Almost Extensivity of Barrow Entropy Favoured by the Full Dynamical and Geometrical Set of Cosmological Data

Mariusz P. Dąbrowski

Institute of Physics, University of Szczecin, Poland &
National Centre for Nuclear Research, Otwock, Poland &
& Copernicus Center for Interdisciplinary Studies, Kraków, Poland

Plan:

- 1. Beyond additivity and extensivity of entropy in gravitational systems.
- 2. Holographic screens as dark energy.
- 3. Barrow/Tsallis holographic horizons.
- 4. Constraining Barrow Nonextensive Holography by kinematic and dynamical data.
- 5. Discussion.

References

- basic idea: J.D. Barrow, Phys. Lett. B 808, 135643 (2020) (ArXiv: 2004.09444).
- MPD and V. Salzano, Geometrical observational bounds on a fractal horizon holographic dark energy, https://www.researchgate.net/publication/342571909 (June 2020); PRD 102, 064047 (2020) (3 Sept 2020), arXiv:2009.08306 (16 Sept 2020).
- Tomasz Denkiewicz, Vincenzo Salzano, MPD - arXiv: 2303.11680.
- also relate to: A. Alonso-Serrano, MPD, H. Gohar - PRD 103, 026021 (2021); I. Çimdiker, H. Gohar, MPD - EPJC 83, 169 (2023), CQG 40, 145001 (2023).
- see also papers on nonextensive thermodynamics + papers on the applications of nonextensive entropies to cosmology (cited later accordingly)

1. Beyond additivity and extensivity of entropy in gravitational systems.

- Gibbs thermodynamics is connected with **ignoring long-range forces** between thermodynamic subsystems i.e. one assumes that the size of the system **exceeds** the range of interaction between its components
- In particular, Gibbs thermodynamics is based on the **additivity** and **extensivity** of entropy
- **Additivity composition rule:**

$$S(A+B) \equiv S(P_A P_B) = S(P_A) + S(P_B) = S(A) + S(B), \tag{1}$$

where A, B are independent, equipped with configurational possibilities Ω_A and Ω_B, and corresponding probabilities P_A and P_B. The composite system $A \cup B$ allows the probability $P_{A \cup B}$ and it is equipped with the set of possibilities $\Omega_{A \cup B}$. Systems' independence give $P_{A \cup B} = P_A P_B$, then. (cf. Çimdiker et al. 2023 a, b)

Beyond additivity and extensivity of entropy.

- **Extensivity:** Assume the set of thermodynamical variables $(X_0, X_1, X_2, \ldots, X_k)$ such that $X_0 = f(X_1, X_2, \ldots, X_k)$. Extensivity means that the function f is homogeneous degree one i.e. that

$$f(aX_1, aX_2, ..., aX_k) = af(X_1, X_2, ..., X_k) \tag{2}$$

for a positive real number a, for all $X_1, X_2, ...X_k$. For entropy S, energy U, volume V, mole number N, we have $S(aU, aV, aN) = aS(U, V, N)$.

- Attention! Extensive quantity can be nonadditive (e.g. if $f(x_1, x_2) = x_1^2/\sqrt{x_1^2 + x_2^2}$) (Landsberg, Braz. J. Phys. 29, 46 (1999)); additive quantity can be nonextensive.

- More common, but not so precise definition states that if a system's total number of microstates, Ω, is proportional to its number of particles or degrees of freedom, the entropy is extensive. For Gibbs entropy one has $S_G(N) = k_B \ln \Omega(N) \propto N$, where N is the total number of particles or degrees of freedom in the system (proportional to its size).

Nonadditivity and nonextensivity of entropy in gravity.

Gravitational systems are long-range interacting and highly nonlinear and so they **cannot be adopted** as thermodynamical systems of Gibbs additive and extensive type. In order to study them, let us first define

Nonadditivity rule:

$$S(A+B) = S(A) + S(B) + \frac{\lambda}{k_B} S(A)S(B). \qquad (3)$$

Taking $\lambda \equiv 1 - q = 0$, one gets additive entropies:

- Gibbs entropy $S_G = -\sum_i p(i) \ln p(i)$ - extensive.
- Rényi entropy $S_R = k \frac{\ln \sum_i p^q(i)}{1-q}$ (Rényi 1959) - nonextensive.

Taking $\lambda \equiv 1 - q \neq 0$, one gets nonadditive entropies such as:

- Tsallis entropy $S_T = S_q = -\sum_i [p(i)]^q \ln_q p(i)$, (Tsallis, J. Stat. Phys. 52, 479 (1988); book of 2009) - in the limit $q \to 1$, gives Gibbs entropy.

where $p(i)$ is the probability distribution defined on a set of microstates Ω, $q \in \mathcal{R}$ is the nonextensivity parameter.

Nonadditivity and nonextensivity of entropy in gravity.

Well-known **Bekenstein-Hawking entropy** of a black hole which is proportional to its mass/length scale $S \propto M^2 \propto L^2$ fulfils another type of nonadditivity rule

$$S(A+B) = S(A) + S(B) + 2\sqrt{S(A)}\sqrt{S(B)}. \qquad (4)$$

This is since $S(A) = M_A^2/4$, $S(B) = M_B^2/4$ and after a merge one has $S(A+B) = (M_A + M_B)^2/4$, which gives an extra term $M_A M_B/2$. Interestingly, one may write more general composition rule (Abé 2001):

$$H[S(A+B)] = H[S(A)] + H[S(B)] + \frac{\lambda}{k_B} H[S(A)]H[S(B)], \qquad (5)$$

where $H(S)$ is differentiable function of entropy, and use this formula to make the nonadditive entropy additive (Tsallis to Rényi case) when considering a general logarithm of the form

$$L(S) = \frac{k_B}{\lambda} \ln\left[1 + \frac{\lambda}{k_B} H(S)\right], \quad \text{giving} \quad L[S(A+B)] = L[S(A)] + L[S(B)]. \qquad (6)$$

Tsallis nonadditive entropy into additive Rényi

The best-known example of the application of the formula (6) is the Tsallis entropy into (additive) Rényi entropy

$$S_R = \frac{k_B}{\lambda} \ln[1 + \frac{\lambda}{k_B} S_T]. \tag{7}$$

Under the assumption that Tsallis entropy is Bekenstein entropy, the former can also be brought into an additive form. However, this also modifies the temperature of a black hole described by Rényi entropy as Rényi temperature

$$T_R = (1 + \frac{\lambda}{k_B} S_B) T_H, \tag{8}$$

where Bekenstein entropy S_B and Hawking temperature T_H are given in a standard way as

$$S_B = \frac{4\pi G k_B M^2}{\hbar c} \quad \text{and} \quad T_H = \frac{\hbar c^3}{8\pi G k_B M}. \tag{9}$$

Sharma-Mittal nonadditive and nonextensive entropy

Sharma-Mittal (Sharma & Mittal, J. Comb. Inf. Syst. Sci. 2, 122 (1977)) generalizes Rényi and Tsallis as follows

$$S_{SM} = \frac{1}{R}\left[\left(\sum_{i=1}^{W} p_i^{1-\lambda}\right)^{\frac{R}{\lambda}} - 1\right], \tag{10}$$

where R is a new parameter. Under the equiprobability condition one gets

$$S_{SM} = \frac{k_B}{R}\left[(1+\frac{\lambda}{k_B}S_T)^{R/\lambda} - 1\right], \tag{11}$$

where $R \to \lambda$ limit yields the Tsallis entropy, and $R \to 0$ yields Rényi entropy. It obeys nonadditive composition rule (3).

Replacing Tsallis S_T by Bekenstein, one has entropy and temperature

$$S_{SM} = \frac{k_B}{R}\left[(1+\frac{\lambda}{k_B}S_B)^{R/\lambda} - 1\right], T_{SM} = T_H(1+\frac{\lambda}{k_B}S_B)^{1-\frac{R}{\lambda}}. \tag{12}$$

Kaniadakis nonadditive and nonextensive entropy.

Kaniadakis entropy (Kaniadakis 2002, 2005) results from Lorentz transformation of special relativity which is a single parameter K deformation of Gibbs entropy given by

$$S_K = k_B \log_K \Omega \quad \text{where} \quad \log_K(\Omega) = \frac{\Omega^K - \Omega^{-K}}{2K}. \tag{13}$$

Considering $S_B = k_B \ln \Omega$, which means that the number of microstates Ω for a black hole is proportional to e^{S_B/k_B}, the entropy and its corresponding temperature can be written as

$$S_K = \frac{k_B}{K} \sinh\left[K \frac{S_B}{k_B}\right], T_K = T_H \,\mathrm{sech}\left[K \frac{S_B}{k_B}\right] \tag{14}$$

where we have used log-equation for the $\sinh x$ function and used the relation $\Omega = e^{S_B/k_B}$. For $K \ll 1$ one gets

$$S_K = S_B + \frac{1}{6}\left(\frac{K}{k_B}\right)^2 S_B^3 + \dots \tag{15}$$

Tsallis-Cirto/Barrow nonadditive and nonextensive entropy.

Tsallis and Cirto (EPJC 73, 2487 (2013)) applied the general definition of Tsallis entropy to the black holes to get (cf. also Çimdiker et al. 2023)

$$S_\delta = k_B \left(\frac{S_B}{k_B}\right)^\delta, T_\delta = \frac{T_H}{\delta}\left(\frac{S_B}{k_B}\right)^{1-\delta} \qquad (16)$$

where $\delta > 0$ is a real parameter. For Schwarzschild black hole with radius r_g, when $\delta = 3/2$, the entropy scales as $S \propto r_g^3 \propto V$ (the volume) and so it **recovers extensivity**. The temperature scales as $T_\delta \propto 1/M^2 \propto r_g^{-2}$.

Taking non-thermodynamical motivation (cf. later), Barrow (2020) has proposed similar relation ($A_{Pl} = 4l_{pl}^2$)

$$S_{Barrow} = k_B \left(\frac{A}{A_{Pl}}\right)^{1+\frac{\Delta}{2}} = k_B \left(\frac{S_B}{k_B}\right)^{1+\frac{\Delta}{2}}, \qquad (17)$$

where $0 \leq \Delta \leq 1$ corresponding to $1 \leq \delta \leq \frac{3}{2}$ since $\delta = 1 + \Delta/2$ (e.g. A. Mamon, 2007.0159, Y. Liu, 2201.00657, MPD & Salzano 2020).

Tsallis-Cirto/Barrow nonadditive and nonextensive entropy.

- Tsallis-Cirto/Barrow entropies obey a nonadditive composition rule given by

$$\left(\frac{S_{\delta AC}}{k_B}\right)^{1/\delta} = \left(\frac{S_{\delta A}}{k_B}\right)^{1/\delta} + \left(\frac{S_{\delta C}}{k_B}\right)^{1/\delta} . \tag{18}$$

- They are nonextensive except the case $\Delta = 1$ or $\delta = 3/2$, since they are based on the Bekenstein entropy which is also nonextensive .
- In some simplified terms (cf. the definition of extensivity (2)) **strong gravity** makes all the degrees of freedom of a black hole to be **distributed on a surface rather than in a volume** so that Bekenstein entropy is proportional to the area A or characteristic length length scale squared L^2 (Schwarzschild radius) and not to the volume V or length scale cube L^3.

2. Holographic screens as dark energy.

An underlying idea of gravitational entropy comes from thermodynamics of black hole horizons (area entropy), but **applied to cosmological horizons**, which basically in the context of string theory are called **holographic screens**.
Translation the contribution of the screens into the dark energy using the distance L as *a horizon length* reads (Wang 2016)

$$\rho_{DE} \propto S_{DE} L^{-4}, \tag{19}$$

where S_{DE} is one of the above mentioned entropies (Bekenstein, Tsallis, Rényi, Barrow etc.) related to a cosmological horizon.
Problem: There is a selection of horizons (and so the distances L) in (19).
Future event horizon:

$$L \equiv a \int_t^\infty \frac{dt'}{a} = a \int_a^\infty \frac{da'}{H(a')a'^2}\,, \tag{20}$$

where a is the scale factor and $H(a)$ the Hubble parameter (Hsu 2004, Li 2004).

Holographic screens as dark energy ctd.

Hubble horizon:

$$L \equiv \frac{c}{H(a)}\,, \tag{21}$$

though it is not the ”true horizon” since it can be crossed (or has been crossed, in fact) (e.g. Pavon 2005). Besides, running chains takes much calculation time (MPD, Salzano 2020).

Horizon with an infrared cut-off:

$$L \equiv c\left[\alpha H^2(a) + \beta \dot{H}(a)\right]^{-1/2}\,, \tag{22}$$

with α, β - free dimensionless parameters (cf. causality violation issue of (20 (Li 2004); Granda 2008, a, b).

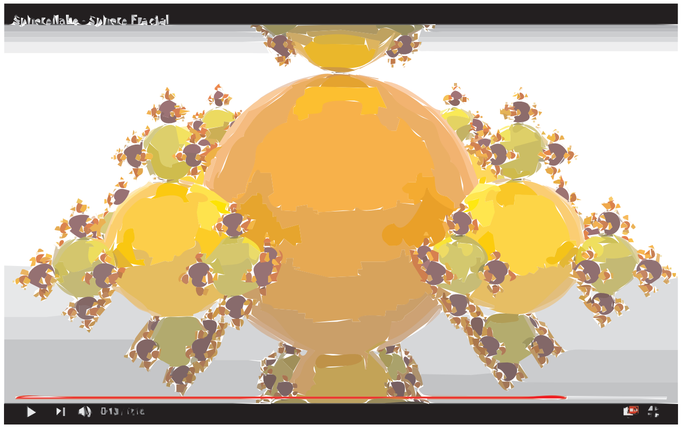
Later, in this talk we will focus on the choice of the future event horizon.

3. Barrow/Tsallis holographic horizons.

- Tsallis entropy refers to proper theory of extended Gibbs thermodynamics (Tsallis 1988, 2009).
- This is unlike Barrow entropy (2020) which idea came from the focus on the black hole/cosmological horizons which has roots in phenomenology of quantum gravity in a way that one takes quantum fluctuations of these horizons which have a fractal (self-similar) nature.
- This is in some analogy to quantum fluctuations of different scales during the inflationary epoch in cosmology.

Barrow fractal horizon idea of entropy.

- Barrow claimed that it was COVID-19 pseudofractal geometrical structure which inspired him
- procedure: 1) take the core sphere of the horizon (**being a surface**); 2) attach number N of heavily packed smaller spheres; 3) make this step by step, latching on a number N of some smaller spheres on and on; 4) this forms a fractal - **”sphereflake”**: https://www.youtube.com/watch?v=pPb5NKEYCD8;



Barrow horizon entropy ctd.

- apply geometrical series with the recurrence formula for the radius r_{n+1} of the $(n+1)$-th sphere is $r_{n+1} = \lambda r_n$, where r_n is the radius of the n-th sphere, and $\lambda < 1$;
- take infinite number of steps $(n \to \infty)$ to count up the effective surface area of all the spheres:

$$A_{eff} = \sum_{n=0}^{\infty} N^n 4\pi(\lambda^n r)^2 = \frac{4\pi r^2}{1 - N\lambda^2} = 4\pi r_{eff}^2; \quad r_{eff} = \frac{r}{\sqrt{1 - N\lambda^2}} \equiv r^{1+\Delta/2}. \tag{23}$$

where $0 \leq \Delta \leq 1$, provided that $N\lambda^2 < 1$. In the extreme case of $\Delta = 1$, it acts **as it was a volume**

- this leads to a change of Bekenstein entropy (making it larger), according to

$$S_{eff} \propto A_{eff} \propto r_{eff}^2 \propto r^{2+\Delta} \propto A^{1+\frac{\Delta}{2}}, \tag{24}$$

where r can be either the black hole Schwarzschild radius r_s or the cosmological horizon length L. The exact formula was given in (17).

Barrow horizon entropy ctd.

- More accurately, the Bekenstein entropy generalizes into

$$S_{Barrow} = k_B \pi^{1+\frac{\Delta}{2}} \left(2 \frac{M}{m_{pl}} \right)^{2+\Delta}, \tag{25}$$

- and the generalized Hawking temperature reads

$$T_{Barrow} = \frac{c^2}{4\pi^{1+\frac{\Delta}{2}} k_B \left(1 + \frac{\Delta}{2}\right)} \frac{m_{pl}^{2+\Delta}}{(2M)^{1+\Delta}}, \tag{26}$$

  where M is a black hole mass, k_B is the Boltzmann constant, c is the speed of light, and $m_{pl}^2 = \hbar c/G$ is the Planck mass.

- In the limit $\Delta \to 0$ they reduce to standard Bekenstein and Hawking formulas for the black holes (9).

Barrow holographic screen as dark energy.

- Now applying Barrow entropy as holographic screen to (19) we have

$$\rho_{DE} \propto S_{eff} L^{-4}, \tag{27}$$

where $S_{eff} \propto A_{eff} \propto L^{2+\Delta}$, one can express **B**arrow **H**olographic dark energy (BH) as (e.g. Saridakis 2020):

$$\rho_{BH} = \frac{3\,C^2}{8\pi G} L^{(\Delta-2)}\,, \tag{28}$$

where C is the *holographic parameter* with dimensions of $[\mathsf{T}]^{-1}[\mathsf{L}]^{1-\Delta/2}$.

- **Observation: ΛCDM ($\Delta = 2$, $\rho_{BH} =$ const.) is excluded in Barrow holography**.

- **Note:** all the calculations are also valid for Tsallis entropy at least in the range of its parameter $1 \leq \delta \leq 3/2$.

Barrow/Tsallis entropy - cosmological equations and quantities.

The Friedmann equation is

$$H^2(a) = \frac{8\pi G}{3}\left(\rho_m(a) + \rho_r(a) + \rho_{BH}(a)\right) , \tag{29}$$

where the suffices m and r refer respectively to matter and radiation.
Standard continuity equation for matter and radiation is still valid, i.e.

$$\dot{\rho}_{m,r}(a) + 3H\left(\rho_{m,r}(a) + \frac{p_{m,r}(a)}{c^2}\right) = 0 , \tag{30}$$

where the pressure $p_i = w_i \rho_i$. We can rewrite (6) as

$$1 = \Omega_m(a) + \Omega_r(a) + \Omega_H(a) , \tag{31}$$

introducing the dimensionless density parameters $\Omega_i(a)$, defined as

$$\Omega_{m,r}(a) = \frac{H_0^2}{H^2(a)}\Omega_{m,r}a^{-3(1+w_{m,r})} , \quad \Omega_{BH}(a) = \frac{C^2}{H^2(a)}L^{(\Delta-2)} . \tag{32}$$

Barrow/Tsallis entropy - cosmological ctd.

Combining above equations one can express the Hubble parameter as

$$H(a) = H_0 \sqrt{\frac{\Omega_m a^{-3} + \Omega_r a^{-4}}{1 - \Omega_{BH}(a)}} \,. \tag{33}$$

Final relation for the Barrow Holographic dark energy reads (prime is derivative with respect to a):

$$\begin{aligned} a\Omega'_{BH}(a) &= \Omega_{BH}(a)\,(1-\Omega_{BH}(a)) \times \\ &\times \left[\left(1+\frac{\Delta}{2}\right)\mathcal{F}_r(a) + (1+\Delta)\,\mathcal{F}_m(a)\right. \\ &+ \left.(1-\Omega_{BH}(a))^{\frac{\Delta}{2(\Delta-2)}}\,\Omega_{BH}(a)^{\frac{1}{2-\Delta}}\,\mathcal{Q}(a)\right] , \end{aligned} \tag{34}$$

with $(\ldots)'$ meaning the derivative w.r.t a.

Barrow/Tsallis entropy - cosmological ctd.

with

$$\mathcal{F}_r(a) = \frac{2\Omega_r a^{-4}}{\Omega_m a^{-3} + \Omega_r a^{-4}}, \quad (35)$$

$$\mathcal{F}_m(a) = \frac{\Omega_m a^{-3}}{\Omega_m a^{-3} + \Omega_r a^{-4}},$$

$$\mathcal{Q}(a) = (2-\Delta)\left(H_0\sqrt{\Omega_m a^{-3} + \Omega_r a^{-4}}\right)^{\frac{\Delta}{2-\Delta}} C^{\frac{2}{\Delta-2}}.$$

This formula contains radiation and in the limit $\Omega_r \to 0$ one retrieves Eq. (14) from (Saridakis 2020).

The application of the Hubble horizon (21) into the dimensionless dark energy density parameter (32) gives yet another option for it which reads

$$\Omega_{BH} = \frac{C^2 c^2}{H^2} H^{2-\Delta} = C^2 c^2 H^{-\Delta}, \quad (36)$$

which was applied in MPD & Salzano 2020.

Barrow/Tsallis - cosmological dynamics.

Growth of both the DM and DE perturbations (e.g. Mehrabi 2018) while in the limit of no perturbations in DE can be described by the scale-invariant equations valid also for ΛCDM cosmology:

$$a^2\mathcal{H}^2\phi'' + (4a\mathcal{H}^2 + a\dot{\mathcal{H}})\phi' + (\mathcal{H}^2 + 2\dot{\mathcal{H}})\phi = 0,$$

$$\delta_{\rm m}'' + \frac{1}{a}\left(2 + \frac{\dot{\mathcal{H}}}{a\mathcal{H}^2}\right)\delta_{\rm m}' = \frac{3\mathcal{H}^2}{2}\Omega_{\rm m}\delta_{\rm m}, \qquad (37)$$

where: $\mathcal{H} = aH$ is the conformal Hubble parameter; δ_m is the density contrast parameter for DM; ϕ is the Bardeen potential coming from metric perturbation. For the choice of the initial conditions we follow Abramo et al. (2008): for the matter density contrast and its derivative respectively, we use

$$\delta_m(a_i) = -2\phi(a_i)\left(1 + \frac{1}{3\mathcal{H}(a_i)^2}\right), \frac{d\delta_m(a_i)}{da} = -\frac{2}{3}\frac{\phi(a_i)}{\mathcal{H}(a_i)^2}. \qquad (38)$$

Dynamical quantities

For the scalar field ϕ we set $\phi(a_i) = -6 \times 10^{-7}$ and assume $\phi'(a_i) = 0$, with $a_i = 0.01$. Our results are unaffected by reasonable changes of the exact initial value of the ϕ (Mehrabi et al. 2015).

The information about the **growth rate** of (matter) perturbations is then encoded in the quantity

$$f(a) = \frac{d \, \ln \delta_m(a)}{d \, \ln a} \,, \tag{39}$$

derived from solving the set of differential equations (37). Observations from galaxy clustering are able to measure the combination $f\sigma_8(a) = f(a) \cdot \sigma_8(a)$, where $\sigma_8(a)$ is the conventionally defined amplitude of the linear power spectrum on the scale of $8 \, h^{-1}$ Mpc:

$$\sigma_8(a) = \sigma_{8,0} \frac{\delta_m(a)}{\delta_m(1)} \,, \tag{40}$$

with $\sigma_{8,0}$ - the normalization factor at present time ($a = 1$ or equivalently $z = 0$).

Dynamical quantities

Comment: redshift space distortions (RSD) measurements are not cosmologically independent, which means that data points are provided for a given fiducial cosmology. Thus, in order to be used by us in our analysis for our Barrow DE model, we **rescaled them**, considering the following relation:

$$[f\sigma_8(z)]_{model} = [f\sigma_8(z)]_{data} \frac{H_{fid,\,data}(z) \cdot D_{A/fid,\,data}(z)}{H_{model}(z) \cdot D_{A/model}(z)}$$

where D_A is the angular diameter distance.

4. Constraining Barrow Nonextensive Holography by kinematic and dynamical data.

Data applied (Denkiewicz et al. 2023):

- Type Ia Supernovae (SNeIa) from the Pantheon sample;
- Cosmic Chronometers (CC);
- the “Mayflower” sample of Gamma Ray Bursts (GRBs);
- latest *Planck* 2018 release for Cosmic Microwave Background radiation (CMB) (shift parameter);
- Baryon Acoustic Oscillations (BAO) from several surveys.

Considered 2 cases:

- “full data”, where we join both early- (CMB + BAO data from SDSS) and late-time observations (SNeIa, CC, GRBs + BAO data from WiggleZ);
- “late-time” data set - includes only late-time data **(after recombination)**.

Data taken into account in Denkiewicz et al. (2023) with references:

Below: "✓" = given data is included in the final χ^2 function; *"Geo"* = "geometrical", only relates to cosmological background; "+*dyn*" includes solution of perturbation equations

name	geo-late	geo-full	geo-late+dyn	geo-full+dyn	ref.
Pantheon SNeIa	✓	✓	✓	✓	Brout et al. 2022
Cosmic Chronometers	✓	✓	✓	✓	Jiao et al. 2022
GRBs	✓	✓	✓	✓	Liu & Wei 2015
CMB	–	✓	–	✓	Zhai et al. 2020
SDSS-IV DR16 ELG	–	✓(BAO)	✓(RSD)	✓(BAO+RSD)	Tamone et al., deMattia et al. 2020
SDSS-III DR12 LRG	–	✓(BAO)	✓(RSD)	✓(BAO+RSD)	Alam et al. 2017
SDSS-IV DR16 LRG	–	✓(BAO)	✓(RSD)	✓(BAO+RSD)	Gil-Marin et al. 2020, Bautista et al. 2020
SDSS-IV DR16 LRG+Void	–	✓(BAO)	✓(RSD)	✓(BAO+RSD)	Nadathur et al. 2020
SDSS-IV DR16 Lyman α	–	✓(BAO)	–	✓(BAO)	du Mas des Bourboux et al. 2020
SDSS-IV DR16 QSO (BAO)	–	✓(BAO)	✓(RSD)	✓(BAO+RSD)	Hou et al. 2020, Neveux et al. 2020
SDSS-IV DR14 QSO (BAO)	–	✓(BAO)	✓(RSD)	✓(BAO+RSD)	Zhao et al. 2019
WiggleZ	✓(BAO)	✓(BAO)	✓(BAO+RSD)	✓(BAO+RSD)	Blake et al. 2012

Samples taken into account in Denkiewicz et al. (2023) with references:

name	geo-late	geo-full	geo-late+dyn	geo-full+dyn	ref.
2dFGRS	–	–	✓(RSD)	✓(RSD)	Song & Percival et al. 2008
6dFGS	–	–	✓(RSD)	✓(RSD)	Achitouv et al. 2017
6dFGS Voids	–	–	✓(RSD)	✓(RSD)	Achitouv et al. 2017
FASTSOUND	–	–	✓(RSD)	✓(RSD)	Okumura et al. 2016
GAMA	–	–	✓(RSD)	✓(RSD)	Blake et al. 2013
BOSS-WiggleZ	–	–	✓(RSD)	✓(RSD)	Marin et al. 2016
BOSS LOWZ	–	–	✓(RSD)	✓(RSD)	Lange et al. 2021
SDSS-IV DR15 LGR-SMALL	–	–	✓(RSD)	✓(RSD)	Chapman et al. 2021
SDSS DR7 MGS	–	–	✓(RSD)	✓(RSD)	Howlett et al. 2015
VIPERS Voids	–	–	✓(RSD)	✓(RSD)	Hawken et al. 2017
VIPERS	–	–	✓(RSD)	✓(RSD)	Mohammad et al. 2018
VIPERS+GGL	–	–	✓(RSD)	✓(RSD)	Jullo et al. 2019

Calculated quantities - explanation of the following table:

In the following table for each parameter we provide the median and the 1σ constraints. The columns show:

1. considered theoretical scenario;
2. dimensionless matter parameter, Ω_m;
3. dimensionless baryonic parameter, Ω_b;
4. dimensionless Hubble constant, h;
5. fiducial absolute magnitude, $\mathcal{M}$;
6. amplitude of the linear power spectrum at present time, $\sigma_{8,0}$;
7. Barrow entropic parameter, Δ;
8. holographic parameter, C;
9. amplitude of the weak lensing measurement (secondary derived parameter), $S_{8,0} = \sigma_{8,0}\sqrt{\Omega_m/0.3}$;
10. logarithm of the Bayes Factor, $\log \mathcal{B}^i_j$.

Results of statistical analysis ("geo" - geometrical, "late" - late time data, "dyn" - dynamical, BH1 - event horizon (not Hubble))

	Ω_m	Ω_b	h	$\mathcal{M}$	$\sigma_{8,0}$	Δ	C	$S_{8,0}$	$\log \mathcal{B}^i_j$
"Revision" of MPD & Salzano 2020									
LCDM (geo-late)	$0.293^{+0.016}_{-0.016}$	–	$0.713^{+0.013}_{-0.013}$	–	–	–	–	–	*0*
LCDM (geo-full)	$0.319^{+0.005}_{-0.005}$	$0.0494^{+0.0004}_{-0.0004}$	$0.673^{+0.003}_{-0.003}$	–	–	–	–	–	*0*
BH1 (geo-late)	$0.290^{+0.020}_{-0.019}$	–	$0.715^{+0.014}_{-0.013}$	–	–	$>$ **0.63**	$3.93^{+1.77}_{-1.88}$	–	$-0.71^{+0.03}_{-0.02}$
BH1 (geo-full)	$0.314^{+0.006}_{-0.006}$	$0.049^{+0.001}_{-0.001}$	$0.676^{+0.007}_{-0.007}$	–	–	$>$ **0.84**	$4.66^{+0.87}_{-1.07}$	–	$-0.05^{+0.03}_{-0.03}$
Updated and newest constraints from Denkiewicz et al. 2023									
LCDM (geo-late)	$0.321^{+0.015}_{-0.015}$	–	$0.730^{+0.010}_{-0.009}$	$-19.263^{+0.028}_{-0.028}$	–	–	–	–	*0*
LCDM (geo-full)	$0.318^{+0.007}_{-0.006}$	$0.0493^{+0.0006}_{-0.0006}$	$0.674^{+0.004}_{-0.004}$	$-19.437^{+0.012}_{-0.012}$	–	–	–	–	*0*
LCDM (geo-late+dyn)	$0.315^{+0.014}_{-0.014}$	–	$0.731^{+0.010}_{-0.010}$	$-19.263^{+0.028}_{-0.028}$	$0.770^{+0.018}_{-0.017}$	–	–	$0.790^{+0.023}_{-0.022}$	*0*
LCDM (geo-full+dyn)	$0.314^{+0.006}_{-0.005}$	$0.0490^{+0.0006}_{-0.0006}$	$0.677^{+0.004}_{-0.004}$	$-19.429^{+0.011}_{-0.011}$	$0.779^{+0.017}_{-0.017}$	–	–	$0.796^{+0.019}_{-0.019}$	*0*
BH1 (geo-late)	$0.300^{+0.020}_{-0.019}$	–	$0.729^{+0.010}_{-0.010}$	$-19.263^{+0.028}_{-0.029}$	–	$>$ **0.63**	$4.50^{+2.20}_{-2.13}$	–	$-0.39^{+0.02}_{-0.04}$
BH1 (geo-full)	$0.311^{+0.006}_{-0.006}$	$0.0486^{+0.0008}_{-0.0008}$	$0.679^{+0.006}_{-0.006}$	$-19.438^{+0.013}_{-0.013}$	–	$>$ **0.82**	$4.58^{+0.90}_{-1.16}$	–	$-2.99^{+0.04}_{-0.04}$
BH1 (geo-late+dyn)	$0.290^{+0.018}_{-0.017}$	–	$0.729^{+0.010}_{-0.010}$	$-19.261^{+0.028}_{-0.028}$	$0.791^{+0.022}_{-0.022}$	$>$ **0.69**	$5.31^{+1.97}_{-2.27}$	$0.778^{+0.021}_{-0.022}$	$-0.35^{+0.03}_{-0.03}$
BH1 (geo-full+dyn)	$0.307^{+0.006}_{-0.006}$	$0.0484^{+0.0009}_{-0.0008}$	$0.681^{+0.006}_{-0.005}$	$-19.431^{+0.013}_{-0.013}$	$0.777^{+0.017}_{-0.017}$	$>$ **0.86**	$4.89^{+0.76}_{-1.03}$	$0.786^{+0.020}_{-0.020}$	$-4.36^{+0.04}_{-0.04}$

Tension with other bounds on Barrow parameter Δ.

Our bound on Barrow parameter $\Delta > 0.86$ strongly points towards its maximum value $\Delta = 1$ in which case Barrow entropy **recovers extensivity** (though still remains nonadditive fulfilling the rule (18)) - this is why we call it **”nearly extensive Gibbs-like entropy”**.

However, this result is **in tension with all the other bounds** made so far in the literature. Possible explanations lie in the fact that we use directly **holographic principle (HP)** while most of the other papers use **gravity-thermodynamics conjecture (GT)** of Jacobson (1995), but not solely.

Jacobson method of obtaining gravity from thermodynamics relies on the fact that the Barrow entropy gives a general relativity-like gravity with a rescaled cosmological constant $\tilde{\Lambda} = \Lambda[(1+\Delta/2)A^{\Delta/2}]^{-1}$, and having the limit $\Delta \to 0$ as standard Λ. In view of that, it is already Λ-term dominated model which solves dark energy problem and is statistically preferable with some ”small correction” coming from the nonextensive entropy. This is what happens in most of these types of bounds which obtain $\Delta \sim 0$.

Tension with other bounds on Barrow parameter Δ.

Other evaluations are, among others:

- From obs. bounds on baryon asymmetry: Barrow entropy $\Delta \sim 0.005 - 0.008$ (Luciano & Giné arXiv: 2210.09755); Tsallis entropy $0.002 \lesssim \mid \delta - 1 \mid = \mid \Delta/2 \mid \lesssim 0.004$ (arXiv: 2204.02723);
- From Big-Bang nucleosynthesis: Tsallis entropy $1 - \delta < 10^{-5}$ (Ghoshal, Lambiase arXiv:2104.11296); Barrow entropy $\Delta \lesssim 1.4 \cdot 10^{-4}$ (Barrow, Basilakos, Saridakis PLB 815, 136134 (2021);
- GT approach applied to Pantheon + BAO (”late-time”) data: $\Delta \sim 10^{-4}$ (Leon et al. JCAP 12, 032 (2021) - radiation is neglected in spite of our formula (35);
- GT approach applied also to cosmol. perturbations with an extra scalar field acting as dark energy which is effectively Λ (Aghari, Sheykhi arXiv:2106.15551) - Tsallis $\delta \approx 0.9997$ ($\Delta = -0.0006 < 0$);
- GT application of Planck data to Barrow restricts to $\Delta \lesssim 10^{-4}$ (though assuming only 30 e-folds) (Luciano arXiv: 2301.12509)

Tension with other bounds on Barrow parameter Δ.

- HP approach; using SNeIa, CC, GRBs they obtain $\Delta \approx -1.68$ which is behind the domain of Barrow parameter though consistent with Tsallis $\delta \approx 0.16$ (Mangoudehi arXiv: 2211.17212);
- HP approach, usage of early-times data (BAO, CMB), no radiation, Tsallis $\delta \approx 1.07$ corresponding Barrow $\Delta \approx 0.14$ (Sadri arXiv: 1905.11210);
- HP approach, late-time data only, Barrow $\Delta \sim 0.09$ (Saridakis et al. JCAP 12, 012 (2018), Anagnostopoulos et al. EPJS 80, 826 (2020);
- HP approach, SNeIa + CC only, Barrow $\Delta \sim 0.06 \div 0.2$ (Adhikary et al. PRD 104, 123519 (2021));
- HP approach, different horizon + interaction between dark matter and dark energy, tested against combined Pantheon, SNIa, BAO, CMB, and GRB, getting $\delta \sim 1.360$ which translates into $\Delta \sim 0.72$ (Mamon arXiv: 2007.01591) - this is closest bound to ours!!!

Results of statistical analysis: H_0 and S_8 tensions.

- H_0-tension: all cases are consistent with *Planck*, except for the "geo-full" case, which also has a lower Ω_m.
- S_8-tension: generally expressed in terms of the quantity

$$S_{8,0} = \sigma_{8,0}\sqrt{\Omega_m/0.3}\,,$$

which for *Planck* is $S_{8,0} = 0.834 \pm 0.016$. Tension comes from late-time data (weak lensing, cosmic shear) which have lower values, in the range $[0.74 - 0.78]$. From our results S_8-tension is "ALLEVIATED": when using late-time we have a bit lower values; when including early-time we have a bit larger ones; but if you take into account the errors, they basically overlap at 1σ.

5. Discussion

- **All our data tests** lead to the conclusion that the Barrow fractal index Δ is **bound from below ($\Delta > 0.86$)** which means that cosmological horizon **should be of the fractal nature** and the entropy is peaking towards $\Delta \to 1$ which **would recover extensivity** but **still keeping nonadditivity**.
- Our results are **in strong tension** with all the other bounds on the Barrow/Tsallis entropy parameter.
- The ''standard'' limit of a non-fractal horizon, i.e. $\Delta = 0$, is excluded by the data.
- ΛCDM ($\Delta = 2$) is also excluded in Barrow holography (though admitted in Tsallis holography $\delta = 1 + \frac{\Delta}{2} = 2$)!
- Hubble measurements are basically consistent with Planck; S_8 tension can be **alleviated** slightly.
- The Bayes factor $\mathcal{B}^i_j$, given Jeffreys' scale $\ln \mathcal{B}^i_j$, the **new data disfavours Barrow entropy models** w.r.t. ΛCDM.