for metric and  $p$ -form gauge field

generalised Lie derivative (algebra of gauge transformations)

$$\mathcal{L}_{\phi_1} \phi_2^N = 2\phi_{[1}^M \partial_M \phi_2^N] + \underbrace{Y^{MN}}_{\text{generalised vector}} \partial_M \phi_1^K \phi_2^L$$

- covariance under duality groups**

e.g. when interpreted as string/M-theory duality groups: **hidden symmetry made explicit**

vector (diff.)       $p$ -form ( $p+1$ -form gauge transformation)

$$\phi = v + \xi \in \Gamma(TM \oplus \bigwedge^p T^*M)$$

generalised vector (diffeomorphism)

G-invariant

# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<i>tensor hierarchy of bundles</i> <ul style="list-style-type: none"> <li>generalised tangent bundle ‘<math>R_1</math>-bundle’: <math>TM \oplus \bigwedge^1 T^*M \oplus \dots</math></li> <li>‘<math>R_2</math>-bundle’: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique) <span style="color: red;">↗ See Larisa’s &amp; Falk’s talks</span>
curvature	Riemann tensor	generalised Riemann tensor (not unique)

- two aims:

- unified treatment of metric and  $p$ -form gauge fields**

e.g. ‘gauge transformations’ for metric and  $p$ -form gauge field

generalised Lie derivative (algebra of gauge transformations)

$$\mathcal{L}_{\phi_1} \phi_2^N = 2\phi_{[1}^M \partial_M \phi_2^N] + \underbrace{Y^{MN}}_{\text{generalised vector (diffeomorphism)}} \partial_M \phi_1^K \phi_2^L$$

- covariance under duality groups**

e.g. when interpreted as string/M-theory duality groups: **hidden symmetry made explicit**

typically: applied as technique in supergravity, **here: for  $\sigma$ -model**


vector (diff.)       $p$ -form ( $p+1$ -form gauge transformation)

$$\phi = v + \zeta \in \Gamma(TM \oplus \bigwedge^p T^*M)$$

generalised vector (diffeomorphism)

G-invariant

# Generalised geometry (overview)

	Riemannian geometry	Generalised Geometry
physical objects	metric $g$	metric $g$ , $(p+1)$ -form gauge fields $C, \dots$ , dilaton
underlying bundle	tangent bundle $TM$	<i>,tensor‘ hierarchy of bundles</i> <ul style="list-style-type: none"> <li>generalised tangent bundle <i>,<math>R_1</math>-bundle‘</i>: <math>TM \oplus \bigwedge^p T^*M \oplus \dots</math></li> <li><i>,<math>R_2</math>-bundle‘</i>: <math>\bigwedge^{p-1} T^*M \oplus \dots</math></li> <li>...</li> </ul>
geometry	metric $g \in \frac{GL(d)}{O(d)}$	generalised metric: $\mathcal{H} \in \frac{G}{H_d}$ $H_d$ : maximal compact subgroup of $G$
structure group	$GL(d)$	duality group $G = O(d,d), E_{d(d)}, \dots$ , action on bundles invariants of $G$ describe algebraic structure of tensor hierarchy
connection	Levi-Civita	generalised Levi-Civita (not unique) 
curvature	Riemann tensor <b>not relevant for talk</b>	generalised Riemann tensor (not unique)

• two aims:

- **unified treatment of metric and  $p$ -form gauge fields**

e.g. *,gauge transformations‘* for metric and  $p$ -form gauge field

generalised Lie derivative (algebra of gauge transformations)

$$\mathcal{L}_{\phi_1} \phi_2^N = 2\phi_{[1}^M \partial_M \phi_2^N] + \underbrace{Y^{MN}}_{\text{generalised vector (diffeomorphism)}} \partial_M \phi_1^K \phi_2^L$$

- **covariance under duality groups**

e.g. when interpreted as string/M-theory duality groups: **hidden symmetry made explicit**

typically: applied as technique in supergravity, **here: for  $\sigma$ -model**

vector (diff.)       $p$ -form ( $p+1$ -form gauge transformation)

$$\phi = v + \zeta \in \Gamma(TM \oplus \bigwedge^p T^*M)$$

generalised vector (diffeomorphism)

G-invariant



# Generalised geometry (slightly more formal)

---

# Generalised geometry (slightly more formal)

---

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^{\mathcal{M}} = (x^m, y^\mu)$

# Generalised geometry (slightly more formal)

---

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^{\mathcal{M}} = (x^m, y^\mu)$   
*internal coordinates (generalised geometry)*      *external coordinates (ord. geometry)*

# Generalised geometry (slightly more formal)

---

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^{\mathcal{M}} = (x^m, y^\mu)$   
*internal coordinates (generalised geometry)*      *external coordinates (ord. geometry)*
- duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$

# Generalised geometry (slightly more formal)

---

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^{\mathcal{M}} = (x^m, y^\mu)$   
*internal coordinates (generalised geometry)*      *external coordinates (ord. geometry)*
- duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$
- tensor hierarchy (& associated bundles) – representations of  $G$ :  $\mathfrak{g}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$

# Generalised geometry (slightly more formal)

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^{\mathcal{M}} = (x^m, y^\mu)$ 
  - internal coordinates (generalised geometry)* (pointing to  $x^m$ )
  - external coordinates (ord. geometry)* (pointing to  $y^\mu$ )
- duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$
- tensor hierarchy (& associated bundles) – representations of  $G$ :  $\mathfrak{g}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$ 
  - algebraic structure:
    - dot product  $\bullet : \mathcal{R}_p \times \mathcal{R}_q \rightarrow \mathcal{R}_{p+q}$
    - derivative  $\partial : \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$  with  $\partial^2 = 0$

# Generalised geometry (slightly more formal)

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^{\mathcal{M}} = (x^m, y^\mu)$ 
  - $x^m$ : internal coordinates (generalised geometry)
  - $y^\mu$ : external coordinates (ord. geometry)
- duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$
- tensor hierarchy (& associated bundles) – representations of  $G$ :  $\mathfrak{g}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$ 
  - algebraic structure:
    - dot product  $\bullet : \mathcal{R}_p \times \mathcal{R}_q \rightarrow \mathcal{R}_{p+q}$
    - derivative  $\partial : \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$  with  $\partial^2 = 0$
  - in indices  $A \in \mathcal{R}_p$ ,  $A^{K_p}$ 
    - $(A \bullet B)^{M_{p+q}} = \eta^{M_{p+q}}{}_{K_p L_q} A^{K_p} B^{L_q}$
    - $(\partial A)^{M_{p-1}} = D_{M_p}{}^{M_{p-1}, L_1} \partial_{L_1} A^{M_p}$

# Generalised geometry (slightly more formal)

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^{\mathcal{M}} = (x^m, y^\mu)$ 
    - $x^m$ : internal coordinates (generalised geometry)
    - $y^\mu$ : external coordinates (ord. geometry)
  - duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$
  - tensor hierarchy (& associated bundles) – representations of  $G$ :  $\mathfrak{g}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$ 
    - algebraic structure:
      - dot product  $\bullet : \mathcal{R}_p \times \mathcal{R}_q \rightarrow \mathcal{R}_{p+q}$
      - derivative  $\partial : \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$  with  $\partial^2 = 0$
    - in indices  $A \in \mathcal{R}_p, A^{K_p}$ 
      - $(A \bullet B)^{M_{p+q}} = \eta^{M_{p+q}}{}_{K_p L_q} A^{K_p} B^{L_q}$
      - $(\partial A)^{M_{p-1}} = D_{M_p}{}^{M_{p-1}, L_1} \partial_{L_1} A^{M_p}$
- $G$ -invariants



# Generalised geometry (slightly more formal)

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^M = (x^m, y^\mu)$ 
  - $x^m$ : internal coordinates (generalised geometry)
  - $y^\mu$ : external coordinates (ord. geometry)
- duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$
- tensor hierarchy (& associated bundles) – representations of  $G$ :  $\mathfrak{g}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$ 
  - algebraic structure:
    - dot product  $\bullet : \mathcal{R}_p \times \mathcal{R}_q \rightarrow \mathcal{R}_{p+q}$
    - derivative  $\partial : \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$  with  $\partial^2 = 0$
  - in indices  $A \in \mathcal{R}_p, A^{K_p}$ 
    - $(A \bullet B)^{M_{p+q}} = \eta^{M_{p+q}}{}_{K_p L_q} A^{K_p} B^{L_q}$
    - $(\partial A)^{M_{p-1}} = D_{M_p}{}^{M_{p-1}, L_1} \partial_{L_1} A^{M_p}$
- consistency conditions:  $\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \dots$  together with  $\bullet$  and  $\partial$  forms diff. graded Lie algebra  
*[Palmkvist, Cederwall/Palmkvist, Lavau/Palmkvist, Bonezzi/Hohm]*

# Generalised geometry (slightly more formal)

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^{\mathcal{M}} = (x^m, y^\mu)$ 
  - $x^m$ : internal coordinates (generalised geometry)
  - $y^\mu$ : external coordinates (ord. geometry)
- duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$
- tensor hierarchy (& associated bundles) – representations of  $G$ :  $\mathfrak{g}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$ 
  - algebraic structure:
    - dot product  $\bullet : \mathcal{R}_p \times \mathcal{R}_q \rightarrow \mathcal{R}_{p+q}$
    - derivative  $\partial : \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$  with  $\partial^2 = 0$
  - in indices  $A \in \mathcal{R}_p, A^{K_p}$ 
    - $(A \bullet B)^{M_{p+q}} = \eta^{M_{p+q}}{}_{K_p L_q} A^{K_p} B^{L_q}$
    - $(\partial A)^{M_{p-1}} = D_{M_p}{}^{M_{p-1}, L_1} \partial_{L_1} A^{M_p}$
- consistency conditions:  $\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \dots$  together with  $\bullet$  and  $\partial$  forms diff. graded Lie algebra  
*[Palmkvist, Cederwall/Palmkvist, Lavau/Palmkvist, Bonezzi/Hohm]*
- extended internal coordinates:  $X^{M_1} = (x^m, \tilde{x}_{m_1 \dots m_p}, \dots)$

# Generalised geometry (slightly more formal)

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^M = (x^m, y^\mu)$ 
  - internal coordinates (generalised geometry)
  - external coordinates (ord. geometry)
- duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$
- tensor hierarchy (& associated bundles) – representations of  $G$ :  $\mathfrak{g}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$ 
  - algebraic structure: dot product  $\bullet : \mathcal{R}_p \times \mathcal{R}_q \rightarrow \mathcal{R}_{p+q}$   
 derivative  $\partial : \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$  with  $\partial^2 = 0$
  - in indices  $A \in \mathcal{R}_p, A^{K_p}$ 

$$(A \bullet B)^{M_{p+q}} = \eta^{M_{p+q}}{}_{K_p L_q} A^{K_p} B^{L_q}$$

$$(\partial A)^{M_{p-1}} = D_{M_p}{}^{M_{p-1}, L_1} \partial_{L_1} A^{M_p}$$

$G$ -invariants
  - consistency conditions:  $\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \dots$  together with  $\bullet$  and  $\partial$  forms diff. graded Lie algebra  
*[Palmkvist, Cederwall/Palmkvist, Lavau/Palmkvist, Bonezzi/Hohm]*
- extended internal coordinates:  $X^{M_1} = (x^m, \tilde{x}_{m_1 \dots m_p}, \dots)$
- section condition:  $D_{M_2}{}^{K_1, L_1} \partial_{K_1} \cdot \partial_{L_1} \cdot = 0$ , solutions:  $\partial_{M_1} = (\partial_m, 0, \dots)$

# Generalised geometry (slightly more formal)

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^M = (x^m, y^\mu)$ 
  - internal coordinates (generalised geometry)
  - external coordinates (ord. geometry)
- duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$
- tensor hierarchy (& associated bundles) – representations of  $G$ :  $\mathfrak{g}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$ 
  - algebraic structure: dot product  $\bullet : \mathcal{R}_p \times \mathcal{R}_q \rightarrow \mathcal{R}_{p+q}$   
 derivative  $\partial : \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$  with  $\partial^2 = 0$
  - in indices  $A \in \mathcal{R}_p$ ,  $A^{K_p}$ 

$$(A \bullet B)^{M_{p+q}} = \eta^{M_{p+q}}{}_{K_p L_q} A^{K_p} B^{L_q}$$

$$(\partial A)^{M_{p-1}} = D_{M_p}{}^{M_{p-1}, L_1} \partial_{L_1} A^{M_p}$$

$G$ -invariants
  - consistency conditions:  $\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \dots$  together with  $\bullet$  and  $\partial$  forms diff. graded Lie algebra  
*[Palmkvist, Cederwall/Palmkvist, Lavau/Palmkvist, Bonezzi/Hohm]*
- extended internal coordinates:  $X^{M_1} = (x^m, \tilde{x}_{m_1 \dots m_p}, \dots)$
- section condition:  $D_{M_2}{}^{K_1, L_1} \partial_{K_1} \cdot \partial_{L_1} \cdot = 0$ , solutions:  $\partial_{M_1} = (\partial_m, 0, \dots)$
- generalised Lie derivative  $\mathcal{L}_V \phi$ ,  $V \in \mathcal{R}_1$ ,  $\phi \in \mathcal{R}_p$ , algebra closes:  $[\mathcal{L}_{V_1}, \mathcal{L}_{V_2}] \phi = \mathcal{L}_{\mathcal{L}_{V_1} V_2} \phi$

# Generalised geometry (slightly more formal)

- d-dimensional internal manifold in splitting  $D = (D-d)+d$ , coordinates  $Y^M = (x^m, y^\mu)$ 
  - internal coordinates (generalised geometry)
  - external coordinates (ord. geometry)
- duality group  $G$ :  $E_{d(d)}$ ,  $O(d,d)$ ,  $O(d,d+n)$ ,  $SL(d+1)$
- tensor hierarchy (& associated bundles) – representations of  $G$ :  $\mathfrak{g}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$ 
  - algebraic structure: dot product  $\bullet : \mathcal{R}_p \times \mathcal{R}_q \rightarrow \mathcal{R}_{p+q}$   
 derivative  $\partial : \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$  with  $\partial^2 = 0$
  - in indices  $A \in \mathcal{R}_p$   $A^{K_p}$ 

$$(A \bullet B)^{M_{p+q}} = \eta^{M_{p+q}}{}_{K_p L_q} A^{K_p} B^{L_q}$$

$$(\partial A)^{M_{p-1}} = D_{M_p}{}^{M_{p-1}, L_1} \partial_{L_1} A^{M_p}$$

$\leftarrow$   $G$ -invariants
  - consistency conditions:  $\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \dots$  together with  $\bullet$  and  $\partial$  forms diff. graded Lie algebra  
*[Palmkvist, Cederwall/Palmkvist, Lavau/Palmkvist, Bonezzi/Hohm]*
- extended internal coordinates:  $X^{M_1} = (x^m, \tilde{x}_{m_1 \dots m_p}, \dots)$
- section condition:  $D_{M_2}{}^{K_1, L_1} \partial_{K_1} \cdot \partial_{L_1} \cdot = 0$ , solutions:  $\partial_{M_1} = (\partial_m, 0, \dots)$
- generalised Lie derivative  $\mathcal{L}_V \phi$ ,  $V \in \mathcal{R}_1$ ,  $\phi \in \mathcal{R}_p$ , algebra closes:  $[\mathcal{L}_{V_1}, \mathcal{L}_{V_2}] \phi = \mathcal{L}_{\mathcal{L}_{V_1} V_2} \phi$
- geometry:
  - generalised metric  $\mathcal{H}^{M_1 N_1}(g, C, \dots) \in \frac{G}{H_d}$
  - external background  $A_{\mu_1 \dots \mu_p}^{M_p} \in \mathcal{R}_p$ ,  $g_{\mu\nu}$  (ext. metric)

# Realisation in $p$ -dim. $\sigma$ -models

---

# Realisation in $p$ -dim. $\sigma$ -models

---

- phase space variables (currents):

# Realisation in $p$ -dim. $\sigma$ -models

---

- phase space variables (currents):

spatial  $(p - q + 1)$ -forms (for all  $q > 0$ ):

$$t^{(q)}(\sigma) \in \mathcal{R}_q$$



# Realisation in $p$ -dim. $\sigma$ -models

---

- phase space variables (currents):

spatial  $(p - q + 1)$ -forms (for all  $q > 0$ ):

$$t^{(q)}(\sigma) \in \mathcal{R}_q$$

- Poisson brackets (realisation of  $\bullet$ -product)

$$\{t_{K_q}^{(q)}(\sigma), t_{L_r}^{(r)}(\sigma')\} = \eta^{M_{q+r}} K_q L_r t_{M_{q+r}}^{(q+r)}(\sigma) \wedge d\delta(\sigma - \sigma')$$

# Realisation in $p$ -dim. $\sigma$ -models

- phase space variables (currents):

spatial  $(p - q + 1)$ -forms (for all  $q > 0$ ):  $t^{(q)}(\sigma) \in \mathcal{R}_q$

- Poisson brackets (realisation of  $\bullet$ -product)

$$\{t_{K_q}^{(q)}(\sigma), t_{L_r}^{(r)}(\sigma')\} = \eta^{M_{q+r}} K_q L_r t_{M_{q+r}}^{(q+r)}(\sigma) \wedge d\delta(\sigma - \sigma')$$

- Hamiltonian and diff. constraints (extension to external space exists)

$$H = \frac{1}{2} \mathcal{H}^{M_1 N_1} t_{M_1}^{(1)} \wedge \star t_{N_1}^{(1)}$$

$$0 = \eta_{M_2}^{K_1 L_1} t_{K_1}^{(1)} \wedge \star t_{L_1}^{(1)}$$

# Realisation in $p$ -dim. $\sigma$ -models

- phase space variables (currents):

spatial  $(p - q + 1)$ -forms (for all  $q > 0$ ):  $t^{(q)}(\sigma) \in \mathcal{R}_q$

- Poisson brackets (realisation of  $\bullet$ -product)

$$\{t_{K_q}^{(q)}(\sigma), t_{L_r}^{(r)}(\sigma')\} = \eta^{M_{q+r}} K_q L_r t_{M_{q+r}}^{(q+r)}(\sigma) \wedge d\delta(\sigma - \sigma')$$

- Hamiltonian and diff. constraints (extension to external space exists)

$$H = \frac{1}{2} \mathcal{H}^{M_1 N_1} t_{M_1}^{(1)} \wedge \star t_{N_1}^{(1)}$$

$$0 = \eta_{M_2} K_1 L_1 t_{K_1}^{(1)} \wedge \star t_{L_1}^{(1)}$$

- example – membrane currents:

$$t_{M_1}^{(1)} = (p_m, dx^m \wedge dx^{m'}, 0, \dots)$$

$$t_{M_2}^{(2)} = (dx^m, 0, \dots)$$

$$t_{M_3}^{(3)} = (1, 0, \dots), \quad t_{M_q}^{(q)} = 0 \quad \text{for } q \geq 4$$

# Generalised Lie bracket and brane charge constraint

---

- **All** diff-invariant  $\sigma$ -models can be realised like that (Claim 1).  
It turns that it is not a very strong statement, concrete choices of  $t^{(q)}(\sigma)$  can break  $G$ (duality) symmetry completely.

# Generalised Lie bracket and brane charge constraint

---

- **All** diff-invariant  $\sigma$ -models can be realised like that (Claim 1).  
It turns that it is not a very strong statement, concrete choices of  $t^{(q)}(\sigma)$  can break  $G$ (duality) symmetry completely.
- Stronger statement: **G-covariance on phase space**  
For this, the *generalised Lie bracket* should be realised on phase space:

# Generalised Lie bracket and brane charge constraint

- **All** diff-invariant  $\sigma$ -models can be realised like that (Claim 1).  
It turns that it is not a very strong statement, concrete choices of  $t^{(q)}(\sigma)$  can break  $G$  (duality) symmetry completely.
- Stronger statement: **G-covariance on phase space**  
For this, the *generalised Lie bracket* should be realised on phase space:

$$V = \int V^{M_1}(x) t_{M_1}^{(1)} \in \mathcal{R}_1, \quad \phi = \int \phi^{M_p}(x) t_{M_p}^{(p)} \in \mathcal{R}_p, \quad \boxed{\{\phi, V\} \stackrel{!}{=} \int (\mathcal{L}_V \phi)^{M_p} t_{M_p}^{(p)}}$$

# Generalised Lie bracket and brane charge constraint

- All diff-invariant  $\sigma$ -models can be realised like that (Claim 1).  
It turns that it is not a very strong statement, concrete choices of  $t^{(q)}(\sigma)$  can break  $G$ (duality) symmetry completely.

- Stronger statement: **G-covariance on phase space**  
For this, the *generalised Lie bracket* should be realised on phase space:

$$V = \int V^{M_1}(x) t_{M_1}^{(1)} \in \mathcal{R}_1, \quad \phi = \int \phi^{M_p}(x) t_{M_p}^{(p)} \in \mathcal{R}_p, \quad \boxed{\{\phi, V\} \stackrel{!}{=} \int (\mathcal{L}_V \phi)^{M_p} t_{M_p}^{(p)}}$$

- This requires a **hierarchy of constraints**:

# Generalised Lie bracket and brane charge constraint

- All diff-invariant  $\sigma$ -models can be realised like that (Claim 1).  
It turns that it is not a very strong statement, concrete choices of  $t^{(q)}(\sigma)$  can break  $G$ (duality) symmetry completely.

- Stronger statement: **G-covariance on phase space**  
For this, the *generalised Lie bracket* should be realised on phase space:

$$V = \int V^{M_1}(x) t_{M_1}^{(1)} \in \mathcal{R}_1, \quad \phi = \int \phi^{M_p}(x) t_{M_p}^{(p)} \in \mathcal{R}_p, \quad \boxed{\{\phi, V\} \stackrel{!}{=} \int (\mathcal{L}_V \phi)^{M_p} t_{M_p}^{(p)}}$$

- This requires a **hierarchy of constraints**:

$$\boxed{t_{K_{p+1}}^{(p+1)} \wedge dX^{L_1} \partial_{L_1} = D_{K_{p+1}}^{M_p, L_1} t_{M_p}^{(p)} \partial_{L_1}}$$



# Generalised Lie bracket and brane charge constraint

- All diff-invariant  $\sigma$ -models can be realised like that (Claim 1).  
It turns that it is not a very strong statement, concrete choices of  $t^{(q)}(\sigma)$  can break  $G$ (duality) symmetry completely.

- Stronger statement: **G-covariance on phase space**  
For this, the *generalised Lie bracket* should be realised on phase space:

$$V = \int V^{M_1}(x) t_{M_1}^{(1)} \in \mathcal{R}_1, \quad \phi = \int \phi^{M_p}(x) t_{M_p}^{(p)} \in \mathcal{R}_p, \quad \boxed{\{\phi, V\} \stackrel{!}{=} \int (\mathcal{L}_V \phi)^{M_p} t_{M_p}^{(p)}}$$

- This requires a **hierarchy of constraints**:

$$\boxed{t_{K_{p+1}}^{(p+1)} \wedge dX^{L_1} \partial_{L_1} = D_{K_{p+1}}^{M_p, L_1} t_{M_p}^{(p)} \partial_{L_1}}$$

- **Lessons/Results** for  $G = E_{d(d)}$ : *[DO, 21, 23, 24]*

# Generalised Lie bracket and brane charge constraint

- All diff-invariant  $\sigma$ -models can be realised like that (Claim 1).  
It turns that it is not a very strong statement, concrete choices of  $t^{(q)}(\sigma)$  can break  $G$ (duality) symmetry completely.

- Stronger statement: **G-covariance on phase space**  
For this, the **generalised Lie bracket** should be realised on phase space:

$$V = \int V^{M_1}(x) t_{M_1}^{(1)} \in \mathcal{R}_1, \quad \phi = \int \phi^{M_p}(x) t_{M_p}^{(p)} \in \mathcal{R}_p, \quad \boxed{\{\phi, V\} \stackrel{!}{=} \int (\mathcal{L}_V \phi)^{M_p} t_{M_p}^{(p)}}$$

- This requires a **hierarchy of constraints**:

$$\boxed{t_{K_{p+1}}^{(p+1)} \wedge dX^{L_1} \partial_{L_1} = D_{K_{p+1}}^{M_p, L_1} t_{M_p}^{(p)} \partial_{L_1}}$$

- **Lessons/Results** for  $G = E_{d(d)}$ : [DO, 21,23,24]

- Constraints are very restrictive:

**only  $\frac{1}{2}$ -BPS branes (string, membrane, D-branes, ...) are allowed.**

# Generalised Lie bracket and brane charge constraint

- All diff-invariant  $\sigma$ -models can be realised like that (Claim 1).  
It turns that it is not a very strong statement, concrete choices of  $t^{(q)}(\sigma)$  can break  $G$ (duality) symmetry completely.

- Stronger statement: **G-covariance on phase space**  
For this, the *generalised Lie bracket* should be realised on phase space:

$$V = \int V^{M_1}(x) t_{M_1}^{(1)} \in \mathcal{R}_1, \quad \phi = \int \phi^{M_p}(x) t_{M_p}^{(p)} \in \mathcal{R}_p, \quad \boxed{\{\phi, V\} \stackrel{!}{=} \int (\mathcal{L}_V \phi)^{M_p} t_{M_p}^{(p)}}$$

- This requires a **hierarchy of constraints**:

$$\boxed{t_{K_{p+1}}^{(p+1)} \wedge dX^{L_1} \partial_{L_1} = D_{K_{p+1}}^{M_p, L_1} t_{M_p}^{(p)} \partial_{L_1}}$$

- **Lessons/Results** for  $G = E_{d(d)}$ : [DO, 21,23,24]

- Constraints are very restrictive:

**only  $\frac{1}{2}$ -BPS branes (string, membrane, D-branes, ...) are allowed.**

- remarkable, as only bosonic part of  $\sigma$ -models
- Still: solutions break  $E_{d(d)} \rightarrow GL(d)$

# Generalised Lie bracket and brane charge constraint

- All diff-invariant  $\sigma$ -models can be realised like that (Claim 1).  
It turns that it is not a very strong statement, concrete choices of  $t^{(q)}(\sigma)$  can break  $G$ (duality) symmetry completely.

- Stronger statement: **G-covariance on phase space**  
For this, the **generalised Lie bracket** should be realised on phase space:

$$V = \int V^{M_1}(x) t_{M_1}^{(1)} \in \mathcal{R}_1, \quad \phi = \int \phi^{M_p}(x) t_{M_p}^{(p)} \in \mathcal{R}_p, \quad \boxed{\{\phi, V\} \stackrel{!}{=} \int (\mathcal{L}_V \phi)^{M_p} t_{M_p}^{(p)}}$$

- This requires a **hierarchy of constraints**:

$$\boxed{t_{K_{p+1}}^{(p+1)} \wedge dX^{L_1} \partial_{L_1} = D_{K_{p+1}}^{M_p, L_1} t_{M_p}^{(p)} \partial_{L_1}}$$

- **Lessons/Results** for  $G = E_{d(d)}$ : [DO, 21,23,24]

- Constraints are very restrictive:

**only  $\frac{1}{2}$ -BPS branes (string, membrane, D-branes, ...) are allowed.**

- remarkable, as only bosonic part of  $\sigma$ -models
- Still: solutions break  $E_{d(d)} \rightarrow GL(d)$

# summary & outlook

---

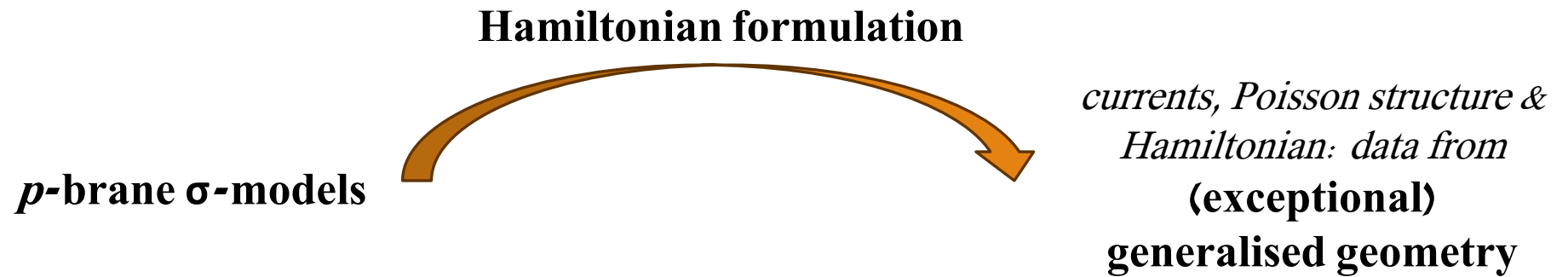
# summary & outlook

---

***p*-brane  $\sigma$ -models**

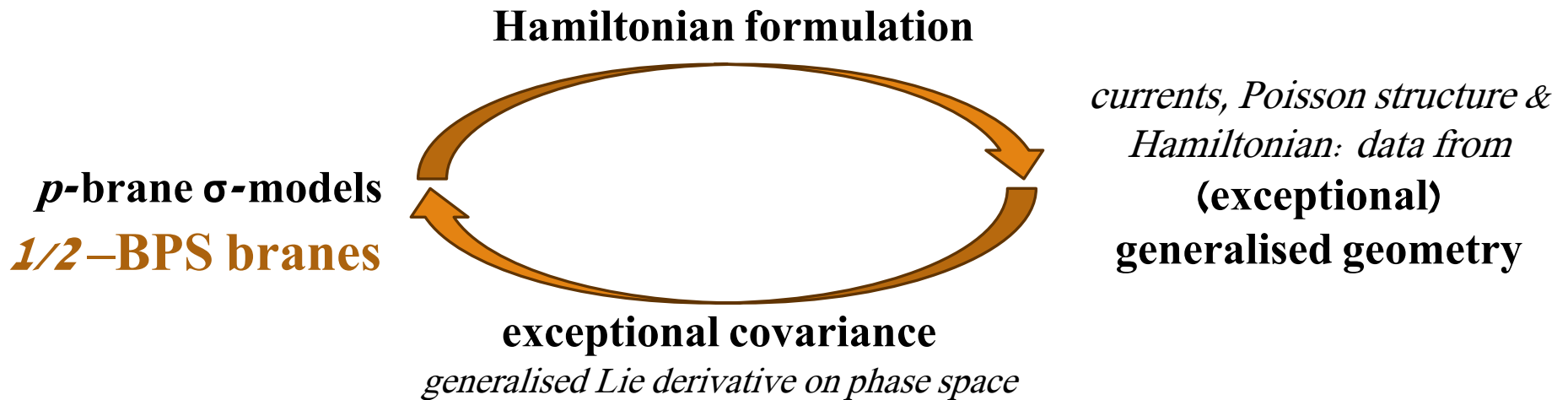
# summary & outlook

---



# summary & outlook

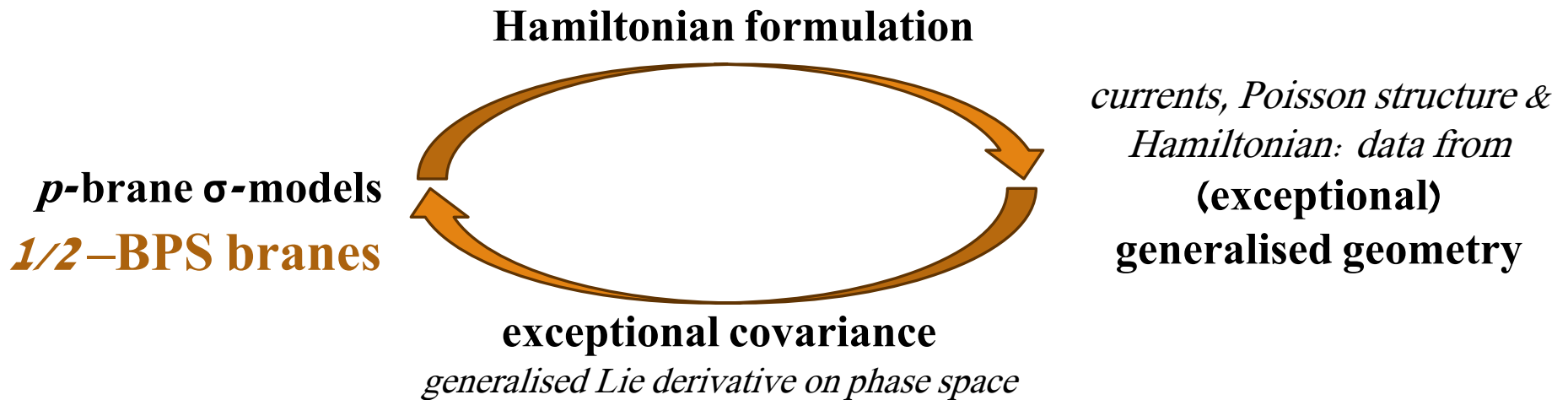
---





# summary & outlook

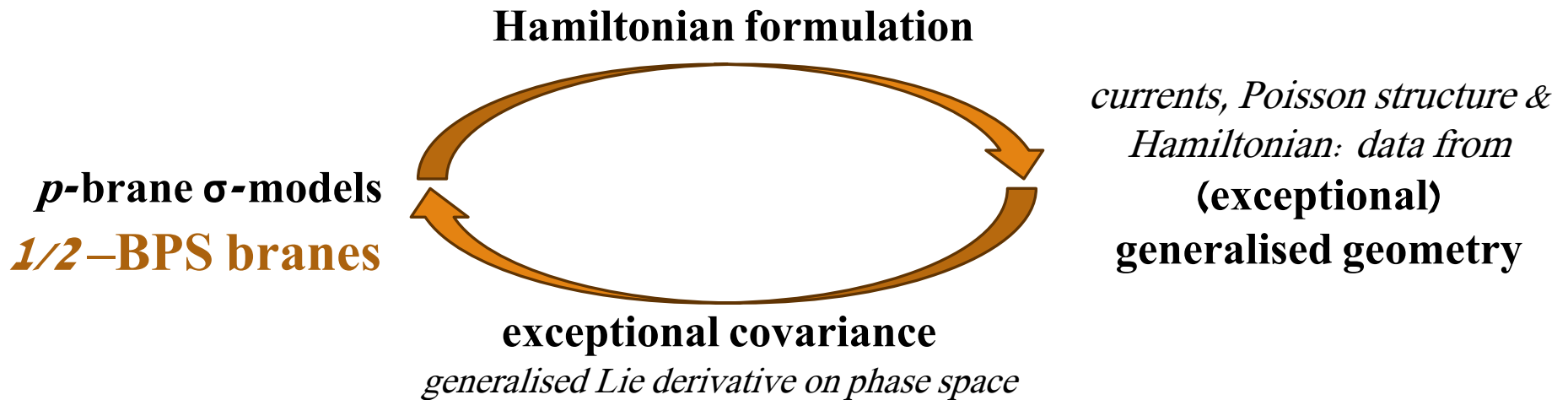
---



- so far for  $E_{d(d)}$  for  $d < 9$ , extension to higher  $d$ , exotic brane  $\sigma$ -models

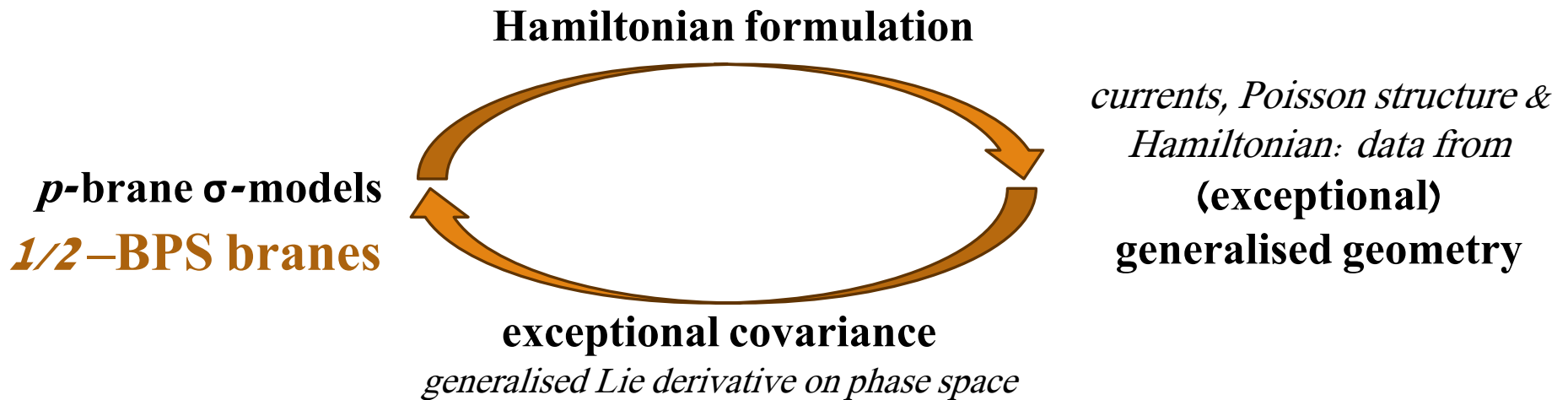
# summary & outlook

---



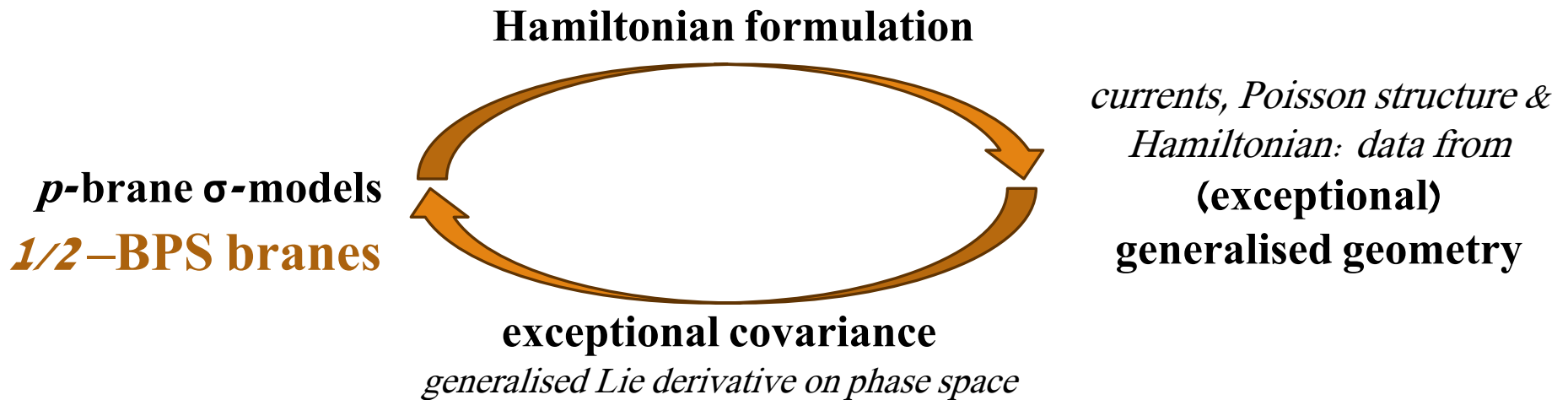
- so far for  $E_{d(d)}$  for  $d < 9$ , extension to higher  $d$ , exotic brane  $\sigma$ -models
- gauge symmetries: exceptional Cartan geometry, cf. *[Hassler, Hulik, DO 24]*

# summary & outlook



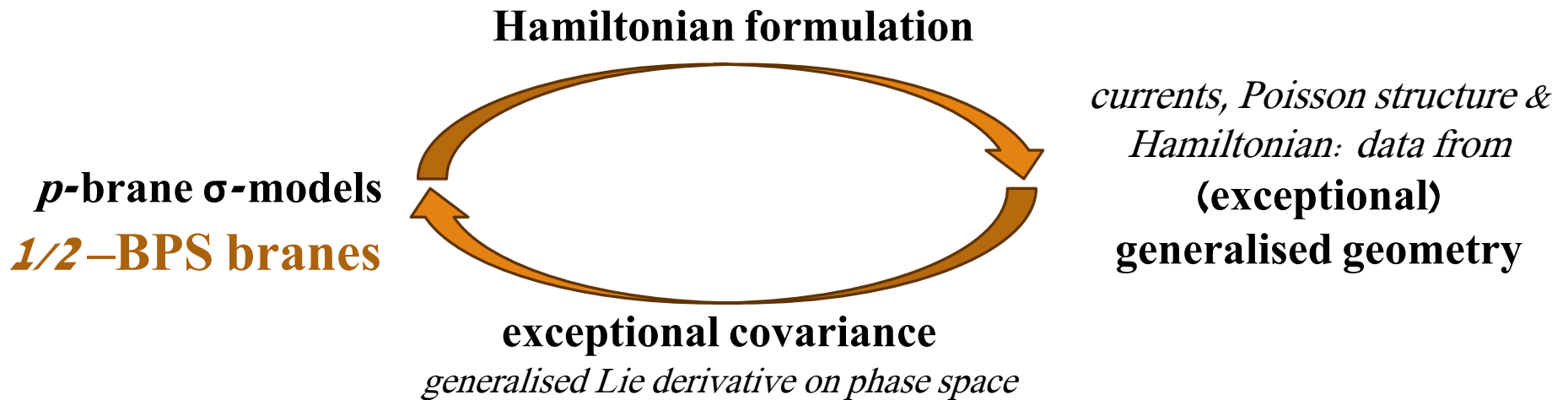
- so far for  $E_{d(d)}$  for  $d < 9$ , extension to higher  $d$ , exotic brane  $\sigma$ -models
- gauge symmetries: exceptional Cartan geometry, cf. *[Hassler, Hulik, DO 24]*
- canonical quantisation? (very speculative)

# summary & outlook



- so far for  $E_{d(d)}$  for  $d < 9$ , extension to higher  $d$ , exotic brane  $\sigma$ -models
- gauge symmetries: exceptional Cartan geometry, cf. *[Hassler, Hulik, DO 24]*
- canonical quantisation? (very speculative)
- classical dynamics: integrability, duality, solutions to membrane dynamics = generalised geodesics in generalised geometry *[Strickland-Constable 21]*  $\longrightarrow$  generalisation

# summary & outlook



- so far for  $E_{d(d)}$  for  $d < 9$ , extension to higher  $d$ , exotic brane  $\sigma$ -models
- gauge symmetries: exceptional Cartan geometry, cf. [\[Hassler, Hulik, DO 24\]](#)
- canonical quantisation? (very speculative)
- classical dynamics: integrability, duality, solutions to membrane dynamics = generalised geodesics in generalised geometry [\[Strickland-Constable 21\]](#)  $\longrightarrow$  generalisation
- A-theory [\[Hatsuda, Hulik, Linch, Siegel, Wang, Wang 23\]](#) : non-conventional brane theories, without requiring exceptional covariance (brane charge constraints)

Thank you for your attention!