Weak chaos in string-black hole scattering: from classical to quantum

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String-black hole scattering

- Black holes, fast scrambling and all that: want to probe black hole dynamics
- Scattering "experiments": true and tried way of inspecting a scatterer (black hole) through its effective potential

Outline

- Classical string motion in black hole geometries: no chaos but thermalization in CFTs [Đukić & Čubrović 2310.15697]
- Chaos from eigenvalue statistics of the string S-matrix [Savić & Čubrović 2401.02211]
- Random matrix theory for the S-matrix – Wishart ensembles [Čubrović 24xx.xxxxx]

Classical diagnostics: open string hanging from boundary to horizon

- Straight static string + small radial and transverse fluctuations
- Integrable subsector of a nonintegrable system
- Probe dynamics in black hole background ***can*** be integrable despite the fast scrambling og black holes degrees of freedom
- Chaotic cases are known (Basu, Pando-Zayas...) but are not the only possibility!

Open string in D1-D5-p background

- The celebrated 3-charge system, microstates counted by Strominger&Vafa 1996
- Compactified on $T^5 = T^4 \otimes S^1$ with p-modes along the circle
- Angular velocity Σ determines the temperature
- Straight static string + small radial and transverse fluctuations: effective Lagrangian

$$L_{\mathrm{eff}} = \frac{1}{\sqrt{f}(R)}\left[-1 + \frac{r_0^2 \cosh^2 \Sigma}{R^2} - \frac{f(R)}{h(R)}(R)'^2 \left(1 + \frac{r_0^2 \sinh^2 \Sigma}{R^2}\right)(X_5)'^2\right]$$

$$f(r) = \left(1 + \frac{r_1^2}{r^2}\right)\left(1 + \frac{r_5^2}{r^2}\right), \quad h(r) = 1 - \frac{r_0^2}{r^2}$$

Dynamics of open strings

- Analytic estimate of the Lyapunov exponent

"dilute gas" approximation

$R^4 \gg V_{T^4}$

$$\lambda_L^{(\mathrm{rot})} = \frac{r_0}{r_1 r_5}\sqrt{1 + \frac{r_0^2(r_1^2 + r_5^2)}{2 r_1^2 r_5^2}\cosh 2\Sigma} \approx \frac{r_0}{r_1 r_5} = \frac{2\pi T}{\sqrt{1 - L^2\Omega^2}} \qquad L\Omega \in [0,1)$$

VĐ, Čubrović 2024

- Compare with results from the literature, obtained from OTOC calculations

$$\lambda_\pm = \frac{2\pi T}{1 \mp L\Omega}$$

Jahnke, Kim, Yoon 2019

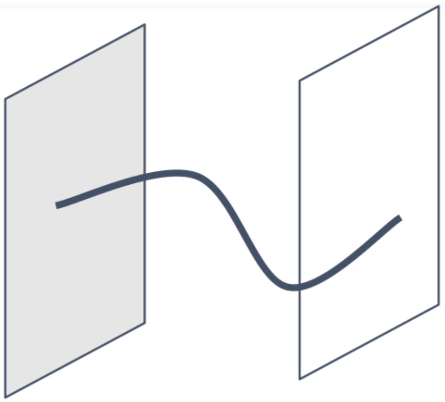
- Our result is an exact geometric mean $\lambda_- < \lambda_L^{(\mathrm{rot})} < \lambda_+$ — corrections from MSS are non-universal!

Retarded Green's function

- Study the dynamics of transverse fluctuations of the string:

$t(\tau,\sigma) = \tau, \quad r(\tau,\sigma) = \sigma \equiv r$ (static gauge)

$x_5 = x(t,r) \qquad x(t,r) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, x_\omega(r)$



- 2-point correlation function in the IR-region: open string in $\mathrm{BTZ} \times S^3$ (ignore rotation for now, $\lambda = 2\pi T$)

$$\frac{h(r)}{r^4}\frac{d}{dr}\left(h(r) r^4 \frac{dx_\omega(r)}{dr}\right) + \frac{L^4\omega^2}{r^4} x_\omega(r) = 0$$

$\xi = h(r) = 1 - r_0^2/r^2$

$$\Rightarrow \; x_\omega(r) = \mathscr{A}\,\xi^{-i\mathfrak{a}}\;{}_2F_1(a,b,c;\xi)$$

$$\sim \mathscr{S}\; r^{-d+\Delta} + \mathscr{F}\; r^{-\Delta}, \qquad r \gg r_0$$

$$d = \Delta = 3$$

$$\mathscr{G}_R^{(T)} \sim \mathscr{F}/\mathscr{S} = \frac{\Gamma\left(1 - i\frac{\omega}{2\lambda}\right)\Gamma\left(\frac{3}{2} - i\frac{\omega}{2\lambda}\right)}{\Gamma\left(-i\frac{\omega}{2\lambda}\right)\Gamma\left(-\frac{1}{2} - i\frac{\omega}{2\lambda}\right)}$$

Signals of instability

- *"Poles in the retarded Green's function give us a spectrum of quasi-normal modes."*

$$\mathcal{G}_R^{(T)} \sim \frac{\Gamma\left(1 - i\frac{\omega}{2\lambda}\right)\Gamma\left(\frac{3}{2} - i\frac{\omega}{2\lambda}\right)}{\Gamma\left(-i\frac{\omega}{2\lambda}\right)\Gamma\left(-\frac{1}{2} - i\frac{\omega}{2\lambda}\right)} \quad \Rightarrow \quad \omega_{\mathfrak{n}} = -\,i(\mathfrak{n}+1)\lambda \qquad \mathfrak{n} \in \mathbb{Z}^+$$

- The true interpretation of the bulk Lyapunov exponent: the timescale of the decay of instabilities on the fundamental strings

- In dual field theory it gives the thermalisation timescale for heavy quarks in thermal plasma

Bulk Lyapunov exponent → thermal instability

- We can always compute the Lyapunov exponent of a bulk probe in black hole background but it has nothing to do with scrambling, black hole chaos etc.
- Instead: bulk Lyapunov exponent ~ QN mode in overdamped regime ~ thermalization
- Can we see actual black hole chaos with a quantum probe?

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Highly excited strings and black holes

- Highly excited string (HES): $N \gg 1$
- Horowitz&Polchinski 1990s: BH/string complementarity

$$M_{\mathrm{BH}} = \frac{r_s^{D-2}}{G_N}, \quad M_{\mathrm{string}} \sim \frac{N}{\alpha'}$$

when string becomes a black hole: $M_{\mathrm{string}} \sim M_{\mathrm{BH}}, l_s = \sqrt{\alpha'} \sim r_s$

- From $G_N = \alpha' g^2$ we get: $N g^4 \sim (\alpha')^{D-3} \quad \Rightarrow \quad N_c \sim 1/g^4$
- HES → Horowitz-Polchinski solution → black hole
- We **cannot** see the black hole regime – beyond the scope of tree-level HES! But can we see a trend as we approach it?

Highly excited string scattering

- Idea: look at the scattering processes HES → HES + t, HES → HES + t + t, HES → HES + γ etc.
- A single string amplitude contains a lot of structure (infinite series of QFT diagrams): is there any chaos?
- Studied by Gross, Rosenhaus, Firrotta, Bianchi, Sonnenschein et al
- Bottom line from the literature:

  - some indications of RMT statistics for the ***spacing of poles in the amplitude***

  - no signs of fractality in the ***positions*** of poles $\theta'(\theta)$

Detecting chaos in HES scattering

- Wigner-Dyson statistics expected for eigenenergies of the Hamiltonian / eigenphases of the S-matrix – how to understand the "pole repulsion" in amplitudes?
- Robust measure proposed by Sonnenschein, Firrotta, Bianchi, Weissmann in 2207.13112, 2303.17233: level spacing ratio r_n
- Eigenphase spacings: $r_n = \frac{\Phi_{n+1} - \Phi_n}{\Phi_n - \Phi_{n-1}}$
- Average values seems to be robust to symmetries and unfolding: average values $\langle r \rangle$ known for GOE, GUE, GSE
- Outcome for HES scattering in 2303.17233: close for some occupation numbers but not for $N \to \infty$

Highly excited string construction

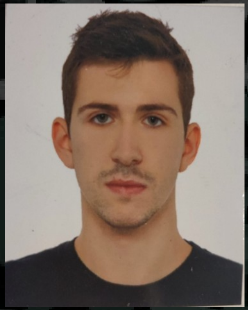
- DDF formalism (Di Vecchia, Del Guidice & Fubini): build a highly excited string (HES) in a controled way
- Start from the tachyon ($N=0$ state) and add to it $J \gg 1$ photons ($N=1$ states) to get a HES with $N \gg 1$
- Outcome: ***all possible states*** (i.e. all states stasfying the Virasoro constraints) with given occupation N:

$$|\mathrm{HES}\rangle \propto \xi^{i_1 \ldots i_J} P\left(\partial X, (\partial X)^2, \ldots (\partial X)^N\right)|0; p\rangle$$

- States enummerated by partitions of N, Hilbert space dimension ~ $\exp(\sqrt{N})$
- State $|\vec{k}\rangle$: $|\vec{k}|=n, \;\; 1 \le n \le N$ $\quad \sum_{a=1}^{n} k_a = N$

Highly excited string scattering

- Our idea: consider a 2→2 process (analogous to the Shenker-Stanford protocol)
- Look at the S-matrix structure rather than individual amplitudes
- Explore the dependence of dynamics on total occupation number N, spin S, and ensemble averaging in order to understand the ***nature*** of chaos (if any)

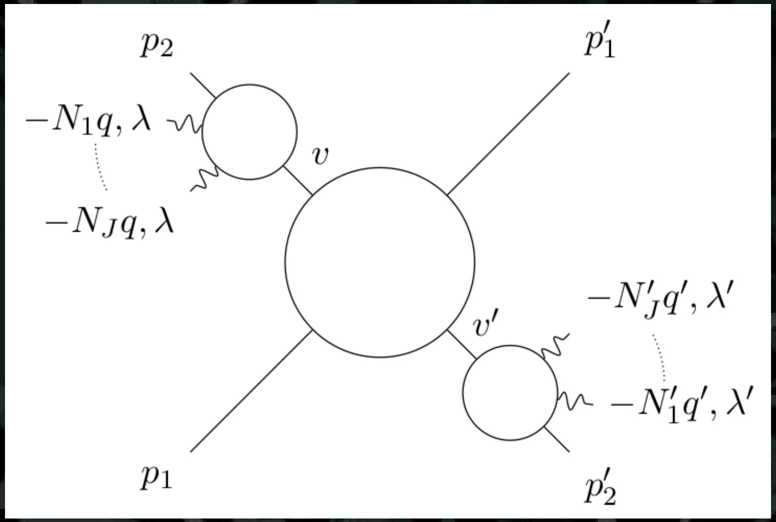


Nikola Savić

Highly excited open string amplitudes

- Setup: HES + tachyon → HES + tachyon
- Scattering amplitude at tree-level is found analytically in Hashimoto et al 2208.08380: $A = A_{st} + A_{tu} + A_{us} + A_{ts} + A_{su} + A_{ut}$

$$A_{st} = \frac{1}{\text{Vol SL}(2,R)} \int DX\, e^{-S_P} \int \prod_i dw_i V_t(w_i, p_i) \int \prod_{a=1}^{J} dz_a V_\gamma(z_a, -N_a q, \lambda) \int \prod_{b=1}^{J'} d z_b V_\gamma(z_b, -N_b q, -\lambda)$$



Taken over from 2208.08380

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w_i - tachyon insertion points $i \in \{1, 2, 1', 2'\}$

z_a - photon insertion points for incoming HES $a \in \{1, \ldots J\}$

z_b - photon insertion points for incoming HES $b \in \{1, \ldots J'\}$

N_a - incoming photon occupation number

N_b - outgoing photon occupation number

λ - incoming photon polarization

$-\lambda$ - outgoing photon polarization

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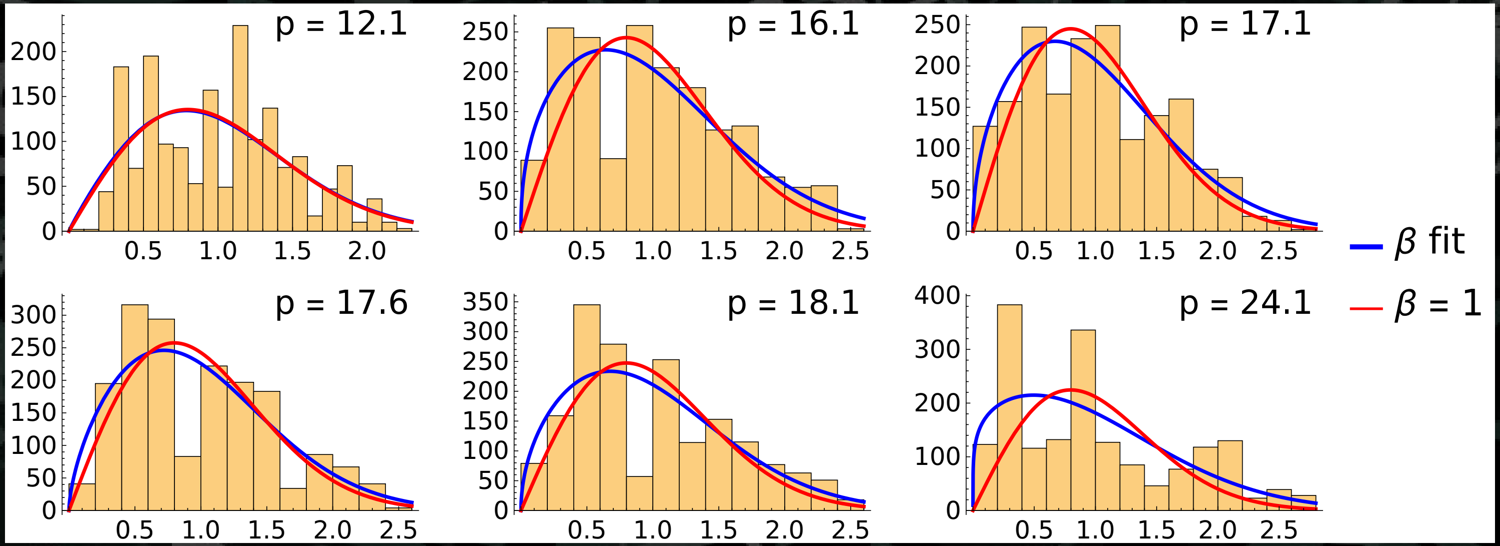
- For the uniform and circular polarization of the DDF photons the worldsheet integrals reduce to gamma and beta functions:

$$A_{st} = 4 \sum_{i,\, j=2,2'} \sum_{\vec{k},\, \vec{l}} \left(\prod_{a=1}^{J} (p_i \cdot \lambda) c^i_{k_a} \right) \left(\prod_{b=1}^{J'} (-p_j \cdot \lambda) c^j_{l_b} \right) B\left(-1 - \frac{s}{2} + k, -1 - \frac{t}{2} + l \right)$$

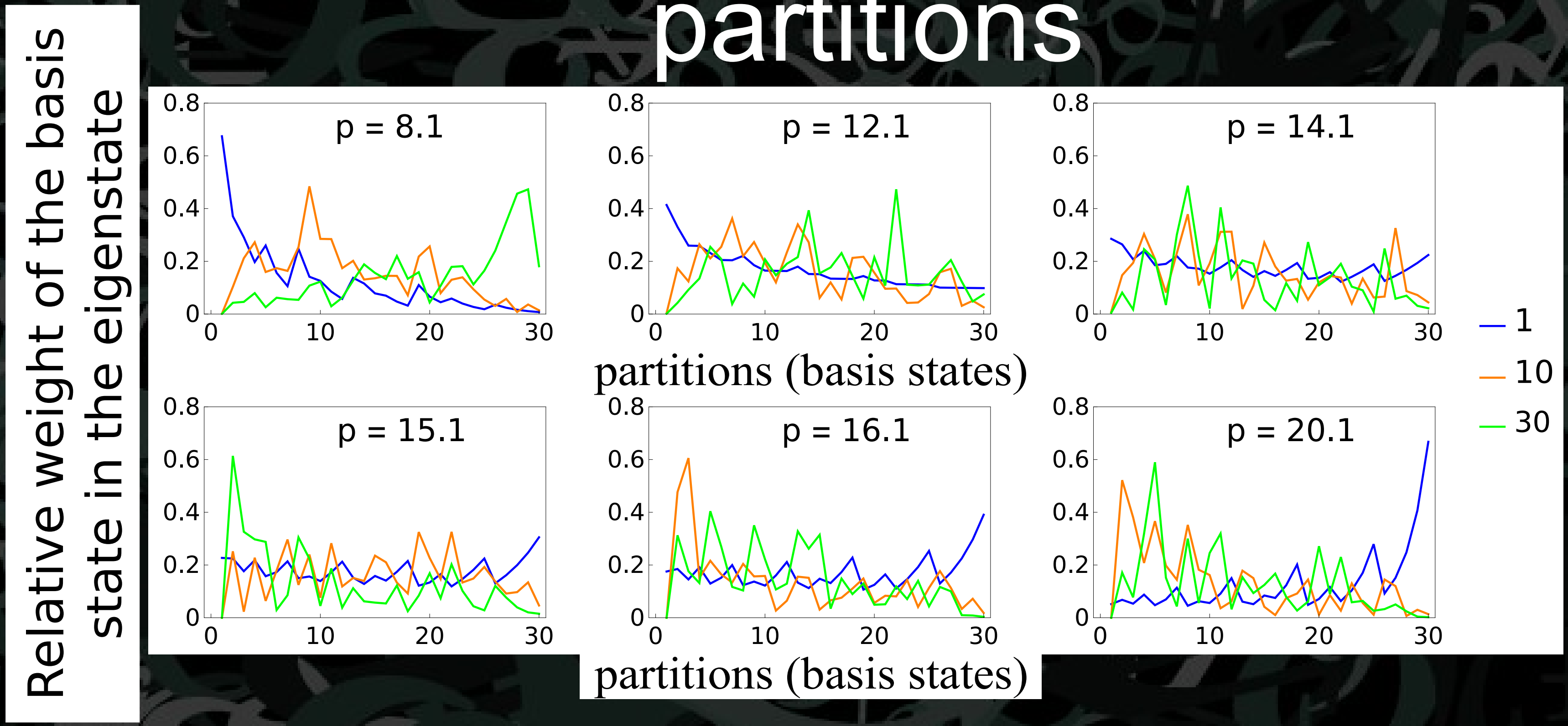
$$c^1_{k_a} = \frac{\Gamma(\alpha_1 + \alpha_2 - k - 1)\Gamma(\alpha'_2 + k)}{\Gamma(\alpha_1)\Gamma(\alpha'_2)\Gamma(\alpha_2 - k)\Gamma(k+1)}$$

Wigner-Dyson statistics in the S-matrix?



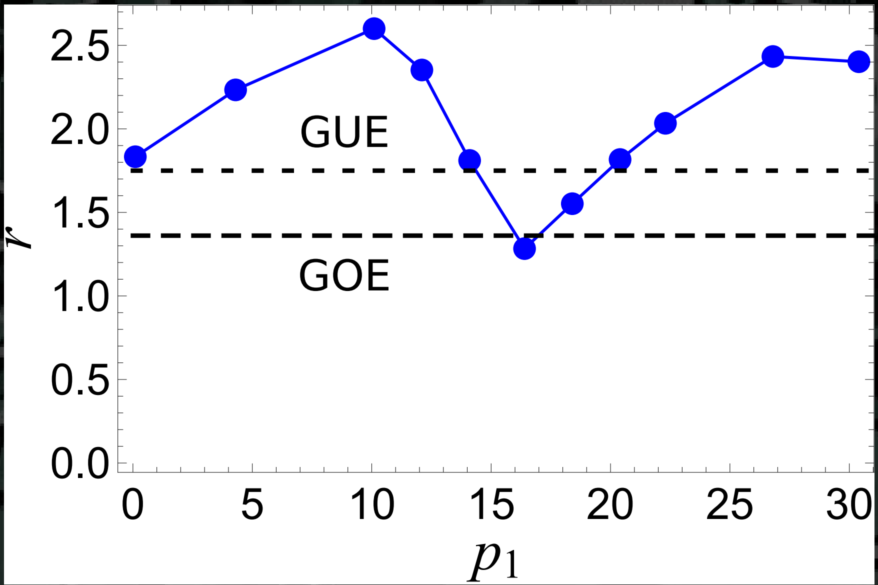
- Textbook test of quantum chaotic scattering: differences between the phases of the S-matrix eigenvalues vs. Random matrix theory (Gaussian orthogonal ensemble)
- There are clear and sometimes large deviations from Wigner-Dyson

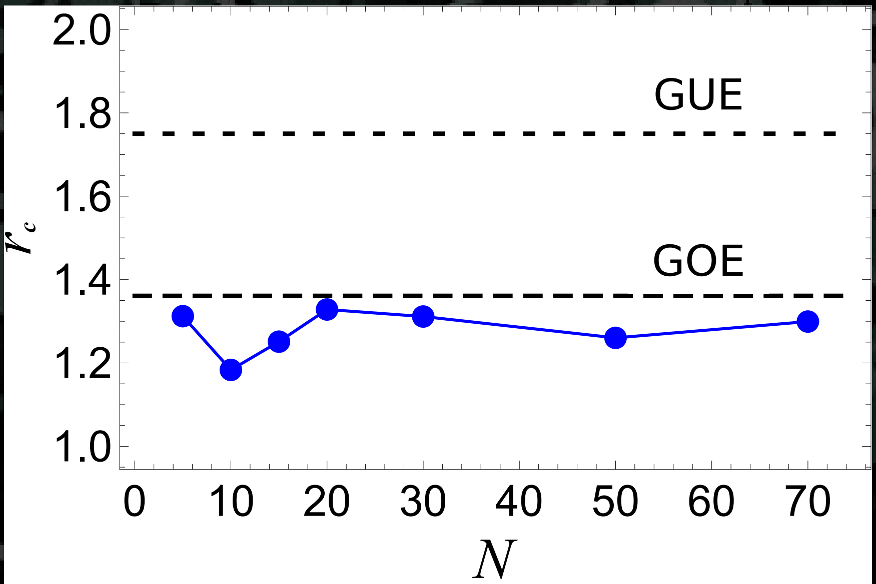
Crossover from sparse to dense partitions



- Eigenvectors of the S-matrix ordered from the largest eigenvalue (n=1, blue) toward smaller eigenvalues (here n=10, red and n=30, green)
- The leading eigenvector (blue) contributes most to the scattering

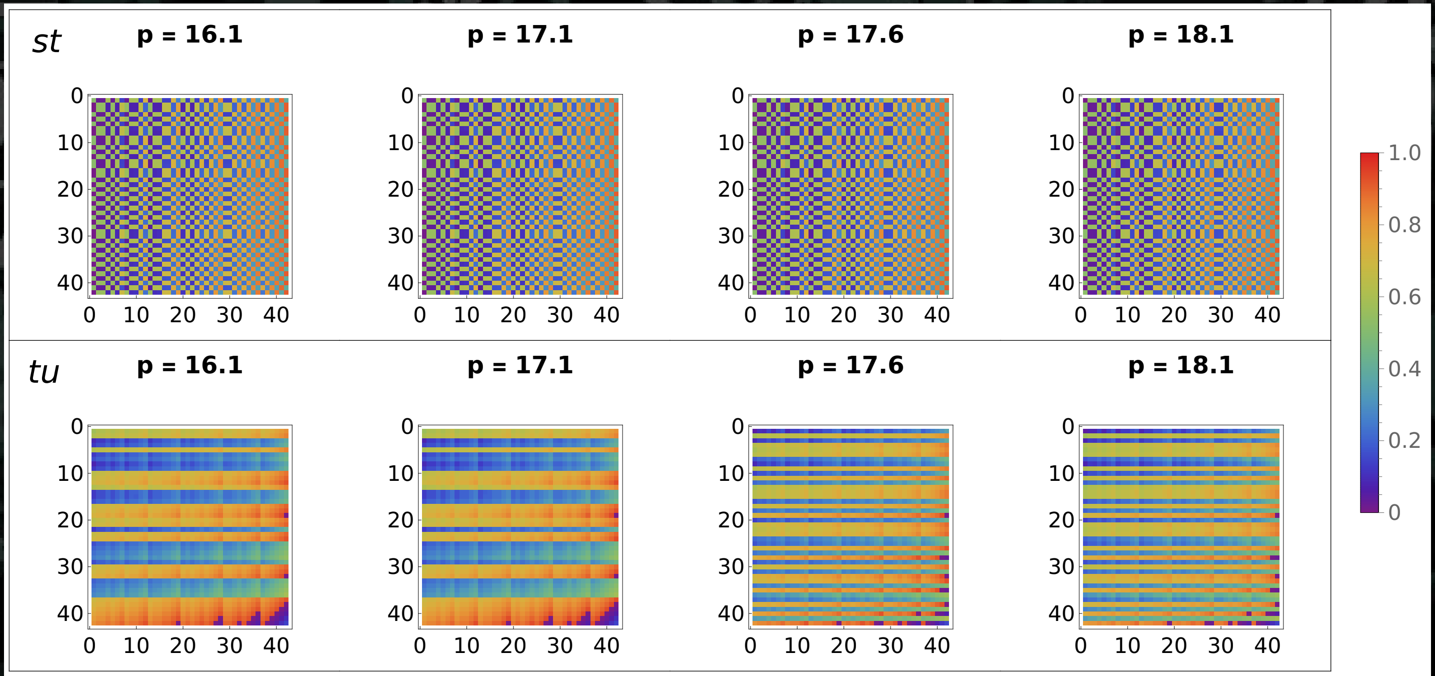
Level-spacing ratio at the crossover





- In general not very close to the value for GOE (or GUE)...
- ... but right around the crossover it comes very close to GOE!
- Only around p_c it may have a definite limit for $N \to \infty$
- In general the amplitudes also do not have a clear $N \to \infty$ limit for $\langle r \rangle$ (discussed in 2303.17233). Makes sense to check for $p = p_c$.

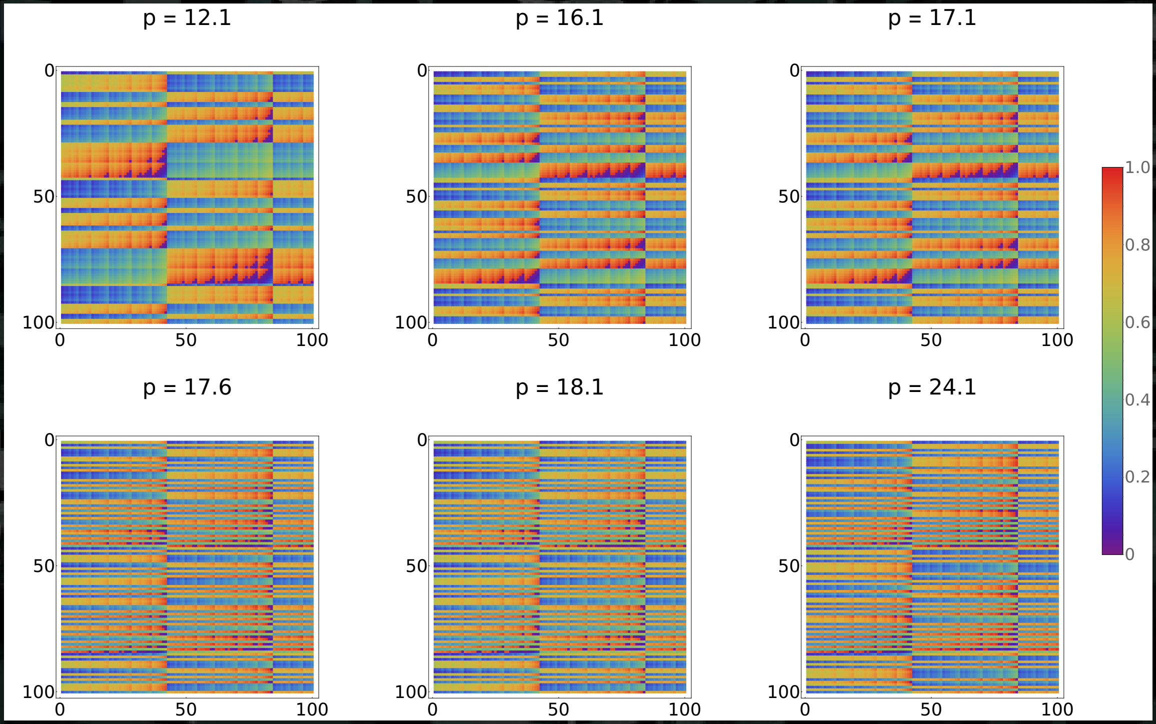
Quasi-invariant states



Phases of the S-matrix in the permutation basis (color code): Wigner-Dyson predicts random structure but in the tu channel we see nearly-invariant states

Persists for large N – what about the black hole?

Quasi-invariant states



Same happens in the tu channel for closed strings.
Amplitudes computed either through KLT relations or directly (brute-force numerics)

Spinful probes and finite-N chaos

- Analytically tractable for circularly polarized photon and graviton. For general polarizations and for $S>2$ hopeless
- Setup: $\text{HES}+\text{photon}\rightarrow\text{HES}+\text{photon}$, photon polarizations $\pm\zeta$

$$A_{st}=\frac{1}{\text{Vol SL}(2,R)}\int DX\, e^{-S_P}\int\prod_i dw_i V_t(w_i,p_i)\int\prod_I V_\gamma(w_I,p_I)\int\prod_{a=1}^{J} dz_a V_\gamma(z_a,-N_a q,\lambda)\int\prod_{b=1}^{J'} dz_b V_\gamma(z_b,-N_b q,-\lambda)$$

$i\in\{2,2'\}$ - probe photons
$I\in\{1,1'\}$ HES tachyons

- No permutation symmetry between different channels

$$A_{st}=A_{st}^{(1)}+A_{st}^{(2)}$$

$$A_{st}^{(1)}=4(-p_2\cdot\zeta)\sum_{i,j}\sum_{\vec{k},\vec{l}}\left(\prod_{a=1}^{J}(p_i\cdot\lambda)c_{k_a}^{i}\right)\left(\prod_{b=1}^{J'}(-p_j\cdot\lambda)c_{l_b}^{j}\right)B\left(-1-\frac{s}{2}+k,-2-\frac{t}{2}+l\right)$$

$$A_{st}^{(2)}=4(-p'_2\cdot\zeta)\sum_{i,j}\sum_{\vec{k},\vec{l}}\left(\prod_{a=1}^{J}(p_i\cdot\lambda)c_{k_a}^{i}\right)\left(\prod_{b=1}^{J'}(-p_j\cdot\lambda)c_{l_b}^{j}\right)B\left(-2-\frac{s}{2}+k,-1-\frac{t}{2}+l\right)$$

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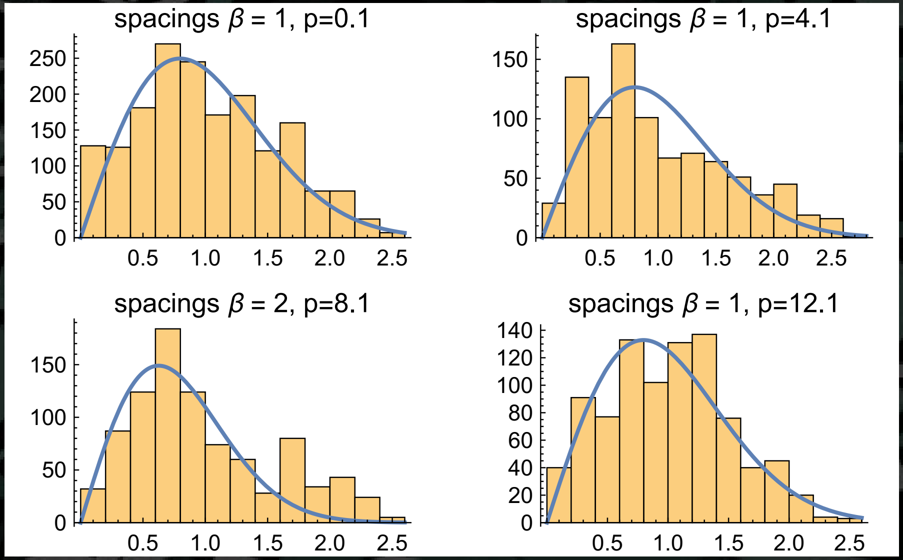
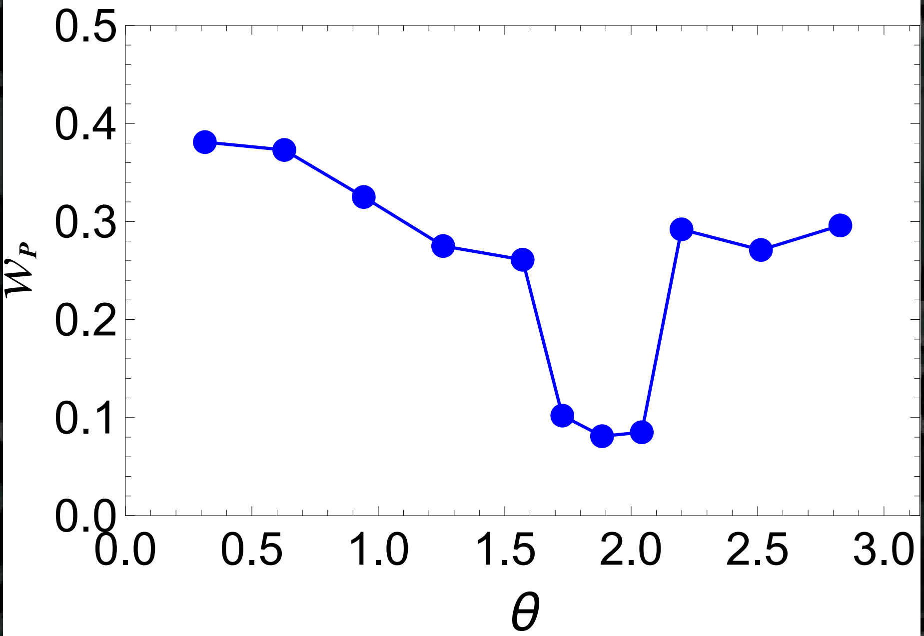


Graviton-HES scattering near the crossover angle θ_c



Mixed Poisson-Wigner-Dyson (Berry-Robnik) fit – strong chaos (w_P small) only for near θ_c

Quasi-invariants and random matrix theory

- Berry-Robnik theory '84: the only rigorous RM theory for a system with classically mixed phase space
- Independent regular and chaotic component (no tunnelling ⇒ **deep in the semiclassical regime**) hence the gap probabilities multiply (works for $\beta=1,2,4$):

$$E_{\mathrm{BR}}(s)=E_{\mathrm{P}}(w_{\mathrm{P}} s)\,E_{\mathrm{WD}\beta}\left((1-w_{\mathrm{P}})s\right)\Rightarrow P_{\mathrm{BR}\beta}(s)=\frac{d^2 E_{\mathrm{BR}}(s)}{ds^2}$$

$$P_{\mathrm{BR}\,1}(s)=\frac{\rho}{\sqrt{\pi}}\left(-\pi(1-\rho)^2 s^2-4\rho s\right)e^{-\frac{\pi}{4}(1-\rho)^2 s^2-\rho s}+\frac{\rho^2(1-\rho)}{2\sqrt{\pi}}e^{\frac{\pi}{4}\rho^2 s^2}\,s\,\gamma\left(\frac{1}{2},\frac{\pi\rho^2}{4}s^2\right)$$

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S-matrix from beta-Wishart matrices

- The S-matrix entries are essentially Veneziano/Virasoro-Shapiro amplitudes consisting of ratios of gamma functions
- Tachyonic amplitude (but similar for any spin):

$$A_{st}=4\sum_{i,\,j=2,2'}\sum_{\vec{k},\,\vec{l}}\left(\prod_{a=1}^{J}(p_i\cdot\lambda)c^{i}_{k_a}\right)\left(\prod_{b=1}^{J'}(-p_j\cdot\lambda)c^{j}_{l_b}\right)B\left(-1-\frac{s}{2}+k,-1-\frac{t}{2}+l\right)$$

-

$$c^{1}_{k_a}=\frac{\Gamma(\alpha_1+\alpha_2-k-1)\Gamma(\alpha'_2+k)}{\Gamma(\alpha_1)\Gamma(\alpha'_2)\Gamma(\alpha_2-k)\Gamma(k+1)}$$

- Gamma-function values the key to statistics – can be found in many places e.g. Online Encyclopedia of Integer Sequences (OEIS)
- Outcome: normal distribution for each partition

Eigenphase statistics of beta-Wishart matrices

- Once we know that each row is normally distributed but with a different variance we can write the S-matrix as a Wishart matrix: $S = T^{\dagger} T$ with T of size $P(N) \times N$
- The theory of Wishart matrices (Dubbs, Edelmann, Koev... e.g. 1305.3561) predicts the distribution of eigenvalues and extreme values but not spacings!
- Eigenvalue distribution:

$$f(\vec{\lambda}) \propto \prod_{m<n} |\lambda_m - \lambda_n|^{\beta} \prod_n \lambda_n^{-(P(N)/2-1)\beta-1} \, {}_0F_0^{(\beta)}\left(-\frac{\beta}{2}\vec{\Lambda}, \Sigma^{-1}\right)$$

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots \lambda_{\mathrm{part}(N)})$$

$$\Sigma = \left(-|\pi_j(N)|\, p \cdot q, -|\pi_j(N)|\right)_{\mathrm{part}(N) \times \mathrm{part}(N)}$$

$${}_0F_0^{(\beta)} - \beta\text{-confluent hypergeometric function}$$

Fixed points of the beta-Wishart ensemble

- For large N the repulsion from the $|\lambda_m - \lambda_n|^\beta$ term dominated by the exponential limit of the confluent hypergeometric function ${}_0F_0^{(\beta)}$
- Outcome: there are always states with eigenvalues $|\lambda - 1| < \text{const.}/N^2$ i.e. ***states that remain almost unchanged (quasi-invariant states!) and accumulate around a fixed point*** ⇒ no repulsion
- In the limit of infinite N we know that Wishart matrices have the Marchenko-Pastur distribution of eigenvalues. Connection to recent work on "bag-of-gold" microstructure by Emparan et al?